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Elliptic partial differential equations involving variable exponential operators with supercritical growth in variable exponential Lebesgue spaces

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Abstract

The article is dedicated to the investigation of existence of weak solutions to a quasilinear elliptic equation involving terms with variable exponents

$$-d/dx_i\ a_i\,(x, u, \nabla u) - \lambda b\,(x)\ |\,u\,|^{p(x)-2}\,u = f\,(x, u)$$

in the variable exponential Sobolev space. Applying the Galerkin method for pseudo-monotone operators, we establish conditions under which there exists at least one weak solution to the boundary problem for such equations. Also, we prove the uniqueness of the weak solution under some additional monotony conditions on operator A0 and function f in functional Sobolev space.

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1. Introduction

Let Ω be a bounded smooth open set in $R^n, n \geq 3$. We are studying the boundary nonstandard elliptic parametrized problems with the critical growth and measure data

$$-\sum_{i=1,\dots,n} \frac{\partial a_i(x,\nabla u)}{\partial x_i} - \lambda b(x)|u|^{\gamma(x)-2}u = \mu$$

for all $x \in \Omega$, and the boundary condition in the following form

$$u = 0$$

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for all $x \in \partial\Omega$, here $x \in \Omega$, and a measure $\boldsymbol{\mu}$ is given, and a real number λ is a small positive parameter such that $\boldsymbol{\lambda} \in \left(\mathbf{0}, \frac{\gamma_m c_p{}^{p_S - \gamma_m}}{4 c_p^{\gamma_m} p_S \|\boldsymbol{b}\|_{L^{\gamma(\cdot)}(\Omega)}}\right)$ and for $p, \gamma \in P^{\log}(\Omega)$, $p_S < \gamma_m \leq \gamma_S < p_m^*$. Here the function $\boldsymbol{a}: \boldsymbol{\Omega} \times \boldsymbol{R^n} \to \boldsymbol{R^n}$ satisfied modified the Leray-Lions type conditions 1) - 4) (see below), which combined with the restriction on the function $\boldsymbol{b}$ provide the possibility of applying the theory of pseudo-monotone operators to approximate problems generated by the elliptic parametrized problem. The assumption $p \in P^{\log}(\Omega)$, $1 < p_m \leq p_S < \infty$ guarantees the Poincare and Sobolev inequalities, which are necessary for proving prior estimates for establishing convergence of sequences of approximate solutions. We assume that function $b \in L^{\gamma(\cdot)}(\Omega)$ is strictly positive and $b \in L^{\frac{p^*(\cdot)}{p^*(\cdot)-\gamma(\cdot)}}(\Omega) \cap L^{\gamma(\cdot)}(\Omega), p^*(x) = \frac{np(x)}{n-p(x)}$ and exponential functions such that p, γ belongs to $P^{\log}$-class of functions, $p_S < \gamma_m \leq \gamma_S < p_m^*$.

The deep interest in the investigation of variable exponent partial differential equations with the critical growth and measure data is due to recent advances in elastic and fluid mechanics; and denoising and restoration image modeling [1, 3, 18]. For instance, a factor $\boldsymbol{b}$ can be considered a term that controls the speed of diffusion processes [3, 18]. In Zhang et al. [17] investigated elliptic systems of variable exponent Laplacian problems connected with denoising image problems, authors studied conditions of the existence and uniqueness of solutions by applying the Schauder fixed-point theorem. Since many models of electro-rheological fluid dynamics involve variable exponent growth conditions, the natural functional spaces to consider this type of problem are variable exponential Lebesgue-Sobolev spaces [5, 6, 7, 8, 16, 17, 22]. To deal with measure data we introduce the notion of variable exponent capacity, which is a generalization of a Newtonian capacity of potential theory, the difference is that a Newtonian capacity is locally defined whereas the Sobolev capacity is global [2, 4, 19, 20, 22]. To deal with such differential problems, we consider less regular entropy and renormalized solutions [19, 20]. Recently, Zhang and Zhang [20] considered the fractional variable Laplacian problem with measure data, authors employed energy methods and proved that the nonnegative renormalized solution is unique. The case $\boldsymbol{\mu} = |\boldsymbol{u}|^{\boldsymbol{p(x)-2}}\boldsymbol{u}$ was studied in [8] by Fan, Zhao and Zhang; the authors investigated the properties of eigenvalues of the Dirichlet problem for $\boldsymbol{p(x)}$-Laplacian with $|\boldsymbol{u}|^{\boldsymbol{p(x)-2}}\boldsymbol{u}$ term in the right side and showed that the eigenvalues set is a nonempty infinite set [10-14].

The main objective of the article is to prove the existence of entropy and renormalized solutions to the boundary nonstandard non-linear parametrized problems with the critical growth $p_S < \gamma_m \leq \gamma_S < p_m^*$ and measure data, with the convection term $b \in L^{\gamma(\cdot)}(\Omega)$, $p, \gamma \in P^{\log}(\Omega)$ with critical growth.

2. Some preliminary information on properties of the variable exponential capacity

Before the studied the existence of renormalized solutions to variable exponent boundary problems with measure data, we present some general notations, for more information, see [9] by Heinonen, Martio, and Kilpelainen.

Definition 1: For a compact set K in Ω, we define the elliptic variable exponential capacity by the following expression

$$Cap_{p(\cdot)}(K) = \inf\{\rho_{p(\cdot)}(\nabla u) : u \in C_0^\infty(\Omega), u \geq 1_K\}. \tag{1}$$

For a a Borelian set B in Ω, we define the elliptic variable exponential capacity by the following expression

$$Cap_{p(\cdot)}(B) = \sup\{Cap_{p(\cdot)}(K) : K \text{ } is \text{ } compact, and \text{ } K \subset B\}. \tag{2}$$

For a Borelian subset B in Ω, the elliptic variable exponential capacity can be obtain by

$$Cap_{p(\cdot)}(B) = \\ = \inf\left\{\rho_{p(\cdot)}(\nabla u), u \in W_{1,0}^{p(\cdot)}(\Omega) : u \overset{Cap}{=} 1 \text{ } on \text{ } B, u \overset{Cap}{\geq} 0 \text{ } on \text{ } \Omega\right\}, \tag{3}$$

where the sign $\overset{\boldsymbol{Cap}}{=}$ means equality of functions $\boldsymbol{Cap}_{\boldsymbol{p}(\cdot)}$ in the sense of quasi-everywhere.

Proposition 1: Let E be a Borelian set of Ω. Then, a function $E \mapsto Cap_{p(\cdot)}(E)$ possesses the following qualities:

1) if $E_1 \subset E_2$ then we have $Cap_{p(\cdot)}(E_1) \leq Cap_{p(\cdot)}(E_2)$;
2) if sets K_1 and K_2 are compact then the we obtain
$$Cap_{p(\cdot)}(K_1 \cup K_2) + Cap_{p(\cdot)}(K_1 \cap K_2) \leq Cap_{p(\cdot)}(K_1) + Cap_{p(\cdot)}(K_2); \tag{4}$$
3) if $E = \bigcup_{j=1,\ldots} E_j$ then we have
$$Cap_{p(\cdot)}(E) \leq \sum_{j=1,\ldots} Cap_{p(\cdot)}(E_j). \tag{5}$$

Proof: Statement 1) is obvious.

Let K_1 and K_2 be compact. A set $W(K)$ consists of all smooth functions $u \in C_0^\infty(\Omega)$ and $u \geq 1_K\}$. For all functions $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{W}(\boldsymbol{K_1})$, we obtain

$$\rho_{p(\cdot)}(\nabla(\max\{u, v\})) + \rho_{p(\cdot)}(\nabla(\min\{u, v\})) = \\ = \rho_{p(\cdot)}(u) + \rho_{p(\cdot)}(v).$$

Since $\boldsymbol{min}\{\boldsymbol{u}, \boldsymbol{v}\}$ and $\boldsymbol{max}\{\boldsymbol{u}, \boldsymbol{v}\}$ are admissible functions for $\boldsymbol{K_1} \cup \boldsymbol{K_2}$ and $\boldsymbol{K_1} \cap \boldsymbol{K_2}$, we obtain

$$Cap_{p(\cdot)}(K_1 \cup K_2) + Cap_{p(\cdot)}(K_1 \cap K_2) \leq \rho_{p(\cdot)}(u) + \rho_{p(\cdot)}(v)$$

and

$$Cap_{p(\cdot)}(K_1 \cup K_2) + Cap_{p(\cdot)}(K_1 \cap K_2) \leq$$
$$\leq Cap_{p(\cdot)}(K_1) + Cap_{p(\cdot)}(K_2).$$

We assume that K, D and F are a compact sets, such that subset K in D, and we obtain

$$Cap_{p(\cdot)}(D \cup F) + Cap_{p(\cdot)}(K) \leq$$
$$\leq Cap_{p(\cdot)}\big(D \cup (K \cup F)\big) + Cap_{p(\cdot)}\big(D \cap (K \cup F)\big) \leq$$
$$\leq Cap_{p(\cdot)}(D) + Cap_{p(\cdot)}(K \cup F)$$

so

$$Cap_{p(\cdot)}(D \cup F) - Cap_{p(\cdot)}(K \cup F) \leq Cap_{p(\cdot)}(D) - Cap_{p(\cdot)}(K).$$

Employing induction process, we deduce the inequality

$$Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} D_j\big) - Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} K_j\big) \leq$$
$$\leq \sum_{j=1}^{m} \Big(Cap_{p(\cdot)}(D_j) - Cap_{p(\cdot)}(K_j)\Big).$$

For open sets E_j and F_j, we assume that $\bigcup_{j=1}^{m} E_j \supset D$ and $F_j \supset K_j$ are compacts, which satisfy condition $\bigcup_{j=1}^{m} K_j \supset D$. We obtain that compact set

$$D_j = D \setminus \cup\ k = 1, \dots, m, k \neq jK_k \subset E_j$$

contains K_j for all $j = 1, .., m$, we have $D = \bigcup_{k=1}^{m} D_k$. Hence, for all open E_j, $j = 1, .., m$, the estimate

$$Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} E_j\big) - Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} F_j\big) \leq$$
$$\leq \sum_{j=1}^{m} \Big(Cap_{p(\cdot)}(E_j) - Cap_{p(\cdot)}(F_j)\Big)$$

holds for all sets $F_j \subset E_j$, $j = 1, \dots, m$, which satisfy the requirement that $Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} F_j\big)$ is finite. Let E_j be such that F_j are subsets of E_j for all $j = 1, \dots, m$ and let the capacity $Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} F_j\big)$ is finite number. Then, we obtain the estimate

$$Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} E_j\big) - Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} F_j\big) \leq$$
$$\leq \sum_{j=1}^{m} \Big(Cap_{p(\cdot)}(E_j) - Cap_{p(\cdot)}(F_j)\Big).$$

So, we have

$$Cap_{p(\cdot)}\big(\bigcup_{j=1}^{m} E_j\big) \leq \sum_{j=1}^{m} Cap_{p(\cdot)}(E_j).$$

Repeating arguments similar to that above, we take the limit as m approaches infinity. So, we obtain the estimate by infinite series

$$Cap_{p(\cdot)}(E) \leq \sum_{j=1}^{\infty} Cap_{p(\cdot)}(E_j).$$

Let $\mathrm{M}_B(\Omega)$ denote the linear space consisting of all bounded measures, and let $\mathrm{M}_B^+(\Omega)$ denote the linear subspace consisting of all non-negative measures of $\mathrm{M}_B(\Omega)$.

Definition 2: We define $\mathrm{M}_0(\Omega)$ as a set of all measures μ of $\mathrm{M}_B(\Omega)$, which satisfy the equality $\mu(E) = 0$ for all sets E in Ω that have zero capacity, namely, $Cap_{p(\cdot)}(E) = 0$. The subset that consists of all non-negative measures of $\mathrm{M}_0(\Omega)$ will be denoted by $\mathrm{M}_0^+(\Omega)$.

Now, we prove the following proposition.

Proposition 2: Assume that $Cap_{p(\cdot)}(E) = 0$, we have $meas(E) = 0$.

Proof: We presume that a set A has zero capacity i.e., $Cap_{p(\cdot)}(E) = 0$. Then, for any $\varepsilon > 0$, we can find an open set $U \supset E$, which satisfies inequality $Cap_{p(\cdot)}(U) < \varepsilon$. We can select a compact K and ψ of the space $W(K)$ consisting of all smooth functions $u \in C_0^\infty(\Omega)$ such that $u \geq 1_K$, and for which we have the following estimate

$$\rho_{p(\cdot)}(\nabla u) \leq Cap_{p(\cdot)}(K) + \varepsilon < 2\varepsilon.$$

By our assumption $p \in P^{\log}(\Omega)$, applying the Poincare Lemma, we deduce

$$meas(K) \leq \rho_{p(\cdot)}(u) \leq c_{\log}(p) d(\Omega) \rho_{p(\cdot)}(\nabla u) < c_{\log}(p) d(\Omega) \varepsilon$$

hence $meas(U) \leq c_{\log}(p) d(\Omega)\varepsilon$, where we denote $d(\Omega)$ the diameter of Ω. We obtain $meas(E) = 0$. Proposition 2 has been proven.

Remark: In case of the general variable exponent, the Poincare Lemma does not have place, therefore, proposition 2 fails in general case.

Theorem 1: *Let μ be a bounded measure on a Borelian σ-algebra. Then, measure μ is positive if and only if μ belongs to $L^1 + W_{-1}^{q(\cdot)}$.*

Proof: Let μ be a Borelian measure from $L^1 + W_{-1}^{q(\cdot)}$, then we obtain that $\mu \in \mathrm{M}_0(\Omega)$. Applying the Hahn theorem, we can presume measure μ is positively defined. There is a compact set sequence $\{K_j\}$ in 2^Ω, K_j in Ω and such that $\Omega = \bigcup_j K_j$. For each $\mu \in \mathrm{M}_0(\Omega)$, we can choose a positively defined Borelian function $\psi \in L^1(\Omega, \eta)$ and measure η from $W_{-1}^{q(\cdot)}$. We define a sequence $\tilde{\mu}_j = T_j\left(1_{K_j}\psi\right)\eta$, where we define the truncated operator $T_j(s) = \max\{\min\{j, s\}, -j\}$ for all $j \geq 0, s \in R$.

For each $j \geq 0$, we take $\mu_j = \tilde{\mu}_j - \tilde{\mu}_{j-1}$ and initial measure $\mu_0 = \tilde{\mu}_0$. Then, we have $\mu = \sum_j \mu_j$ belongs to $\mathrm{M}_B(\Omega)$ hence the series $\sum_{j\in N} \left\|\mu_j\right\|_{\mathrm{M}_B(\Omega)}$. is finite.

Let $\{\Upsilon_j\}$ be a sequence of mollifiers, which satisfy the limiting condition $\Upsilon_j * \mu_m \underset{j\to\infty}{\to} \mu_m$ in the topology of $W_{-1}^{q(\cdot)}(\Omega)$. We select a number j such that functions $\Upsilon_j * \mu_m$ belong to the space $C_0^\infty(\Omega)$ and

$$\left\|\mu_m - \Upsilon_j * \mu_m\right\|_{W_{-1}^{q(\cdot)}} \leq \frac{1}{2^m}$$

hold. Thus, we can present $\mu_m = f_m + g_m$, where $f_m = \Upsilon_j * \mu_m$ and $g_m = \mu_m - \Upsilon_j * \mu_m$. Thus, we deduce that function $g = \sum_j g_j$ belongs to $W_{-1}^{q(\cdot)}(\Omega)$. We have

$$\|f_m\|_{L^1(\Omega)} = \left\|\Upsilon_j * \mu_m\right\|_{L^1(\Omega)} \leq \left\|\mu_j\right\|_{\mathrm{M}_B(\Omega)},$$

and we deduce $f = \sum_{j \in N} f_j \in L^1(\Omega)$. Thus, there exists a sum representation $\mu = f + g$. Theorem 1 has been proven.

Theorem 2: *We assume that a Radon measure* μ *belongs to* $M_B^+(\Omega)$. *Then, there exist a measure* μ_0 *of* $W_{-1}^{q(\cdot)}(\Omega)$, *a function* F *of* $L^1(\Omega)$, *a measure* μ_2 *of* $M_B^+(\Omega)$ *such that, for all* μ*-measurable* E, *the equality*

$$\mu_2(E) = \mu(E \cap N)$$

holds with some Borel N *that has zero capacity.*

Proof: Assuming measure μ be finite, we define a finite number $\alpha = \sup_N\{\mu(N): Cap_{p(\cdot)}(N) = 0\}$. We obtain that there is a decomposition representation $\tilde{\mu} = \tilde{\mu}_0 + \tilde{\mu}_1$, where measure $\tilde{\mu}_0$ of $\mathrm{M}_0(\Omega)$ and measure $\tilde{\mu}_1$ satisfies the theorem condition: $\tilde{\mu}_1(E) = \tilde{\mu}(E \cap N)$, where a Borel set N with zero capacity.

Let $\{E_j\}$ be a sequence of Borel sets such that each set E_i in set E_j for $i \geq j$ and

$$Cap_{p(\cdot)}(E_j) = 0,$$

and the sequence $\tilde{\mu}(E_j)$ tends to α as j tends to infinity. Thus, the union $\bigcup_{j=1,\dots,\dots} E_j$ is Borelian, and we get

$$Cap_{p(\cdot)}\left(\bigcup_{j \in N} E_j\right) = 0$$

and

$$\tilde{\mu}\left(\bigcup_{j \in N} E_j\right) = \alpha.$$

Therefore,

$$\tilde{\mu}\left(E \setminus \bigcup_{j \in N} E_j\right) = 0,$$

where E is a Borel set such that $\tilde{\mu}(E) = 0$. Therefore, we take $\tilde{\mu}_0 = 1_{\Omega \setminus \bigcup_{j \in N} E_j}\tilde{\mu}$ and $\tilde{\mu}_1 = 1_{\bigcup_{j \in N} E_j}\tilde{\mu}$, where $N = \bigcup_{j \in N} E_j$. Thus, Theorem 1 yields Theorem 2.

3. Boundary elliptic problem for the differential equation with reactional term with critical growth and measure data

We consider the boundary problem for the equation involving terms with variable exponent

$$-\sum_{i=1,\dots,n} \frac{\partial a_i(x,\nabla u)}{\partial x_i} - \lambda b(x) u|u|^{\gamma(x)-2} = \mu, \tag{6}$$

$$u|_{\partial\Omega} = 0, \tag{7}$$

for all $x \in \Omega$, where boundary of the domain Ω is smooth and $\mu(E) = 0$ for all sets E of zero capacity, and for some positive λ. Assume $p, \gamma \in P^{\log}(\Omega)$, $1 < p_m \leq p_S < \infty$ and $0 < \varepsilon \leq b(x)$, $b \in L^{\frac{p^*(\cdot)}{p^*(\cdot)-\gamma(\cdot)}}(\Omega) \cap L^{\gamma(\cdot)}(\Omega), p^*(x) = \frac{np(x)}{n-p(x)}$ if $p(x) < n$ and $p^*(x) = \infty$ otherwise; also we assume $p_S < \gamma_m \leq \gamma_S < p_m^*$. A small parameter must be chosen such that

$$\lambda \in \left(0, \frac{\gamma_m c_\rho{}^{p_S-\gamma_m}}{4c_p^{\gamma_m} p_S \|b\|_{L^{\gamma(\cdot)}(\Omega)}}\right).$$

These conditions are generalizations of M. Ding, C. Zhang, and S. Zhou conditions [5], also [15 -21].

We presume the following conditions:

1) $a: \Omega \times R^n \to R^n$ is a vector-value function, belongs to Caratheodory's class of functions;
2) the elliptic inequality $\sum_i^n a_i(x,\xi)\xi_i \geq \nu|\xi|^{p(x)}$ holds for all $x \in \Omega$, $\xi \in R^n$ for $\nu > p_m = \inf p(x)$;
3) the estimate above
$$|a(x,\xi)| \leq \alpha|\xi|^{p(x)-1} + \alpha_1(x);$$
and
4) the monotony condition
$$(a(x,\xi_1) - a(x,\xi_2), \xi_1 - \xi_2) > \nu_1|\xi_1 - \xi_2|^{p(x)}$$
hold for almost all $x \in \Omega$, for every pair of distinct different vectors ξ_1 and ξ_2 of R^n, and some $\nu > p_m$ and $\alpha > 0$, $\nu_1 > 0$, and $\alpha_1 \in L^{q(\cdot)}$ such that $\alpha(\cdot) \geq 0$.

4. Formalization of definitions of entropy and renormalized solutions to our problem

To define entropy and renormalized solutions, we generalized the concept of gradient operator. Assume $T_j(u) \in W_{1,0}^{p(\cdot)}(\Omega)$ for index $j > 0$ and a measurable function u, which is bounded almost everywhere. We define the gradient operator ∇u mapping from Ω to R^n of a function u by the identity

$$\nabla T_j(u) \overset{a.e.}{=} 1_{\{x:\,|u(x)|<j\}} \nabla u \tag{8}$$

for positive j.

Definition 3: Assume measure μ belongs to M_B. A measurable function u is said to be an entropy solution to (6), (7) if:

1) $T_j(u) \in W_{1,0}^{p(\cdot)}(\Omega)$ for all $j > 0$, and $|\boldsymbol{\nabla} u|^{p(\cdot)-1} \in L^{q(\cdot)}(\Omega)$ where $p(\cdot) + q(\cdot) = p(\cdot)q(\cdot)$;

2) the identity

$$\int_\Omega \sum_{i=1}^n a_i\left(x, \nabla T_j(u)\right) \nabla_i T_j(u-\varphi)dx \overset{def}{=}$$
$$\overset{def}{=} \int_\Omega \sum_{i=1}^n a_i\left(x, \nabla T_j(u)\right) \nabla_i T_j(u-\varphi)dx = \lambda \int_\Omega bu|u|^{\gamma(x)-2} T_j(u-\varphi)dx +$$
$$+\int_\Omega T_j(u-\varphi)Fdx + \int_\Omega \sum_{i=1}^n \Upsilon_i \nabla_i T_j(u-\varphi)dx \tag{9}$$

true for all $\varphi \in C_C^\infty(\Omega)$, $\mu \in \mathrm{M}_B(\Omega)$ and

$$\mu = F - \sum_{i=1}^n \frac{\partial \Upsilon_i}{\partial x_i},$$

here $\Upsilon \in \left(L^{q(\cdot)}(\Omega)\right)^n$, $F \in L^1(\Omega)$, for each $j > 0$.

Definition 4: Assume measure μ belongs to M_B. A measurable function u is said to be a renormalized solution to (6), (7) if:

1) for each $j > 0$, we have $T_j(u) \in W_{1,0}^{p(\cdot)}(\Omega)$ $|\boldsymbol{\nabla} u|^{p(\cdot)-1} \in L^{q(\cdot)}(\Omega)$ where $p(\cdot) + q(\cdot) = p(\cdot)q(\cdot)$; and

$$\lim_{j\to\infty} \int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} a(x, \nabla u)\nabla u dx = 0 \tag{10}$$

and local integrable condition

$$b(\cdot)|u|^{\gamma(\cdot)-1} \in L^1_{Loc}; \tag{11}$$

2) for each element h of $W^{2,\infty}(R)$, whose support is compact set, we have

$$\sum_{i=1}^n \int_\Omega a_i(x,\nabla u)h'(u)\nabla_i \varphi dx + \sum_{i=1}^n \int_\Omega a_i(x,\nabla u)h''(u)\varphi \nabla_i u dx =$$
$$= \lambda \int_\Omega bu|u|^{\gamma(x)-2} h'(u)\varphi dx +$$
$$+\int_\Omega h'(u)F\varphi dx + \sum_{i=1}^n \int_\Omega \Upsilon_i h'(u)\nabla_i \varphi dx + \sum_{i=1}^n \int_\Omega \Upsilon_i h''(u)\nabla_i u dx$$

for all $\varphi \in C_C^\infty(\Omega)$.

5. The establishing of the existence of entropy and renormalized solutions to the boundary problem

We will establish the existence of entropy and renormalized solutions to equations involving variable exponent and with the right part.

Theorem 3: *Let $p \in P^{log}(\Omega)$ and $1 < p_m \le p_S < \infty$. Let conditions 1) - 4) be satisfied. Assume measure μ belongs to M_B, and does not charge sets of zero $p(\cdot)$- capacity.*

Then, there is at least one entropy solution u to (6), (7) for all

$$\lambda \in \left(0, \frac{\gamma_m {c_\rho}^{p_S-\gamma_m}}{4c_p^{\gamma_m} p_S \|b\|_{L^{\gamma(\cdot)}(\Omega)}}\right).$$

Proof: We introduce the approximate sequence of (6) and (7) by

$$-\sum_{i=1,\dots,n} \frac{\partial a_i(x,\nabla u_k)}{\partial x_i} = \lambda b(x)u_k|u_k|^{\gamma(x)-2} + F_k - \sum_{i=1,\dots,n} \frac{\partial (\Upsilon_k)_i}{\partial x_i}, \tag{13}$$

$$u_k(x) = 0 \tag{14}$$

on the boundary $x \in \partial\Omega$, here F_k of $C_C^\infty(\Omega)$ and Υ_k of $\left(C_C^\infty(\Omega)\right)^n$, for all $k \in N$, such that $F_k \underset{k\to\infty}{\overset{L^1(\Omega)}{\to}} F$, $\Upsilon_k \underset{k\to\infty}{\overset{\left(L^{q(\cdot)}(\Omega)\right)^n}{\to}} \Upsilon$ and $\|F_k\|_{L^1(\Omega)} \le \|F\|_{L^1(\Omega)}$, $\rho_{q(\cdot)}(\Upsilon_k) \le \rho_{q(\cdot)}(\Upsilon)$.

We assume $p_S < \gamma_m \le \gamma_S < p_m^* < \infty$, $p^*(x) = \frac{np(x)}{n-p(x)}$ for all $p(x) < n$ or $p^*(x) = \infty$ for all $p(x) \ge n$.

If $p_S < n$ then there is a constant $c_p > 0$ such that the inequality $\|u\|_{L^{\gamma(\cdot)}(\Omega)} \le c_p \|u\|_{W_1^{p(\cdot)}(\Omega)}$ holds for all $u \in W_{1,0}^{p(\cdot)}(\Omega)$.

For the approximate sequence, we formulate the theorem.

Theorem (modified Lions) 4: *Let* $p \in P^{log}$ *and* $1 < p_m \le p_S < \infty$. *Let* $\gamma \in C_+\left(clos(\Omega)\right)$ *and* $p_S < \gamma_m \le \gamma_S < p_m^*, p_S < n$. *For all*

$$\lambda \in \left(0, \frac{\gamma_m c_\rho{}^{p_S-\gamma_m}}{4c_p^{\gamma_m} p_S \|b\|_{L^{\gamma(\cdot)}(\Omega)}}\right)$$

with $c_\rho \in (0,1)$, *the approximate sequence*

$$-\sum_{i=1,\dots,n} \frac{\partial a_i(x,\nabla u_k)}{\partial x_i} = \lambda b u_k |u_k|^{\gamma(x)-2} + F_k - \sum_{i=1,\dots,n} \frac{\partial (\Upsilon_k)_i}{\partial x_i},$$

$$u_k(x) = 0$$

on the boundary $x \in \boldsymbol{\partial}\Omega$, *have a nontrivial weak solution* $u_k \in W_{1,0}^{p(\cdot)}(\Omega)$ *for all* $k \in N$.

We can rewrite the approximate problems in the form

$$-\sum_{i=1,\dots,n} \frac{\partial a_i}{\partial x_i} - \lambda b(x) u_k |u_k|^{\gamma(x)-2} = F_k - \sum_{i=1,\dots,n} \frac{\partial (\Upsilon_k)_i}{\partial x_i} \tag{15}$$

for some positive $\lambda \in (0, \lambda_0)$ and some $F_k \in C_C^\infty(\Omega)$, $\Upsilon_k \in \left(C_C^\infty(\Omega)\right)^n$. The proof of the existence of a weak solution to the approximate problems is guaranteed by the application of the following proposition.

Proposition (modified Minty) 3: Assume X is a separable reflexive Banach space, and operator $A: X \to X^*$ is a bounded, coercive, and pseudo-monotone. Then, the equation $A(u) = w$ has at least one solution $u \in X$ for each $w \in X^*$.

The operator $A: X \to X^*$ is said to be pseudo-monotone if from $u_i \underset{i\to\infty}{\overset{weakly\ in\ X}{\to}} u$ and $\limsup_{i\to\infty} \langle A(u_i), u_i - u\rangle \le 0$ follows $A(u_i) \underset{i\to\infty}{\overset{weakly\ in\ X^*}{\to}} A(u)$ and $\langle A(u_i), u_i\rangle \overset{i\to\infty}{\to} \langle A(u), u\rangle$.

We assume $A(u) = -\sum_{i=1,\dots,n} \frac{\partial a_i(x,\nabla u_k)}{\partial x_i} - \lambda b(x) u |u|^{\gamma(x)-2}$ then the pseudo-monotony of A follows from the following estimates

$$\begin{aligned} &\int_\Omega b(x)|u|^{\gamma(x)-2} u |u-w| dx \le \\ &\le c\|b\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-\gamma(\cdot)}}} \|u\|_{L^{p^*(\cdot)}}^{\gamma_S-1} \|u-w\|_{L^{p^*(\cdot)}}, \end{aligned} \tag{16}$$

which holds for some constant $c > 0$ if we take $w = u_i$. Similarly, we obtain the following estimate

$$\begin{aligned}&\int_\Omega b\left(u|u|^{\gamma(x)-2} - w|w|^{\gamma(x)-2}\right)|u-w|dx \leq \\ &\leq c\|b\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-\gamma(\cdot)}}} \|u\|^{\gamma_S-1}_{L^{p^*(\cdot)}} \|u-w\|_{L^{p^*(\cdot)}} + \\ &+c\|b\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-\gamma(\cdot)}}} \|w\|^{\gamma_S-1}_{L^{p^*(\cdot)}} \|u-w\|_{L^{p^*(\cdot)}}\end{aligned} \tag{17}$$

for some constant $c > 0$ and we can apply the modified Minty proposition, from which follows the modified Lions theorem.

This modified Lion's theorem establishes the existence of a weak solution $u_k \in W^{p(\cdot)}_{1,0}(\Omega)$ to the approximate problem sequence these solutions are obviously entropy solutions for all $k \in N$. So, we have to show that there is a subsequence $\{u_{k(j)}\}$, denoted still as $\{u_k\}$, of the sequence $\{u_k\} \subset W^{p(\cdot)}_{1,0}(\Omega)$ of the sequence of approximate solutions that $\{u_k\}$ converge to a function u that is an entropy solution to the problem (6), (7).

We choose the test function $\varphi = T_j(u_k)$ and we calculate

$$\begin{aligned}&\sum_{i=1}^n \int a_i\left(x, \nabla T_j(u_k)\right) \nabla_i T_j(u_k) dx = \\ &= \int \left(\lambda b(x)|u_k|^{\gamma(x)-2} u_k T_j(u_k) + F_k T_j(u_k) + \sum_{i=1}^n (\Upsilon_k)_i \nabla_i T_j(u_k)\right) dx\end{aligned}$$

applying the Young inequality and structural conditions, we have

$$\begin{aligned}&\nu \rho_{p(\cdot)}\left(\nabla T_j(u_k)\right) \leq \\ &\leq \int_\Omega \left(\lambda b(x) u_k |u_k|^{\gamma(x)-2} T_j(u_k) + F_k T_j(u_k) + \sum_{i=1}^n (\Upsilon_k)_i \nabla_i T_j(u_k)\right) dx \leq \\ &\leq j\|F_k\|_{L^1(\Omega)} + \rho_{p(\cdot)}\left(\frac{\nabla T_j(u_k)}{p(\cdot)}\right) + \rho_{q(\cdot)}\left(\frac{\Upsilon_k}{q(\cdot)}\right) + \\ &+\lambda\left(\Upsilon \rho_{\frac{p(x)}{p(x)-\gamma(x)}}(b) + \frac{c}{\Upsilon}\rho_{p(x)}\left(T_j(u_k)\right)\right).\end{aligned}$$

since

$$\begin{aligned}&\int_\Omega b(x)\left|T_j(u_k)\right|^{\gamma(x)} dx \leq \frac{1}{\Upsilon}\int_\Omega |b(x)|^{\frac{p(x)}{p(x)-\gamma(x)}} dx + c\Upsilon \int_\Omega \left|T_j(u_k)\right|^{p(x)} dx \equiv \\ &\equiv \frac{1}{\Upsilon}\rho_{\frac{p(x)}{p(x)-\gamma(x)}}(b) + c\Upsilon \rho_{p(x)}\left(T_j(u_k)\right).\end{aligned}$$

Therefore, we have the gradient inequality

$$\begin{aligned}&\rho_{p(\cdot)}\left(\nabla T_j(u_k)\right) \leq \\ &\leq \left(\nu - \frac{1}{p_m} - \lambda\Upsilon c(n,p,\Omega)\right)^{-1}\left(j\|F_k\|_{L^1(\Omega)} + \rho_{q(\cdot)}\left(\frac{\Upsilon_k}{q(\cdot)}\right) + \lambda\frac{1}{\Upsilon}\rho_{\frac{p(x)}{p(x)-\gamma(x)}}(b)\right),\end{aligned}$$

where we employ the Poincare inequality statement: assume $p \in P^{\log}$, then obtain the inequality

$$\|u\|_{L^{p(\cdot)}} \leq c(n,p)d(\Omega)\|\nabla u\|_{L^{p(\cdot)}}$$

for all $u \in W_{1,0}^{p(\cdot)}(\Omega)$.

From the gradient inequality, we have that the subsequence $\{T_1(u_k)\}$ is bounded in $W_{1,0}^{p(\cdot)}(\Omega)$ so there are a subsequence $\{{u_k}^1\}$ of $\{u_k\}$ and a function $w_1 \in L^{q(\cdot)}(\Omega)$ such that $|w_1| \leq 1$ and $T_1({u_k}^1) \overset{k\to\infty}{\to} w_1$ in the topology of $L^{q(\cdot)}(\Omega)$-norm. Similarly, the sequence $\{T_2({u_k}^1)\}$ is bounded in $W_{1,0}^{p(\cdot)}(\Omega)$ therefore there are a subsequence $\{{u_k}^2\}$ of $\{{u_k}^1\}$ and a function $w_2 \in L^{q(\cdot)}(\Omega)$ such that $|w_1| \leq 2$ and $T_2({u_k}^2) \overset{k\to\infty}{\to} w_2$ in the topology of $L^{q(\cdot)}(\Omega)$-norm. Applying recursion, for each $j \in N$, we select $\{{u_k}^j\}$ of $\{{u_k}^{j-1}\}$ and a function $w_j \in L^{q(\cdot)}(\Omega)$, which $|w_j| \leq j$ and the limit $T_j({u_k}^j) \overset{k\to\infty}{\to} w_j$ converges in the topology of $L^{q(\cdot)}(\Omega)$ -norm. We introduce notation $u_k = {u_k}^k$ and obtain $T_j(u_k) \overset{k\to\infty}{\to} w_j$ in the topology of $L^{q(\cdot)}(\Omega)$ -norm, for all $j \in N$.

We assume $m < j, m, j \in N$ then we have $T_m\left(T_j(u_k)\right) = T_m(u_k)$ and $T_m\left(T_j(u_k)\right) \overset{k\to\infty}{\to} T_m(w_j) = w_m$.

Let u be a function defined by the following identity

$$u(x) = w_j(x)$$

for all $x \in \Omega$ such that $j - 1 \leq |w_j(x)| < j$ for each $j \in N$, and u equals zero on a sets of zero measure. Then, we have $w_j = T_j(u)$ for every $j \in N$.

In the definition, we select a function $\varphi = 0$, and calculate

$$\begin{aligned}&\sum_{i=1}^{n} \int_\Omega a_i\left(x, \nabla T_j(u)\right) \nabla_i T_j(u)dx = \\ &= \int_\Omega (\lambda b(x)u|u|^{\gamma(x)-2}T_j(u) + FT_j(u) + \sum_{i=1}^{n} \Upsilon_i \nabla_i T_j(u))dx,\end{aligned}$$

therefore, we conclude

$$\begin{aligned}&\rho_{p(\cdot)}\left(\nabla T_j(u_k)\right) \leq \\ &\leq c_\nu \left(j\|F_k\|_{L^1(\Omega)} + \rho_{q(\cdot)}\left(\frac{\Upsilon_k}{q(\cdot)}\right) + \lambda \frac{1}{\Upsilon} \rho_{\frac{p(x)}{p(x)-\gamma(x)}}(b)\right) \\ &\leq c_\nu \left(j\|F\|_{L^1(\Omega)} + \rho_{q(\cdot)}\left(\frac{\Upsilon}{q(\cdot)}\right) + \lambda \frac{1}{\Upsilon} \rho_{\frac{p(x)}{p(x)-\gamma(x)}}(b)\right) \equiv c(j, F, \Upsilon, b),\end{aligned}$$

where $c_\nu \equiv \left(\nu - \frac{1}{p_m} - \lambda \Upsilon c(n,p,\Omega)\right)^{-1}$.

By the Sobolev theorem and set identity $\{j \leq |u|\} = \{j \leq |T_j(u)|\}$, we deduce

$$meas\{x \in \Omega :\ j \leq |u(x)|\} \leq c(j, F, \Upsilon, b) j^{p_m^*(p_m^{-1}-1)}.$$

We obtain

$$\limsup_{i,l\to\infty} meas\{|u_l(x)-u_i(x)|\geq\varepsilon\}\leq c(F,j,b,\Upsilon)j^{p_m^*\frac{1-p_m}{p_m}}$$

for every $j\in N$, and we calulate

$$\limsup_{i,l\to\infty} meas\{|u_l(x)-u_i(x)|\geq\varepsilon\}=0.$$

Thus, there exists $\{u_k\}$ convergent almost everywhere in Ω to u as k approaches infinity.

For every $m>j$, we introduce notation $v_k=T_{2j}\left(u_k+T_j(u_k)-T_j(u)-T_m(u_k)\right)$. Assuming $d=4j+m$ so we have $\nabla v_k=0$, $|u_k|>j$. We choose v_k as a test function, so we obtain

$$\sum_{i=1}^n\int_\Omega a_i\left(x,\nabla T_d(u_k)\right)\nabla_i v_k dx=$$
$$=\int_\Omega\left(\lambda b u_k v_k|u_k|^{\gamma(x)-2}+F_k v_k+\sum_{i=1}^n(\Upsilon_k)_i\nabla_i v_k\right)dx.$$

Thus, we deduce

$$\sum_{i=1}^n\int_\Omega\left(a_i\left(x,\nabla T_j(u_k)\right)-a_i\left(x,\nabla T_j(u)\right)\right)\nabla_i\left(T_j(u_k)-T_j(u)\right)dx\leq$$
$$\leq\int_\Omega\lambda b(x)u_k v_k|u_k|^{\gamma(x)-2}dx+\int_{\{|u_k(x)|>j\}}\left|a\left(x,\nabla T_d(u_k)\right)\right|\left|\nabla T_j(u)\right|dx-$$
$$-\sum_{i=1}^n\int_\Omega a_i\left(x,\nabla T_j(u_k)\right)\nabla_i\left(T_j(u_k)-T_j(u)\right))dx+$$
$$+\int_\Omega\left(F_k v_k+\sum_{i=1}^n(\Upsilon_k)_i\nabla_i v_k\right)dx,$$

let $m=m(\varepsilon)$ be such large that the inclusions $\left|a\left(x,\boldsymbol{\nabla}T_m(u_k)\right)\right|\in L^{q(\cdot)}(\Omega)$ and $1_{x\in\Omega:\{|u_k|>j\}}\left|\boldsymbol{\nabla}T_j(u)\right|\overset{L^{p(\cdot)}(\Omega)}{\underset{k\to\infty}{\to}}0$ holds for all $j\in N$ then

$$\lim_{k\to\infty}\int_{\{|u_k|>j\}}\left|a\left(x,\nabla T_d(u_k)\right)\right|\left|\nabla T_j(u)\right|dx=0.$$

By taking v_k as $T_{2j}\left(u_k-T_m(u_k)\right)$, so we have

$$\sum_{i=1}^n\int a_i\left(x,\nabla T_d(u_k)\right)\left(\nabla_i\left(T_{2j}\left(u_k-T_m(u_k)\right)\right)\right)dx=$$
$$=\int\lambda b(x)|u_k|^{\gamma(x)-2}u_k T_{2j}\left(u_k-T_m(u_k)\right)dx+\int T_{2j}\left(u_k-T_m(u_k)\right)F_k dx+$$
$$+\sum_{i=1}^n\int(\Upsilon_k)_i\nabla_i\left(T_{2j}\left(u_k-T_m(u_k)\right)\right)dx,$$

therefore, the value of $\rho_{p(\cdot)}\left(\nabla\left(T_{2j}\left(u_k-T_m(u_k)\right)\right)\right)$ can be estimated above by $c_1(1+2j)$, where constant $c_1>0$ is depending on b but independent on m. Hence $T_{2j}\left(u_k-T_m(u_k)\right)$ weakly converges to $T_{2j}\left(u-T_m(u)\right)$ as $k\to\infty$, we obtain

$$\rho_{p(\cdot)}\left(\nabla\left(T_{2j}\left(u-T_m(u)\right)\right)\right)\leq c_1(1+2j)$$

therefore

$$\int_\Omega|\Upsilon|\left|\nabla\left(T_{2j}\left(u-T_m(u)\right)\right)\right|dx\leq const(j)\int_{\{x\in\Omega:\,|u|\geq m\}}|\Upsilon|^{q(x)}dx,$$

so that

$$\sum_{i=1}^{n} \int_{\Omega} \Upsilon_i \nabla_i \left(T_{2j}\big(u - T_m(u)\big) \right) dx \overset{m\to\infty}{\to} 0.$$

From the dominated convergence Lebesgue lemma, we obtain

$$\int_{\Omega} F T_{2j}\big(u - T_m(u)\big) dx \overset{m\to\infty}{\to} 0.$$

Hence $T_{2j}\left(u_k - T_m(u_k) + T_j(u_k) - T_j(u)\right)$ weakly converges to $T_{2j}\big(u - T_m(u)\big)$ as k approaches infinity, the inequality

$$\int_{\Omega} \sum_{i=1}^{n} \left(a_i\left(x, \nabla T_j(u_k)\right) - a_i\left(x, \nabla T_j(u)\right) \right) \nabla_i \left(T_j(u_k) - T_j(u) \right) dx \leq$$
$$\leq \int_{\Omega} \lambda b(x) u_k T_{2j}\big(u_k - T_m(u_k)\big) |u_k|^{\gamma(x)-2} dx +$$
$$+ \int_{\Omega} T_{2j}\big(u - T_m(u)\big) F dx + \int_{\Omega} \sum_{i=1}^{n} \Upsilon_i \nabla_i T_{2j}\big(u - T_m(u)\big) dx \leq \varepsilon$$

holds for large enough numbers k so we deduce

$$\sum_{i=1}^{n} \int_{\Omega} \left(a_i\left(\nabla T_j(u_k)\right) - a_i\left(\nabla T_j(u)\right) \right) \nabla_i \left(T_j(u_k) - T_j(u) \right) dx \overset{k\to\infty}{\to} 0.$$

Application of the Vitali theorem yields $\nabla u_k \overset{k\to\infty}{\to} \nabla u$ in the topology of $\left(L^{p(\cdot)}(\Omega)\right)^n$. Therefore, we $T_j(u_k) \overset{k\to\infty}{\to} T_j(u)$ in the topology of $W_{1,0}^{p(\cdot)}(\Omega)$ for all $j \in N$. Thus, we have obtained that the sequence $\{u_k\}$ almost everywhere converges to u as k approach to infinity, and the sequence $\{T_j(u_k)\}$ converges to u as k approach to infinity, in the topology of $W_{1,0}^{p(\cdot)}(\Omega)$-norm.

In the approximate problems

$$- \sum_{i=1}^{n} \frac{\partial a_i(x, \nabla u_k)}{\partial x_i} = \lambda b(x) u_k |u_k|^{\gamma(x)-2} + F_k - \sum_{i=1}^{n} \frac{\partial (\Upsilon_k)_i}{\partial x_i}, \tag{18}$$

we choose test element w_k as $T_j(u_k - \varphi)$, $j \in N$, where $\varphi \in W_{1,0}^{p(\cdot)} \cap L^{\infty}$ and assume $s = j + \|\varphi\|_{L^{\infty}(\Omega)}, k > s$ so that

$$\sum_{i=1}^{n} \int_{\Omega} a_i(x, \nabla u_k) \nabla_i T_j(u_k - \varphi) dx = \sum_{i=1}^{n} \int_{\Omega} a_i\big(x, \nabla T_s(u_k)\big) \nabla_i T_j(u_k - \varphi) dx$$

and

$$\sum_{i=1}^{n} \int_{\Omega} a_i\big(\nabla T_s(u_k)\big) \nabla_i \left(T_j(u_k - \varphi) \right) dx =$$
$$= \int_{\Omega} \lambda b(x) |u_k|^{\gamma(x)-2} u_k T_j(u_k - \varphi) dx +$$
$$+ \int_{\Omega} T_j(u_k - \varphi) F_k dx + \sum_{i=1}^{n} \int_{\Omega} (\Upsilon_k)_i \nabla_i \left(T_j(u_k - \varphi) \right) dx$$

and we assume that k approaches infinity, we have

$$\int_{\Omega} \sum_{i=1}^{n} a_i(x, \nabla u) \nabla_i \left(T_j(u - \varphi) \right) dx = \int_{\Omega} \lambda b(x) u T_j(u - \varphi) |u|^{\gamma(x)-2} dx +$$
$$+ \int_{\Omega} T_j(u - \varphi) F dx + \int_{\Omega} \sum_{i=1}^{n} \Upsilon_i \nabla_i \left(T_j(u - \varphi) \right) dx$$

for all functions $\varphi \in W_{1,0}^{p(\cdot)} \cap L^{\infty}$ and all $j \in N$.

Theorem 5: *Let $p \in P^{log}$. Assume $\mu \in M_B$ not charging sets of zero variable exponent capacity. Let conditions 1) - 4) be satisfied. Let*

$$\lambda \in \left(0, \frac{\gamma_m c_\rho{}^{ps-\gamma_m}}{4c_p^{\gamma_m} ps\|b\|_{L^{\gamma(\cdot)}(\Omega)}}\right).$$

Then, there exists at least one renormalized solution u to (6), (7), this renormalized solution coincides with the entropy solution.

Proof: Applying the results of theorem 4, we can assume that u is an entropy solution to (6), (7) so that $T_j(u) \in W_{1,0}^{p(\cdot)}(\Omega)$, $j \in N$, so we obtain that the sequence $\left\{\int_{\{x\in\Omega:\ j\le|u|\le j+1\}} |\nabla u|^{p(x)}dx\right\}$ converges to zero as j approaches to infinity, since the inequalities

$$\nu \int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} |\nabla u|^{p(x)}dx \le$$
$$\le \int_{\{x\in\Omega:\ j<|u(x)|\}} Fdx +$$
$$+\int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} \frac{|\nabla u|^{p(x)}}{p(\cdot)}dx + \int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} \frac{|\Upsilon|^{q(x)}}{q(\cdot)}dx +$$
$$+\lambda\left(\frac{1}{\upsilon}\int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} |b(x)|^{\frac{p(x)}{p(x)-\gamma(x)}}dx + c\upsilon\int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} |\nabla u|^{p(x)}dx\right)$$

and

$$\int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} |\nabla u|^{p(x)}dx \le$$
$$\le \frac{p_m}{\nu p_m - \lambda c\upsilon p_m - 1}\left(\int_{\{x\in\Omega:\ j<|u(x)|\}} Fdx + \int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} \frac{|\Upsilon|^{q(x)}}{q(\cdot)}dx +\right.$$
$$\left.+\lambda\frac{1}{\upsilon}\int_{\{x\in\Omega:\ j\le|u(x)|\le j+1\}} |b(x)|^{1+\frac{\gamma(x)}{p(x)-\gamma(x)}}dx\right)$$

hold with dual variable exponent $q(x) = \frac{p(x)}{p(x)-1}$.

Assume a proximation sequence $\{u_k\}$ belongs to $W_{1,0}^{p(\cdot)}(\Omega)$ then $T_j(u_k) \overset{k\to\infty}{\to} T_j(u)$ in the topology of $W_{1,0}^{p(\cdot)}(\Omega)$ for each $j \in N$. We assume that a function $h \in W^{1,\infty}(R)$ has a compact support on $[-M, M]$, where M is a positive number, then the identity

$$\int_\Omega \left(\sum_{i=1}^n a_i(x,\nabla u_k)h'(u_k)(u_k)\nabla_i\varphi + \sum_{i=1}^n a_i(x,\nabla u_k)h''(u_k)\varphi\nabla_i u\right)dx =$$
$$= \int_\Omega \lambda b|u_k|^{\gamma(x)-2}u_k h'(u_k)\varphi + \int_\Omega \varphi h'(u_k)F_k dx +$$
$$+\int_\Omega \sum_{i=1}^n (\Upsilon_k)_i h'(u_k)\nabla_i\varphi dx + \int_\Omega \sum_{i=1}^n \Upsilon_i h''(u)\nabla_i u dx.$$

holds for all $\varphi \in C_C^\infty(\Omega)$.

Since, we have

$$h'(u_k)a(\cdot,\nabla u_k) = h'(u_k)a(\cdot,\nabla T_M(u_k)),$$
$$h'(u)a(\cdot,\nabla T_M(u)) = h'(u)a(\cdot,\nabla u),$$

and

$$h''(u_k)a(\cdot,\nabla u_k) = h''(u_k)a(\cdot,\nabla T_M(u_k)),$$

$$h''(u)a\big(\cdot,\nabla T_M(u)\big) = h''(u)a(\cdot,\nabla u),$$

and we obtain the limits

$$u_k \overset{k\to\infty}{\to} u$$

almost everywhere in Ω,

$$T_j(u_k) \overset{k\to\infty}{\to} T_j(u)$$

in the topology of $W_{1,0}^{p(\cdot)}$-norm,

$$\left|\nabla T_j(u_k)\right|^{p(x)-2}\nabla T_j(u_k) \overset{k\to\infty}{\to} \left|\nabla T_j(u)\right|^{p(x)-2}\nabla T_j(u)$$

in the topology of $\left(L^{q(\cdot)}\right)^n$-norm. Thus, we obtain

$$h'(u_k)a\big(\cdot,\nabla T_M(u_k)\big) \overset{k\to\infty}{\to} h'(u)a\big(\cdot,\nabla T_M(u)\big)$$

in the topology of $\left(L^{q(\cdot)}\right)^n$-norm,

$$h''(u_k)a\big(\cdot,\nabla T_M(u_k)\big) \overset{k\to\infty}{\to} h''(u)a\big(\cdot,\nabla T_M(u)\big)$$

in the topology of L^1-norm, and

$$h'(u_k)a(\cdot,\nabla u_k) \overset{k\to\infty}{\to} h'(u)a(\cdot,\nabla u)$$

in the topology of $\left(L^{q(\cdot)}\right)^n$-norm, we obtain

$$h''(u_k)a(\cdot,\nabla u_k) \overset{k\to\infty}{\to} h''(u)a(\cdot,\nabla u)$$

in the topology of L^1-norm.

Thus, we conclude

$$\begin{aligned}&\textstyle\sum_{i=1}^n \int_\Omega \left(a_i(\nabla u)h'(u)\nabla_i\varphi + a_i(\nabla u)h''(u)\varphi\nabla_i u\right)dx = \\ &= \textstyle\int_\Omega \left(\lambda b|u|^{\gamma(x)-2}uh'(u)\varphi + \varphi h'(u)F\right)dx + \\ &+ \textstyle\sum_{i=1}^n \int_\Omega \Upsilon_i h'(u)\nabla_i\varphi dx + \sum_{i=1}^n \int_\Omega \Upsilon_i h''(u)\nabla_i u dx.\end{aligned}$$

We have obtained that the entropy solution is a renormalized solution.

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