

Dragomir-Agarwal type of inequalities involving fractional integrals of Atangana-Baleanu and generalized M-convex function

Ibrahim Slimane *

Faculty of Exact Sciences and Informatics

Abdelhamid Ibn Badis University

P. O. Box 227

27000 Mostaganem

Algeria

Muhammad Uzair Awan [†]

Zarnab Aslam [§]

Department of Mathematics

GC University Faisalabad

Faisalabad

Pakistan

Abstract

The major goal of the current article is to establish various novel fractional Dragomir-Agarwal inequalities utilizing Atangana-Baleanu fractional integral operator. We primarily employ the class of generalized M -convex functions to reach the major conclusions of the study. In the form of corollaries, we also gain some consequences from the major results.

Subject Classification: 26A33, 26D10, 26D15, 26A51.

Keywords: Atangana-Baleanu fractional integral, Hermite-Hadamard inequality, Dragomir-Agarwal, Generalized M -convex.

1. Introduction and basic concepts

Classical convexity and its properties have always remained an active area of research in mathematical analysis. This might be ascribed due to its several great implications in other subjects of applied sciences, such as optimization theory, management sciences, operator theory, etc. Although a mathematical model depending upon convexity have significance, a variety of physical

* E-mail: slimaneibrahim@outlook.fr (Corresponding Author)

† E-mail: awan.uzair@gmail.com

§ E-mail: zarnabaslam98@gmail.com

problems also exist, for which we have to use a non-convex mathematical model. Now, we define a convex set and convex function.

Let $\mathcal{P} \subset \mathbb{R}$, then it is called convex, if

$$(1 - \kappa)a_1 + \kappa a_2 \in \mathcal{P},$$

for all $a_1, a_2 \in \mathcal{P}$ and $0 \leq \kappa \leq 1$. Similarly a function $\mathcal{U} : \mathcal{P} \rightarrow \mathbb{R}$ is called convex, if

$$\mathcal{U}((1 - \kappa)a_1 + \kappa a_2) \leq (1 - \kappa)\mathcal{U}(a_1) + \kappa\mathcal{U}(a_2)$$

for all $a_1, a_2 \in \mathcal{P}$ and $\kappa \in [0, 1]$.

Consequently, broader extensions and numerous generalizations of traditional concepts related to convexity were constructed in existing research. It is worth mentioning here that sometimes the formulation of new general concepts of the usual convexity notion also enjoy several good features which usual convexity notion has. For some recent studies regarding convexity and its generalizations, see [8, 9, 19, 27]. If we look at the definition of convex functions, it is the comparison of means. Awan and his coauthors [4] proposed the concept of functions that satisfy M -convexity as:

Definition 1: ([4]) A mapping $\mathcal{U} : D \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$, is called M -convex, if

$$\mathcal{U}((1 - \kappa)Y + \kappa\Psi) \leq \mathcal{U}(Y) + (1 - \kappa)\mathcal{U}(\Psi)$$

for any $a, b \in D$ and $\kappa \in [0, 1]$, where $Y = \sqrt{ab}$ and $\Psi = \frac{a+b}{2}$.

Definition 2: ([4]) The mapping J exhibits log- M -convexity, if

$$\mathcal{U}(Y(-\kappa + 1) + \Psi\kappa) \leq \mathcal{U}(Y)^\kappa + \mathcal{U}(\Psi)^{-\kappa+1}$$

for all $a, b \in D$ and $0 \leq \kappa \leq 1$.

Definition 3: ([4]) The mapping $\mathcal{U}$ exhibits quasi- M -convexity, if

$$\mathcal{U}((1 - \kappa)Y + \kappa\Psi) \leq \max\{\mathcal{U}(Y), \mathcal{U}(\Psi)\}$$

for all $a, b \in D$ and $\kappa \in [0, 1]$.

It is clear from the notion of convexity that they are closely linked to the notion of inequalities. The idea of convex functions influenced the evolution of the analysis of inequalities significantly. Applying the convexity feature of the functions, we can readily generate some inequalities that we've seen in the literature. Hadamard's Hermite-inequality is a famous result in this area. It gives us a criterion for a function to be convex that is both necessary and sufficient. It is defined as:

Let us consider a convex function $\mathcal{U} : I = [d_1, d_2] \rightarrow \mathbb{R}$, then

$$\mathcal{U}\left(\frac{d_1+d_2}{2}\right) \leq \frac{1}{d_2-d_1} \int_{d_1}^{d_2} \mathcal{U}(u) du \leq \frac{\mathcal{U}(d_1) + \mathcal{U}(d_2)}{2}.$$

Pearce et al. prepared a really interesting book on certain recent advances and uses of Hadamard's Hermite-inequality. In the realm of numerical analysis and the notion of mean, the Hermite-Hadamard class of inequalities has several

implications [12]. There are several studies on Hadamard's Hermite-inequality [3, 9, 10, 11, 15, 16, 20, 28, 6, 29, 30].

Fractional calculus (FC) is an area of mathematics that investigates various real and or complex number powers of differential and integral operators. FC has become a multidisciplinary subject because of its various uses in engineering sciences. Several additional extensions of traditional FC ideas have been developed in the literature in recent decades (see [18]). Distinct fractional operators with some singular or non-singular kernels have been seen to play a vital role in various real-world challenges. Using Riemann-Liouville (RL) fractional integrals, Sarikaya et al. [22] found a fractional analog of Hermite-and Hadamard's Dragomir-type Agarwal's inequalities. This piqued scholars' interest, and as a result, numerous novel fractional analogs of Hadamard's Hermite-inequality were discovered through new and creative methods. For example, see [19, 17, 5, 14, 25, 31].

Before going any further, let's review certain formerly learned FC for nonsingular kernels. The FC literature contains some operators. The Caputo-Fabrizio (CF) derivative is useful for analyzing various physical problems [7], but it has the shortcoming of not being reducible to the original function. In 2016, Atangana and Baleanu were able to solve this issue (AB-operator for short). They suggested a modified fractional operator and an accompanying integral operator in the kernel, including the function of Mittag-Leffler, because the ML function is better suited to simulating nature and physical processes. The operator AB has been employed to analyze different models. Some applications of the AB-operators are listed in [1, 13, 23, 21, 24, 26]. Now, let us recall the AB-operators definition:

Definition 4: ([2]) Let $\mathcal{U} \in H^1(c, d)$, $d > c$ and $\mathcal{A} \in (0, 1]$ then, AB derivative of $\mathcal{U}$ in the Caputo sense is expressed as:

$${}_0^{ABC}D_k^{\mathcal{A}}[\mathcal{U}(k)] = \frac{\phi(\mathcal{A})}{1-\mathcal{A}} \int_0^k \mathcal{U}'(x) E_{\mathcal{A}} \left[-\mathcal{A} \frac{(k-x)^{\mathcal{A}}}{(1-\mathcal{A})} \right] dx. \quad (1)$$

Definition 5: ([2]) Let $\mathcal{U} \in H^1(c, d)$, $d > c$ then AB operator of $\mathcal{U}$ in the RL-case is expressed as:

$${}_0^{ABR}D_k^{\mathcal{A}}[\mathcal{U}(k)] = \frac{\phi(\mathcal{A})}{1-\mathcal{A}} \frac{d}{dk} \int_0^k \mathcal{U}(x) E_{\mathcal{A}} \left[-\mathcal{A} \frac{(k-x)^{\mathcal{A}}}{(1-\mathcal{A})} \right] dx. \quad (2)$$

Where $\phi(\mathcal{A})$ is a normalization function satisfying $\phi(0) = \phi(1) = 1$

Definition 6: ([2]) The AB fractional integral is expressed as

$${}_0^{AB}I_k^{\mathcal{A}}\mathcal{U}(k) = \frac{1-\mathcal{A}}{\phi(\mathcal{A})}\mathcal{U}(k) + \frac{\mathcal{A}}{\phi(\mathcal{A})\gamma(\mathcal{A})} \int_0^k \mathcal{U}(y)(k-y)^{\mathcal{A}-1} dy.$$

The motivation of the manuscript is to deduce some novel outcomes of fractional Dragomir-Agarwal type of inequalities involving AB fractional integral operators essentially utilizing the generalization of M -convexity. We also go through a few more particular cases of the acquired outcomes, demonstrating that they are extremely clarifying.

2. Main Results

We go over our major results in this part. Through out the paper, we assum that the mapping J is a differentiable on I , where $\mathcal{U} : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with $d > c$ and $\mathcal{U}'' \in L_1[c, d]$.

2.1 Generalized M -convexity

Let us introduce a class of functions that characterizes generalized M -convexity and provide its special forms.

Definition 7: (Generalized M -convexity) Consider a mapping $h: (0,1) \rightarrow \mathbb{R}$. Then we call $\mathcal{U} : D \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a M -convex function, if

$$\mathcal{U}((1-\kappa)Y + \kappa\psi) \leq h(\kappa)\kappa(\psi) + h(1-\kappa)\mathcal{U}(Y)$$

for all $a, b \in D$ and $\kappa \in (0,1)$, where $Y = \sqrt{ab}$ and $\psi = \frac{a+b}{2}$.

The following are the consequences of the above definition.

- I. If $h(\kappa) = \kappa$, then one achieve Definition 1 from Definition 7.
- II. Let $h(\kappa) = \kappa^s$, one gets the Breckner M -convex functions class .

Definition 8 (Breckner type of M -convex function) Let $\mathcal{U} : D \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$, the map $\mathcal{U}$ is considered to be Breckner M -convex function, if

$$\mathcal{U}((1-\kappa)Y + \kappa\psi) \leq (-\kappa + 1)^s \mathcal{U}(Y) + \kappa^s \mathcal{U}(\psi)$$

for any $a, b \in D$ and $\kappa \in [0,1], s \in (0,1]$.

- III. Put $h(\kappa) = \kappa^{-s}$ in Definition 7, one may achieve class of Godunova-Levin-Dragomir type of M -convex functions.

Definition 9: (Godunova-Levin-Dragomir type of M -convex function) We call the mapping $\mathcal{U}$ defined as above, a M -convex function in Breckner sense, if

$$\mathcal{U}((1-\kappa)Y + \kappa\psi) \leq (-\kappa + 1)^{-s} \mathcal{U}(Y) + \kappa^{-s} \mathcal{U}(\psi)$$

for every $a, b \in D$ and $\kappa \in (0,1), s \in [0,1]$.

- IV. Let us put $h(\kappa) = 1$ in Definition 7 one achieves the class of P type of M -convex functions.

Definition 10: (P type of M -convex function) The map $\mathcal{U}$ as defined in the preceding definition, is called a M -convex function in P sense, if

$$\mathcal{U}((1-\kappa)Y + \kappa\psi) \leq \mathcal{U}(Y) + \mathcal{U}(\psi)$$

for any $a, b \in D$ and $\kappa \in (0,1), s \in (0,1]$.

- V. In Definition 7, We substitute $h(\kappa) = \kappa(1-\kappa)$ to get the class of M -convex functions in tgs type.

Definition 11: (tgs type of M -convex function) We refer to the mapping $\mathcal{U}$ defined as in the previous definition, as M -convex function in tgs sense, if

$$\mathcal{U}((1-\kappa)Y + \kappa\psi) \leq (-\kappa + 1)\kappa[\mathcal{U}(\psi) + \mathcal{U}(Y)]$$

for any $a, b \in D$ and $\kappa \in (0,1)$.

2.2 Auxiliary Result

This part is devoted to the derivation of an auxiliary result. Let FI denotes the fractional integral.

Lemma 12: Let $\mathcal{U} : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function, where I is an interval with $c < d$. For AB FI operators, the following principle holds. $\mathcal{S}$

$$\begin{aligned} & {}^{AB}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) + {}^{AB}I_{\mathcal{V}_2}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) - \frac{2(1-\mathcal{A})\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} - \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})} \\ &= \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}+1}}{(\mathcal{A}+1)\phi(\mathcal{A})\Gamma(\mathcal{A})}\mathcal{U}'(\mathcal{V}_1) + \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})}\int_0^1 (-\mathfrak{w}+1)^{\mathcal{A}+1}\mathcal{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w}+1)\mathcal{V}_1)d\mathfrak{w} \\ &\quad - \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}+1}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})}\mathcal{U}'(\mathcal{V}_2) + \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})}\int_0^1 \mathfrak{w}^{\mathcal{A}+1}\mathcal{U}''(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w}+1)\mathcal{U})d\mathfrak{w}, \end{aligned}$$

where $\mathcal{A} \in [0,1]$, $\mathcal{U} \in I$, and $\Gamma(\cdot)$ denotes the Gamma function.

Proof: We can solve the problem by employing integration by parts to it. So

$$\begin{aligned} & \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}+1}}{\phi(\mathcal{A})\Gamma(\mathcal{A})}\int_0^1 (-\mathfrak{w}+1)^{\mathcal{A}}\mathcal{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w}+1)\mathcal{V}_1)d\mathfrak{w} \\ &= \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}+1}}{\phi(\mathcal{A})\Gamma(\mathcal{A})}\left[\frac{1}{\mathcal{A}+1}\mathcal{U}'(\mathcal{V}_1) + \frac{(-\mathcal{V}_1+\mathcal{U})}{\mathcal{A}+1}\int_0^1 (-\mathfrak{w}+1)^{\mathcal{A}+1}\mathcal{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w}+1)\mathcal{V}_1)d\mathfrak{w}\right]. \end{aligned} \quad (3)$$

Similarly

$$\begin{aligned} & -\frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}+1}}{\phi(\mathcal{A})\Gamma(\mathcal{A})}\int_0^1 \mathfrak{w}^{\mathcal{A}}\mathcal{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w}+1)\mathcal{V}_1)d\mathfrak{w} \\ &= \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}+1}}{\phi(\mathcal{A})\Gamma(\mathcal{A})}\left[\frac{1}{\mathcal{A}+1}\mathcal{U}'(\mathcal{V}_2) - \frac{(-\mathcal{V}_2+\mathcal{U})}{\mathcal{A}+1}\int_0^1 \mathfrak{w}^{\mathcal{A}+1}\mathcal{U}''(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w}+1)\mathcal{U})d\mathfrak{w}\right]. \end{aligned} \quad (4)$$

Addition of (3) and (4) gives:

$$\begin{aligned} & {}^{AB}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) + {}^{AB}I_{\mathcal{V}_2}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \\ &= \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}+1}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})}\mathcal{U}'(\mathcal{V}_1) + \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})}\int_0^1 (-\mathfrak{w}+1)^{\mathcal{A}+1}\mathcal{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w}+1)\mathcal{V}_1)d\mathfrak{w} \\ &\quad - \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}+1}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})}\mathcal{U}'(\mathcal{V}_2) + \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})}\int_0^1 \mathfrak{w}^{\mathcal{A}+1}\mathcal{U}''(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w}+1)\mathcal{U})d\mathfrak{w}. \end{aligned}$$

So, the proof is completed.

2.3 Results

In this portion, we infer certain MN -convex function outcomes incorporating the AB FI.

Theorem 13: If $|\mathcal{U}''|$ is MN -convex mapping, we have

$$\left| {}^{AB}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) + {}^{AB}I_{\mathcal{V}_2}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2+\mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})} \right|$$

$$\begin{aligned}
& - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+1} \mathcal{U}'(\mathcal{V}_1) - (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+1} \mathcal{U}'(\mathcal{V}_2)}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \Big| \\
& \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 (-\mathfrak{w} + 1)^{\mathcal{A}+1} [h(\mathfrak{w})|\mathcal{U}''(\mathcal{U})| + h(-\mathfrak{w} + 1)|\mathcal{U}''(\mathcal{V}_1)|] d\mathfrak{w} \\
& + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 \mathfrak{w}^{\mathcal{A}+1} [h(\mathfrak{w})|\mathcal{U}''(\mathcal{V}_2)| + |\mathcal{U}''(\mathcal{U})|h(-\mathfrak{w} + 1)] d\mathfrak{w},
\end{aligned}$$

where $\mathcal{U} \in I$ and $\mathcal{A} \in [0, 1]$.

Proof: Utilizing the Lemma 12, we can write

$$\begin{aligned}
& \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\
& = \left| \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 (-\mathfrak{w} + 1)^{\mathcal{A}+1} \mathcal{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w} + 1)\mathcal{V}_1) d\mathfrak{w} \right. \\
& + \left. \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 \mathfrak{w}^{\mathcal{A}+1} \mathcal{U}''(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w} + 1)\mathcal{U}) d\mathfrak{w} \right| \\
& \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 (-\mathfrak{w} + 1)^{\mathcal{A}+1} |\mathcal{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w} + 1)\mathcal{V}_1)| d\mathfrak{w} \\
& + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 \mathfrak{w}^{\mathcal{A}+1} |\mathcal{U}''(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w} + 1)\mathcal{U})| d\mathfrak{w}.
\end{aligned}$$

By using convexity of $|\mathcal{U}''|$, it yields

$$\begin{aligned}
& \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 (-\mathfrak{w} + 1)^{\mathcal{A}+1} [h(\mathfrak{w})|\mathcal{U}''(\mathcal{U})| + h(-\mathfrak{w} + 1)|\mathcal{U}''(\mathcal{V}_1)|] d\mathfrak{w} \\
& + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 \mathfrak{w}^{\mathcal{A}+1} [h(\mathfrak{w})|\mathcal{U}''(\mathcal{V}_2)| + h(-\mathfrak{w} + 1)|\mathcal{U}''(\mathcal{U})|] d\mathfrak{w}.
\end{aligned}$$

and hence the proof is finished.

The following are some consequences of Theorem 13.

I. Let $h(\mathfrak{w}) = \mathfrak{w}$, then

Corollary 14: Let the hypothesis of Theorem 13 holds, if $|\mathcal{U}''|^q$ is MN -convex, then

$$\begin{aligned}
& \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + \mathcal{U}(\mathcal{V}_2)(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}}}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\frac{|\mathcal{U}''(\mathcal{U})|}{(\mathcal{A}+2)(\mathcal{A}+3)} + \frac{|\mathcal{U}''(\mathcal{V}_1)|}{(\mathcal{A}+3)} \right] \\
& + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\frac{|\mathcal{U}''(\mathcal{V}_2)|}{(\mathcal{A}+3)} + \frac{|\mathcal{U}''(\mathcal{U})|}{(\mathcal{A}+2)(\mathcal{A}+3)} \right].
\end{aligned}$$

II. If $h(w) = w^s$, then

Corollary 15: If $|\mathcal{U}''|^q$ is Breckner type of MN -convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_u^{\mathcal{E}} \mathcal{U}(\mathcal{U}) + {}^AB I_u^{\mathcal{E}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} [\phi(s+1, \mathcal{E}+2)|\mathcal{U}''(\mathcal{U})| + \phi(1, \mathcal{E}+s+2)|\mathcal{U}''(\mathcal{V}_1)|] \\ & + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} [\phi(\mathcal{E}+s+2, 1)|\mathcal{U}''(\mathcal{V}_2)| + \phi(\mathcal{E}+2, s+1)|\mathcal{U}''(\mathcal{U})|]. \end{aligned}$$

III. If $h(w) = 1$, then

Corollary 16: If $|\mathcal{U}''|^q$ is P - MN -convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_u^{\mathcal{E}} \mathcal{U}(\mathcal{U}) + {}^AB I_u^{\mathcal{E}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2\mathcal{U}(\mathcal{U})(-\mathcal{E}+1)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\frac{1}{2+\mathcal{E}} [|\mathcal{U}''(\mathcal{V}_1)| + |\mathcal{U}''(\mathcal{U})|] \right] \\ & + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\frac{1}{\mathcal{E}+2} [|\mathcal{U}''(\mathcal{V}_1)| + |\mathcal{U}''(\mathcal{U})|] \right]. \end{aligned}$$

IV. Let $h(w) = w(-w+1)$, one gets:

Corollary 17: If $|\mathcal{U}''|^q$ is tg s- MN -convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_u^{\mathcal{E}} \mathcal{U}(\mathcal{U}) + {}^AB I_u^{\mathcal{E}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\frac{1}{(\mathcal{E}+3)(\mathcal{E}+4)} [|\mathcal{U}''(\mathcal{U})| + |\mathcal{U}''(\mathcal{V}_1)|] \right] \\ & + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\frac{1}{(\mathcal{E}+3)(\mathcal{E}+4)} [|\mathcal{U}''(\mathcal{U})| + |\mathcal{U}''(\mathcal{V}_1)|] \right]. \end{aligned}$$

Theorem 18: If $|\mathcal{U}''|^q$ is MN -convex mapping, one achieves:

$$\begin{aligned} & \left| \frac{{}^AB I_u^{\mathcal{E}} \mathcal{U}(\mathcal{U}) + {}^AB I_u^{\mathcal{E}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left(\frac{1}{(\mathcal{E}p+p+1)} \right)^{\frac{1}{p}} \left(\int_0^1 [h(w)|\mathcal{U}''(\mathcal{U})|^q + h(-w+1)|\mathcal{U}''(\mathcal{V}_1)|^q] dw \right)^{\frac{1}{q}} \\ & \leq \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left(\frac{1}{(\mathcal{E}p+p+1)} \right)^{\frac{1}{p}} \left(\int_0^1 [h(w)|\mathcal{U}''(\mathcal{V}_2)|^q + h(-w+1)|\mathcal{U}''(\mathcal{U})|^q] dw \right)^{\frac{1}{q}} \end{aligned}$$

where for $q > 1$, $p^{-1} + \frac{1}{q} = 1$, $\mathcal{U} \in I$.

Proof: With aid of Lemma 12, one reach to:

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\mathcal{U}}^{\mathcal{E}} \mathcal{U}(\mathcal{U}) + \frac{AB}{\mathcal{V}_2} I_{\mathcal{U}}^{\mathcal{E}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})} \right. \\ & \quad \left. - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+1} \mathcal{U}'(\mathcal{V}_1) - (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+1} \mathcal{U}'(\mathcal{V}_2)}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2\mathcal{U}(\mathcal{U})(-\mathcal{E}+1)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \int_0^1 (-w + 1)^{\mathcal{E}+1} |\mathcal{U}''(w\mathcal{U} + (-w + 1)\mathcal{V}_1)| dw \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \int_0^1 w^{\mathcal{E}+1} |\mathcal{U}''(w\mathcal{V}_2 + (-w + 1)\mathcal{U})| dw. \end{aligned}$$

By applying Hölder inequality, we get

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\mathcal{U}}^{\mathcal{E}} \mathcal{U}(\mathcal{U}) + \frac{AB}{\mathcal{V}_2} I_{\mathcal{U}}^{\mathcal{E}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})} \right. \\ & \quad \left. - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+1} \mathcal{U}'(\mathcal{V}_1) - (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+1} \mathcal{U}'(\mathcal{V}_2)}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\left(\int_0^1 (-w + 1)^{(\mathcal{E}+1)p} dw \right)^{\frac{1}{p}} \left(\int_0^1 |\mathcal{U}''(w\mathcal{U} + (-w + 1)\mathcal{V}_1)|^q dw \right)^{\frac{1}{q}} \right] \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\left(\int_0^1 w^{(\mathcal{E}+1)p} dw \right)^{\frac{1}{p}} \left(\int_0^1 |\mathcal{U}''(w\mathcal{V}_2 + (-w + 1)\mathcal{U})|^q dw \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Using notion of MN -convexity of $|\mathcal{U}''|^q$, one may conclude

$$\begin{aligned} \int_0^1 |\mathcal{U}''(w\mathcal{U} + (-w + 1)\mathcal{V}_1)|^q dw &= \int_0^1 [h(w)|\mathcal{U}''(\mathcal{U})|^q + h(-w + 1)|\mathcal{U}''(\mathcal{V}_1)|^q] dw \\ \int_0^1 |\mathcal{U}''(w\mathcal{V}_2 + (-w + 1)\mathcal{U})|^q dw &= \int_0^1 [h(-w + 1)|\mathcal{U}''(\mathcal{U})|^q + |\mathcal{U}''(\mathcal{V}_2)|^q h(w)] dw \end{aligned}$$

and hence we finished the proof.

The following are the consequences of Theorem 18. Let the hypothesis of Theorem 18 holds for all cases.

I. Let $h(w) = w$, one reach to the result as:

Corollary 19: If $|\mathcal{U}''|^q$ is MN -convex, then

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\mathcal{U}}^{\mathcal{E}} \mathcal{U}(\mathcal{U}) + \frac{AB}{\mathcal{V}_2} I_{\mathcal{U}}^{\mathcal{E}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})} \right. \\ & \quad \left. - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+1} \mathcal{U}'(\mathcal{V}_1) - (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+1} \mathcal{U}'(\mathcal{V}_2)}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left(\frac{1}{(\mathcal{E}p + p + 1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{U}''(\mathcal{U})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{2} \right)^{\frac{1}{q}} \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left(\frac{1}{(\mathcal{E}p + p + 1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\mathcal{U})|^q}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

II. If $h(w) = w^s$, then

Corollary 20: If $|\mathcal{U}''|^q$ is Breckner type of MN-convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{(\mathcal{A}p+p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{U}''(\mathcal{U})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{s+1} \right)^{\frac{1}{q}} \\ & + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{(\mathcal{A}p+p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\mathcal{U})|^q}{s+1} \right)^{\frac{1}{q}}. \end{aligned}$$

III. For $h(t) = 1$, one gets:

Corollary 21: If $|\mathcal{U}''|^q$ is P-MN-convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{(\mathcal{A}p+p+1)} \right)^{\frac{1}{p}} (|\mathcal{U}''(\mathcal{U})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q)^{\frac{1}{q}} \\ & + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{(\mathcal{A}p+p+1)} \right)^{\frac{1}{p}} |\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\mathcal{U})|^q)^{\frac{1}{q}}. \end{aligned}$$

IV. For $h(w) = w(-w+1)$, one achieves:

Corollary 22: If $|\mathcal{U}''|^q$ is tgs-MN-convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{(\mathcal{A}p+p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{U}''(\mathcal{U})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{6} \right)^{\frac{1}{q}} \\ & + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{(\mathcal{A}p+p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\mathcal{U})|^q}{6} \right)^{\frac{1}{q}}. \end{aligned}$$

Theorem 23: If $|\mathcal{U}''|^q$ is MN-convex mapping, we reach

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{J}(\mathcal{U}) + {}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{J}(\mathcal{U})}{\phi(\mathcal{A})} \right| \end{aligned}$$

$$\leq \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left(\frac{1}{p(\mathcal{E}p+p+1)} + \int_0^1 [h(\mathfrak{w})|\mathfrak{U}''(\mathcal{U})|^q + h(-\mathfrak{w}+1)|\mathfrak{U}''(\mathcal{V}_1)|^q]d\mathfrak{w} \right) \\ + \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left(\frac{1}{p(\mathcal{E}p+p+1)} + \int_0^1 [h(\mathfrak{w})|\mathfrak{U}''(\mathcal{V}_2)|^q + h(-\mathfrak{w}+1)|\mathfrak{U}''(\mathcal{U})|^q]d\mathfrak{w} \right),$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\mathcal{U} \in I$ and $q > 1$.

Proof: Utilizing Lemma 12, one gets

$$\left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_1) + (-\mathcal{V}_2+\mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathfrak{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ \leq \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \int_0^1 (-\mathfrak{w}+1)^{\mathcal{E}+1} |\mathfrak{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w}+1)\mathcal{V}_1)|d\mathfrak{w} \\ + \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \int_0^1 \mathfrak{w}^{\mathcal{E}+1} |\mathfrak{U}''(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w}+1)\mathcal{U})|d\mathfrak{w}.$$

From the Young inequality: $\mathcal{U}y \leq \frac{1}{p}\mathcal{U}^p + \frac{1}{q}y^q$, one gets

$$\left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_1) + (-\mathcal{V}_2+\mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathfrak{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ \leq \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\frac{1}{p} \int_0^1 (-\mathfrak{w}+1)^{(\mathcal{E}+1)p} d\mathfrak{w} + \frac{1}{q} \int_0^1 |\mathfrak{U}''(\mathfrak{w}\mathcal{U} + (-\mathfrak{w}+1)\mathcal{V}_1)|^q d\mathfrak{w} \right] \\ + \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\frac{1}{p} \int_0^1 \mathfrak{w}^{(\mathcal{E}+1)p} d\mathfrak{w} + \frac{1}{q} \int_0^1 |\mathfrak{U}''(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w}+1)\mathcal{U})|^q d\mathfrak{w} \right].$$

With use of MN -convexity of $|\mathfrak{U}''|^q$ and doing simple calculation, we archive the desired theorem.

We provide some corollaries which can be easily proved using Theorem 23.

I. For $h(\mathfrak{U}) = \mathfrak{U}$, one acquire

Corollary 24: If $|\mathfrak{U}''|^q$ is MN -convex, then

$$\left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_1) + (-\mathcal{V}_2+\mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathfrak{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ \leq \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left(\frac{1}{p(\mathcal{E}p+p+1)} + \frac{|\mathfrak{U}''(\mathcal{U})|^q + |\mathfrak{U}''(\mathcal{V}_1)|^q}{2q} \right) \\ + \frac{(-\mathcal{V}_2+\mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left(\frac{1}{p(\mathcal{E}p+p+1)} + \frac{|\mathfrak{U}''(\mathcal{V}_2)|^q + |\mathfrak{U}''(\mathcal{U})|^q}{2q} \right).$$

II. For $h(\mathfrak{w}) = \mathfrak{w}^s$, one obtain the corollary as follows:

Corollary 25: If $|\mathfrak{U}''|^q$ is Breckner type of MN -convex, then

$$\left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1+\mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_1) + (-\mathcal{V}_2+\mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathfrak{U}(\mathcal{U})}{\phi(\mathcal{E})} \right|$$

$$\begin{aligned}
& - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+1} \mathcal{U}'(\mathcal{V}_1) - (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+1} \mathcal{U}'(\mathcal{V}_2)}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{J}(\mathcal{U})}{\phi(\mathcal{A})} \Big| \\
& \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{p(\mathcal{A}p+p+1)} + \frac{|\mathcal{U}''(\mathcal{U})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{(s+1)q} \right) \\
& + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{p(\mathcal{A}p+p+1)} + \frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\mathcal{U})|^q}{(s+1)q} \right).
\end{aligned}$$

III. For $h(\mathfrak{w}) = 1$, we obtain the corollary as follows:

Corollary 26: *If $|\mathcal{U}''|^q$ is P -MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^{\mathcal{A}\mathcal{B}}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) + {}^{\mathcal{A}\mathcal{B}}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{p(\mathcal{A}p+p+1)} + |\mathcal{U}''(\mathcal{U})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q \right) \\
& + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{p(\mathcal{A}p+p+1)} + |\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\mathcal{U})|^q \right).
\end{aligned}$$

IV. For $h(\mathfrak{w}) = \mathfrak{w}(-\mathfrak{w} + 1)$, the result is:

Corollary 27: *If $|\mathcal{U}''|^q$ is tgs -MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^{\mathcal{A}\mathcal{B}}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) + {}^{\mathcal{A}\mathcal{B}}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{p(\mathcal{A}p+p+1)} + \frac{|\mathcal{U}''(\mathcal{U})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{6q} \right) \\
& + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{p(\mathcal{A}p+p+1)} + \frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\mathcal{U})|^q}{6q} \right).
\end{aligned}$$

Theorem 28: *If $|\mathcal{U}''|$ is MN-convex mapping, we have*

$$\begin{aligned}
& \left| \frac{{}^{\mathcal{A}\mathcal{B}}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) + {}^{\mathcal{A}\mathcal{B}}I_{\mathcal{U}}^{\mathcal{A}}\mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}}\mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{J}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\
& \times \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\int_0^1 (-\mathfrak{w} + 1)^{\mathcal{A}+1} [h(\mathfrak{w})|\mathcal{U}''(\mathcal{U})| + h(-\mathfrak{w} + 1)|\mathcal{U}''(\mathcal{V}_1)|] d\mathfrak{w} \right)^{\frac{1}{q}} \right] \\
& + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\
& \times \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\int_0^1 \mathfrak{w}^{\mathcal{A}+1} [h(\mathfrak{w})|\mathcal{U}''(\mathcal{V}_2)| + h(-\mathfrak{w} + 1)|\mathcal{U}''(\mathcal{U})|] d\mathfrak{w} \right)^{\frac{1}{q}} \right],
\end{aligned}$$

where $\mathcal{U} \in I$ and $q > 1$.

Proof: Utilizing Lemma 12, one have

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 (-w+1)^{\mathcal{A}+1} |\mathcal{U}''(w\mathcal{U} + (-w+1)\mathcal{V}_1)| dw \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \int_0^1 w^{\mathcal{A}+1} |\mathcal{U}''(w\mathcal{V}_2 + (-w+1)\mathcal{U})| dw. \end{aligned}$$

With the use of power mean inequality, we achieve

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\ & \quad \times \left[\left(\int_0^1 (-w+1)^{\mathcal{A}+1} dw \right)^{1-\frac{1}{q}} \left(\int_0^1 (-w+1)^{\mathcal{A}+1} |\mathcal{U}''(w\mathcal{U} + (-w+1)\mathcal{V}_1)|^q dw \right)^{\frac{1}{q}} \right] \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\left(\int_0^1 w^{\mathcal{A}+1} dw \right)^{1-\frac{1}{q}} \left(\int_0^1 w^{\mathcal{A}+1} |\mathcal{U}''(w\mathcal{V}_2 + (-w+1)\mathcal{U})|^q dw \right)^{\frac{1}{q}} \right]. \end{aligned}$$

By using MN -convexity of $|\mathcal{U}''|^q$, we obtain

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathcal{U}}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) + {}^AB I_{\mathcal{V}_2}^{\mathcal{A}} \mathcal{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathcal{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathcal{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\ & \quad \times \left[\left(\int_0^1 (-w+1)^{\mathcal{A}+1} dw \right)^{1-\frac{1}{q}} \left(\int_0^1 (-w+1)^{\mathcal{A}+1} [h(w)|\mathcal{U}''(\mathcal{U})|^q \right. \right. \\ & \quad \left. \left. + h(-w+1)|\mathcal{U}''(\mathcal{V}_1)|^q] dw \right)^{\frac{1}{q}} \right] \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\ & \quad \times \left[\left(\int_0^1 w^{\mathcal{A}+1} dw \right)^{1-\frac{1}{q}} \left(\int_0^1 w^{\mathcal{A}+1} [h(w)|\mathcal{U}''(\mathcal{U})|^q + h(-w+1)|\mathcal{U}''(\mathcal{V}_1)|^q] dw \right)^{\frac{1}{q}} \right]. \end{aligned}$$

So, the proof is completed.

We give some consequences of Theorem 28. Let the hypothesis of Theorem 28 holds.

I. Let $h(\mathfrak{w}) = \mathfrak{w}$, then

Corollary 29: If $|\mathfrak{U}''|^q$ is MN -convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathfrak{U}}^{\mathcal{A}} \mathfrak{U}(\mathcal{U}) + {}^AB I_{\mathfrak{U}}^{\mathcal{A}} \mathfrak{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathfrak{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathfrak{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathfrak{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\frac{|\mathfrak{U}''(\mathcal{U})|^q}{(\mathcal{A}+2)(\mathcal{A}+3)} + \frac{|\mathfrak{U}''(\mathcal{V}_1)|^q}{(\mathcal{A}+3)} \right)^{\frac{1}{q}} \right] \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\frac{|\mathfrak{U}''(\mathcal{V}_2)|^q}{(\mathcal{A}+3)} + \frac{|\mathfrak{U}''(\mathcal{U})|^q}{(\mathcal{A}+2)(\mathcal{A}+3)} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

(2): If $h(\mathfrak{w}) = \mathfrak{w}^s$, then

Corollary 30: If $|\mathfrak{U}''|^q$ is Breckner type of MN -convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathfrak{U}}^{\mathcal{A}} \mathfrak{U}(\mathcal{U}) + {}^AB I_{\mathfrak{U}}^{\mathcal{A}} \mathfrak{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathfrak{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathfrak{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathfrak{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\ & \quad \times \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} (\phi(s+1, \mathcal{A}+2)|\mathfrak{U}''(\mathcal{U})|^q + \phi(1, \mathcal{A}+s+2)|\mathfrak{U}''(\mathcal{V}_1)|^q)^{\frac{1}{q}} \right] \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\ & \quad \times \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} (\phi(\mathcal{A}+s+2, 1)|\mathfrak{U}''(\mathcal{V}_2)|^q + \phi(\mathcal{A}+2, s+1)|\mathfrak{U}''(\mathcal{U})|^q)^{\frac{1}{q}} \right]. \end{aligned}$$

IV. If $h(\mathfrak{w}) = 1$, then

Corollary 31: If $|\mathfrak{U}''|^q$ is P - MN -convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\mathfrak{U}}^{\mathcal{A}} \mathfrak{U}(\mathcal{U}) + {}^AB I_{\mathfrak{U}}^{\mathcal{A}} \mathfrak{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}} \mathfrak{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}} \mathfrak{U}(\mathcal{V}_2)}{\phi(\mathcal{A})\Gamma(\mathcal{A})}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} - \frac{2(-\mathcal{A}+1)\mathfrak{U}(\mathcal{U})}{\phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\frac{1}{\mathcal{A}+2} [|\mathfrak{U}''(\mathcal{U})|^q + |\mathfrak{U}''(\mathcal{V}_1)|^q] \right)^{\frac{1}{q}} \right] \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{A}+2}}{\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\frac{1}{\mathcal{A}+2} [|\mathfrak{U}''(\mathcal{U})|^q + |\mathfrak{U}''(\mathcal{V}_1)|^q] \right)^{\frac{1}{q}} \right]. \end{aligned}$$

IV. For $h(\mathfrak{w}) = \mathfrak{w}(-\mathfrak{w} + 1)$, one receive

Corollary 32: *If $|\mathfrak{U}''|^q$ is tgs-MN-convex, then*

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\mathcal{U}}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) + \frac{AB}{\mathcal{V}_2} I_{\mathcal{U}}^{\mathcal{E}} \mathfrak{U}(\mathcal{U}) - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_1) + (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}} \mathfrak{U}(\mathcal{V}_2)}{\phi(\mathcal{E})\Gamma(\mathcal{E})} \right. \\ & \quad \left. - \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+1} \mathfrak{U}'(\mathcal{V}_1) - (-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+1} \mathfrak{U}'(\mathcal{V}_2)}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} - \frac{2(-\mathcal{E}+1)\mathfrak{U}(\mathcal{U})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\left(\frac{1}{\mathcal{E}+2} \right)^{1-\frac{1}{q}} \left(\frac{1}{(\mathcal{E}+3)(\mathcal{E}+4)} [|\mathfrak{U}''(\mathcal{U})|^q + |\mathfrak{U}''(\mathcal{V}_1)|^q] \right)^{\frac{1}{q}} \right] \\ & \quad + \frac{(-\mathcal{V}_2 + \mathcal{U})^{\mathcal{E}+2}}{\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\left(\frac{1}{\mathcal{E}+2} \right)^{1-\frac{1}{q}} \left(\frac{1}{(\mathcal{E}+3)(\mathcal{E}+4)} [|\mathfrak{U}''(\mathcal{V}_2)|^q + |\mathfrak{U}''(\mathcal{U})|^q] \right)^{\frac{1}{q}} \right]. \end{aligned}$$

2.4 More result for AB FI operators in convex functions

Now we'll apply a new variation of Lemma 12 that takes into consideration a different way for removing the function's first derivative. We get novel results and estimates if we use $\mathcal{U} = \frac{M+N}{2}$ in Lemma 12.

Lemma 33: *Let I be an interval and $\mathfrak{U} : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable function. Then*

$$\begin{aligned} & \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}}\phi(\mathcal{E})\Gamma(\mathcal{E})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \\ & \quad - \frac{2(-\mathcal{E}+1)\mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\phi(\mathcal{E})} \\ & = \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2}\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\int_0^1 ((-\mathfrak{w}+1)^{\mathcal{E}+1} - 1) \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1)\mathcal{V}_1 \right) d\mathfrak{w} \right. \\ & \quad \left. + \int_0^1 (\mathfrak{w}^{\mathcal{E}+1} - 1) \mathfrak{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) d\mathfrak{w} \right], \end{aligned}$$

where $\mathcal{U} \in I$.

Proof: Set $\mathcal{U} = \frac{\mathcal{V}_2+\mathcal{V}_1}{2}$ in the Lemma 12, one may get

$$\begin{aligned} & \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}}\phi(\mathcal{E})\Gamma(\mathcal{E})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \\ & \quad - \frac{2(-\mathcal{E}+1)\mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\phi(\mathcal{E})} \\ & = \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+1}}{2^{\mathcal{E}+1}\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} [\mathfrak{U}'(\mathcal{V}_1) - \mathfrak{U}'(\mathcal{V}_2)] \tag{5} \\ & \quad + \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2}\Gamma(\mathcal{E})\phi(\mathcal{E})(1+\mathcal{E})} \left[\int_0^1 (-\mathfrak{w}+1)^{\mathcal{E}+1} \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1)\mathcal{V}_1 \right) d\mathfrak{w} \right. \\ & \quad \left. + \int_0^1 \mathfrak{w}^{\mathcal{E}+1} \mathfrak{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) d\mathfrak{w} \right]. \end{aligned}$$

Then

$$[\mathfrak{U}'(\mathcal{V}_1) + \mathfrak{U}'(\mathcal{V}_2)] = -\int_M^N \mathfrak{U}''(y) dy = \left[\frac{-\mathcal{V}_1 + \mathcal{V}_2}{2} \int_0^1 \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} + (-\mathfrak{w} + 1)\mathcal{V}_1 \right) d\mathfrak{w} \right. \\ \left. + \frac{-\mathcal{V}_1 + \mathcal{V}_2}{2} \int_0^1 \mathfrak{U}'' \left(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w} + 1) \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) d\mathfrak{w} \right]. \quad (6)$$

From (5) and (6), one reach

$$\begin{aligned} & \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \\ & - \frac{2(-\mathcal{E}+1)\mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right)}{\phi(\mathcal{E})} \\ & = \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left[\int_0^1 ((-\mathfrak{w} + 1)^{\mathcal{E}+1} - 1) \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} + (-\mathfrak{w} + 1)\mathcal{V}_1 \right) d\mathfrak{w} \right. \\ & \left. + \int_0^1 (\mathfrak{w}^{\mathcal{E}+1} - 1) \mathfrak{U}'' \left(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w} + 1) \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) d\mathfrak{w} \right]. \end{aligned}$$

Hence, we finished the proof.

Theorem 34: If $|\mathfrak{U}''|$ is MN -convex mapping, then

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\ & \left. - \frac{2(-\mathcal{E}+1)\mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \\ & \times \left[\int_0^1 ((-\mathfrak{w} + 1)^{\mathcal{E}+1} - 1) \left(h(\mathfrak{w}) \left| \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) \right| + h(-\mathfrak{w} + 1) |\mathfrak{U}''(\mathcal{V}_1)| \right) d\mathfrak{w} \right. \\ & \left. + \int_0^1 (\mathfrak{w}^{\mathcal{E}+1} - 1) \left(h(\mathfrak{w}) |\mathfrak{U}''(\mathcal{V}_2)| + h(-\mathfrak{w} + 1) \left| \mathfrak{U}'' \left(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w} + 1) \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) \right| \right) d\mathfrak{w} \right], \end{aligned}$$

where $\mathcal{U} \in I$.

Proof: Utilizing Lemma 33, one may achieve

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\ & \left. - \frac{2(-\mathcal{E}+1)\mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right)}{\phi(\mathcal{E})} \right| \\ & = \left| \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left[\int_0^1 ((-\mathfrak{w} + 1)^{\mathcal{E}+1} - 1) \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} + (-\mathfrak{w} + 1)\mathcal{V}_1 \right) d\mathfrak{w} \right. \right. \\ & \left. \left. + \int_0^1 (\mathfrak{w}^{\mathcal{E}+1} - 1) \mathfrak{U}'' \left(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w} + 1) \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) d\mathfrak{w} \right] \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left[\int_0^1 ((-\mathfrak{w} + 1)^{\mathcal{E}+1} - 1) \left| \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} + (-\mathfrak{w} + 1)\mathcal{V}_1 \right) \right| d\mathfrak{w} \right. \\ & \left. + \int_0^1 (\mathfrak{w}^{\mathcal{E}+1} - 1) \left| \mathfrak{U}'' \left(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w} + 1) \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) \right| d\mathfrak{w} \right]. \end{aligned}$$

By using MN -convexity of $|\mathfrak{U}''|$, we get

$$\left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right|$$

$$\begin{aligned}
& - \frac{2(-\mathcal{A}+1)\mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right)}{\phi(\mathcal{A})} \Big| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2}\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\
& \times \left[\int_0^1 ((-\mathfrak{w}+1)^{\mathcal{A}+1}-1) \left(h(\mathfrak{w}) \left| \mathfrak{U}''\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) \right| + h(-\mathfrak{w}+1) |\mathfrak{U}''(\mathcal{V}_1)| \right) d\mathfrak{w} \right. \\
& \left. + \int_0^1 (\mathfrak{w}^{\mathcal{A}+1}-1) \left(h(\mathfrak{w}) |\mathfrak{U}''(\mathcal{V}_2)| + h(-\mathfrak{w}+1) \left| \mathfrak{U}''\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) \right| \right) d\mathfrak{w} \right]
\end{aligned}$$

and the proof is completed.

Next, we give some corollaries which can be easily proved using Theorem 34.

I. If $h(\mathfrak{w}) = \mathfrak{w}$, then we arrive to the following consequence:

Corollary 35: *If $|\mathfrak{U}''|^q$ is MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^{AB}\mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) + {}^{AB}\mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}}\phi(\mathcal{A})\Gamma(\mathcal{A})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\
& \left. - \frac{2(-\mathcal{A}+1)\mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right)}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2}\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[(|\mathfrak{U}''(\mathcal{V}_1)| + |\mathfrak{U}''(\mathcal{V}_2)|) \left(\frac{1}{\mathcal{A}+3} - \frac{1}{2} \right) \right. \\
& \left. + 2 \left| \mathfrak{U}''\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) \right| \left(\frac{1}{(\mathcal{A}+2)(\mathcal{A}+3)} - \frac{1}{2} \right) \right].
\end{aligned}$$

II. Let $h(\mathfrak{w}) = \mathfrak{w}^s$, Then we arrive to the following consequence:

Corollary 36: *If $|\mathfrak{U}''|^q$ is Breckner type of MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^{AB}\mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) + {}^{AB}\mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}}\phi(\mathcal{A})\Gamma(\mathcal{A})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\
& \left. - \frac{2(-\mathcal{A}+1)\mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right)}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2}\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[(|\mathfrak{U}''(\mathcal{V}_1)| + |\mathfrak{U}''(\mathcal{V}_2)|) \left(\frac{1}{\mathcal{A}+s+2} - \frac{1}{s+1} \right) \right. \\
& \left. + 2 \left| \mathfrak{U}''\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) \right| \left(\phi(s+1, \mathcal{A}+2) - \frac{1}{s+1} \right) \right].
\end{aligned}$$

III. If $h(\mathfrak{w}) = 1$, then we arrive to the following corollary:

Corollary 37: *If $|\mathfrak{U}''|^q$ is P-MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^{AB}\mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) + {}^{AB}\mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}}\phi(\mathcal{A})\Gamma(\mathcal{A})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\
& \left. - \frac{2(-\mathcal{A}+1)\mathfrak{U}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right)}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2}\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left(\frac{1}{\mathcal{A}+2} - 1 \right) \left[2 \left| \mathfrak{U}''\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right) \right| + |\mathfrak{U}''(\mathcal{V}_1)| + |\mathfrak{U}''(\mathcal{V}_2)| \right].
\end{aligned}$$

IV. Let $h(w) = w(-w + 1)$, then we arrive to the following consequence.

Corollary 38: *If $|\mathcal{U}''|^q$ is tgs-MN-convex, then*

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathcal{U}(\mathcal{V}_1) + \mathcal{U}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E}+1) \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left(\frac{1}{(\mathcal{E}+3)(\mathcal{E}+4)} - \frac{1}{6} \right) \\ & \quad \times \left[2 \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| + |\mathcal{U}''(\mathcal{V}_1)| + |\mathcal{U}''(\mathcal{V}_2)| \right]. \end{aligned}$$

Theorem 39: *If $|\mathcal{U}''|$ is MN-convex mapping, then we have*

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathcal{U}(\mathcal{V}_1) + \mathcal{U}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E}+1) \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left(\left(\frac{1}{\mathcal{E}p+p+1} \right)^{\frac{1}{p}} - 1 \right) \\ & \quad \times \left[\left(\int_0^1 \left[h(w) \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q + h(-w+1) |\mathcal{U}''(\mathcal{V}_1)|^q \right] dw \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 \left[h(w) |\mathcal{U}''(\mathcal{V}_2)|^q + h(-w+1) \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q \right] dw \right)^{\frac{1}{q}} \right], \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathcal{U} \in I$.

Proof: With the use of Lemma 33, we can write

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathcal{U}(\mathcal{V}_1) + \mathcal{U}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E}+1) \mathcal{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left[\int_0^1 ((-w+1)^{\mathcal{E}+1} - 1) \left| \mathcal{U}'' \left(w \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-w+1) \mathcal{V}_1 \right) \right| dw \right. \\ & \quad \left. + \int_0^1 (w^{\mathcal{E}+1} - 1) \left| \mathcal{U}'' \left(t \mathcal{V}_2 + (-w+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| dw \right] \\ & = \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left[\int_0^1 (-w+1)^{\mathcal{E}+1} \left| \mathcal{U}'' \left(w \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-w+1) \mathcal{V}_1 \right) \right| dw \right. \\ & \quad \left. - \int_0^1 \left| \mathcal{U}'' \left(w \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-w+1) \mathcal{V}_1 \right) \right| dw \right. \\ & \quad \left. + \int_0^1 w^{\mathcal{E}+1} \left| \mathcal{U}'' \left(w \mathcal{V}_2 + (-w+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| dw - \int_0^1 \left| \mathcal{U}'' \left(t \mathcal{V}_2 + (-w+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| dw \right]. \end{aligned}$$

Using Hölder's inequality, we reach

$$\begin{aligned}
& \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}} \Phi(\mathcal{A}) \Gamma(\mathcal{A})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\
& \quad \left. - \frac{2(-\mathcal{A}+1) \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\Phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \Phi(\mathcal{A}) (1+\mathcal{A})} \\
& \quad \times \left[\left(\int_0^1 (-\mathfrak{w}+1)^{\mathcal{A}+1} - 1 \right)^{\frac{1}{p}} \left(\int_0^1 \left| \mathfrak{U}'' \left(t \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1) \mathcal{V}_1 \right) \right|^q d\mathfrak{w} \right)^{\frac{1}{q}} \right. \\
& \quad - \left(\int_0^1 (\mathfrak{w}^0)^p d\mathfrak{w} \right)^{\frac{1}{p}} \left(\int_0^1 \left| \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1) \mathcal{V}_1 \right) \right|^q d\mathfrak{w} \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \mathfrak{w}^{(\mathcal{A}+1)p} d\mathfrak{w} \right)^{\frac{1}{p}} \left(\int_0^1 \left| \mathfrak{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q d\mathfrak{w} \right)^{\frac{1}{q}} \\
& \quad \left. + \left(\int_0^1 (\mathfrak{w}^0)^p d\mathfrak{w} \right)^{\frac{1}{p}} \left(\int_0^1 \left| \mathfrak{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q d\mathfrak{w} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

By using MN -convexity of $|\mathfrak{U}''|^q$, we obtain

$$\begin{aligned}
& \int_0^1 \left| \mathfrak{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1) \mathcal{V}_1 \right) \right|^q d\mathfrak{w} \\
& \leq \int_0^1 \left[h(\mathfrak{w}) \left| \mathfrak{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q + h(-\mathfrak{w}+1) |\mathfrak{U}''(\mathcal{V}_1)|^q \right] d\mathfrak{w}
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \left| \mathfrak{U}'' \left(t \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q dt \\
& \leq \int_0^1 \left[h(\mathfrak{w}) |\mathfrak{U}''(\mathcal{V}_2)|^q + h(-\mathfrak{w}+1) \left| \mathfrak{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q \right] dt.
\end{aligned}$$

After some simple calculations, we get desired result.

Further, we provide some consequences which can be easily proved using Theorem 39.

I. If $h(\mathfrak{w}) = \mathfrak{w}$, then we reach to the following consequence:

Corollary 40: If $|\mathfrak{U}''|^q$ is MN -convex, then

$$\begin{aligned}
& \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}} \Phi(\mathcal{A}) \Gamma(\mathcal{A})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\
& \quad \left. - \frac{2(-\mathcal{A}+1) \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\Phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \Phi(\mathcal{A}) (1+\mathcal{A})} \left(\left(\frac{1}{(\mathcal{A}p+p+1)} \right)^{\frac{1}{p}} - 1 \right)
\end{aligned}$$

$$\times \left[\left(\frac{|\mathfrak{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q + |\mathfrak{U}''(\mathcal{V}_1)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|\mathfrak{U}''(\mathcal{V}_2)|^q + |\mathfrak{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{2} \right)^{\frac{1}{q}} \right]$$

II. If $h(\mathfrak{w}) = \mathfrak{w}^s$, then we come to the following consequence:

Corollary 41: If $|\mathfrak{U}''|^q$ is Breckner type of MN-convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + {}^AB I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E}+1)\mathfrak{U}(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left(\left(\frac{1}{\mathcal{E}p+p+1} \right)^{\frac{1}{p}} - 1 \right) \\ & \times \left[\left(\frac{|\mathfrak{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q + |\mathfrak{U}''(\mathcal{V}_1)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|\mathfrak{U}''(\mathcal{V}_2)|^q + |\mathfrak{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{s+1} \right)^{\frac{1}{q}} \right] \end{aligned}$$

III. If $h(\mathfrak{w}) = 1$, then we reach to the following deduction:

Corollary 42: If $|\mathfrak{U}''|^q$ is P-MN-convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + {}^AB I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathfrak{J}(\mathcal{V}_1) + \mathfrak{J}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E}+1)\mathfrak{J}(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left(\left(\frac{1}{\mathcal{E}p+p+1} \right)^{\frac{1}{p}} - 1 \right) \\ & \times \left[\left(|\mathfrak{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q + |\mathfrak{U}''(\mathcal{V}_1)|^q \right)^{\frac{1}{q}} + \left(|\mathfrak{U}''(\mathcal{V}_2)|^q + |\mathfrak{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

IV. If $h(\mathfrak{w}) = \mathfrak{w}(-\mathfrak{w}+1)$, then we arrive to the following consequence:

Corollary 43: If $|\mathfrak{U}''|^q$ is tgs-MN-convex, then

$$\begin{aligned} & \left| \frac{{}^AB I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + {}^AB I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} \mathfrak{U} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [\mathfrak{U}(\mathcal{V}_1) + \mathfrak{U}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E}+1)\mathfrak{U}(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1+\mathcal{E})} \left(\left(\frac{1}{\mathcal{E}p+p+1} \right)^{\frac{1}{p}} - 1 \right) \end{aligned}$$

$$\times \left[\left(\frac{|\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{6} \right)^{\frac{1}{q}} \right].$$

Theorem 44: If $|\mathcal{U}''|$ is MN-convex mapping, we have

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}} \Phi(\mathcal{A}) \Gamma(\mathcal{A})} [\mathcal{J}(\mathcal{V}_1) + \mathcal{J}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{A}+1) \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\Phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \Phi(\mathcal{A}) (1+\mathcal{A})} \left[\frac{2}{p(\mathcal{A}p+p+1)} - \frac{2}{p} \right], \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 1$.

Proof: Utilizing 33, we can write

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}} \Phi(\mathcal{A}) \Gamma(\mathcal{A})} [\mathcal{J}(\mathcal{V}_1) + \mathcal{J}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{A}+1) \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\Phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \Phi(\mathcal{A}) (1+\mathcal{A})} \left[\int_0^1 ((-\mathfrak{w}+1)^{\mathcal{A}+1} - 1) \left| \mathcal{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1) \mathcal{V}_1 \right) \right| d\mathfrak{w} \right. \\ & \quad \left. + \int_0^1 (\mathfrak{w}^{\mathcal{A}+1} - 1) \left| \mathcal{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| d\mathfrak{w} \right] \\ & = \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \Phi(\mathcal{A}) (1+\mathcal{A})} \left[\int_0^1 (-\mathfrak{w}+1)^{\mathcal{A}+1} \left| \mathcal{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1) \mathcal{V}_1 \right) \right| d\mathfrak{w} \right. \\ & \quad - \int_0^1 \left| \mathcal{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1) \mathcal{V}_1 \right) \right| d\mathfrak{w} \\ & \quad + \int_0^1 \mathfrak{w}^{\mathcal{A}+1} \left| \mathcal{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| d\mathfrak{w} \\ & \quad \left. - \int_0^1 \left| \mathcal{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| d\mathfrak{w} \right]. \end{aligned}$$

By using the Young inequality as $\mathcal{U}y \leq \frac{1}{p} \mathcal{U}^p + \frac{1}{q} y^q$

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}} \Phi(\mathcal{A}) \Gamma(\mathcal{A})} [\mathcal{J}(\mathcal{V}_1) + \mathcal{J}(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{A}+1) \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\Phi(\mathcal{A})} \right| \\ & \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \Phi(\mathcal{A}) (1+\mathcal{A})} \left[\frac{1}{p} \int_0^1 (-\mathfrak{w}+1)^{(\mathcal{A}+1)p} d\mathfrak{w} + \frac{1}{q} \int_0^1 \left| \mathcal{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1) \mathcal{V}_1 \right) \right|^q d\mathfrak{w} \right. \\ & \quad \left. - \frac{1}{p} \int_0^1 (\mathfrak{w}^0)^p d\mathfrak{w} - \frac{1}{q} \int_0^1 \left| \mathcal{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1) \mathcal{V}_1 \right) \right|^q d\mathfrak{w} \right. \\ & \quad \left. + \frac{1}{p} \int_0^1 \mathfrak{w}^{(\mathcal{A}+1)p} d\mathfrak{w} + \frac{1}{q} \int_0^1 \left| \mathcal{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q d\mathfrak{w} \right] \end{aligned}$$

$$+ \frac{1}{p} \int_0^1 \mathfrak{w}^0 d\mathfrak{w} + \frac{1}{q} \int_0^1 \left| \mathfrak{U}'' \left(\mathfrak{w} \mathcal{V}_2 + (-\mathfrak{w} + 1) \frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) \right|^q d\mathfrak{w}].$$

With aid of convexity of $|J''|^q$ and doing simple manipulation, we obtain the theorem.

Further, we present some deductions from the Theorem 13.

I. If $h(W) = t$, then we come to the consequence as follows:

Corollary 45: If $|\mathfrak{U}''|^q$ is *MN-convex*, then

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [J(\mathcal{V}_1) + J(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E} + 1)J\left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}\right)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E} + 2}}{2^{\mathcal{E} + 2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1 + \mathcal{E})} \left[\frac{2}{p(\mathcal{E}p + p + 1)} - \frac{2}{p} \right]. \end{aligned}$$

II. Let $h(W) = W^s$, then we arrive to the following consequence:

Corollary 46: If $|\mathfrak{U}''|^q$ is *Breckner type of MN-convex*, then

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [J(\mathcal{V}_1) + J(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E} + 1)J\left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}\right)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E} + 2}}{2^{\mathcal{E} + 2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1 + \mathcal{E})} \left[\frac{2}{p(\mathcal{E}p + p + 1)} - \frac{2}{p} \right]. \end{aligned}$$

III. Let $h(\mathfrak{w}) = 1$, then we come to the following deduction:

Corollary 47: If $|\mathfrak{U}''|^q$ is *P-MN-convex*, then

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [J(\mathcal{V}_1) + J(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E} + 1)J\left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}\right)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E} + 2}}{2^{\mathcal{E} + 2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1 + \mathcal{E})} \left[\frac{2}{p(\mathcal{E}p + p + 1)} - \frac{2}{p} \right]. \end{aligned}$$

IV. If $h(W) = W(-W + 1)$, then one come to the following corollary:

Corollary 48: If $|\mathfrak{U}''|^q$ is *tgs-MN-convex*, then

$$\begin{aligned} & \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}}^{\mathcal{E}} \left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \phi(\mathcal{E}) \Gamma(\mathcal{E})} [J(\mathcal{V}_1) + J(\mathcal{V}_2)] \right. \\ & \quad \left. - \frac{2(-\mathcal{E} + 1)J\left(\frac{\mathcal{V}_2 + \mathcal{V}_1}{2}\right)}{\phi(\mathcal{E})} \right| \\ & \leq \frac{(-\mathcal{V}_1 + \mathcal{V}_2)^{\mathcal{E} + 2}}{2^{\mathcal{E} + 2} \Gamma(\mathcal{E}) \phi(\mathcal{E}) (1 + \mathcal{E})} \left[\frac{2}{p(\mathcal{E}p + p + 1)} - \frac{2}{p} \right]. \end{aligned}$$

Theorem 49: If $|\mathcal{U}''|$ is MN-convex mapping, we have

$$\begin{aligned}
& \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}}\phi(\mathcal{A})\Gamma(\mathcal{A})} [\mathcal{J}(\mathcal{V}_1) + \mathcal{J}(\mathcal{V}_2)] \right. \\
& \quad \left. - \frac{2(-\mathcal{A}+1)\mathcal{J}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right)}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2}\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \\
& \quad \times \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\int_0^1 (-\mathfrak{w}+1)^{\mathcal{A}+1} \left[h(\mathfrak{w}) \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q + h(-\mathfrak{w}+1) |\mathcal{U}''(\mathcal{V}_1)|^q \right] d\mathfrak{w} \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\int_0^1 \mathfrak{w}^{\mathcal{A}+1} \left[h(\mathfrak{w}) |\mathcal{U}''(\mathcal{V}_2)|^q + h(-\mathfrak{w}+1) \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q \right] d\mathfrak{w} \right)^{\frac{1}{q}} \\
& \quad - \left(\int_0^1 h(\mathfrak{w}) \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q + h(-\mathfrak{w}+1) |\mathcal{U}''(\mathcal{V}_1)|^q d\mathfrak{w} \right)^{\frac{1}{q}} \\
& \quad \left. - \left(\int_0^1 h(\mathfrak{w}) |\mathcal{U}''(\mathcal{V}_2)|^q + h(-\mathfrak{w}+1) \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q d\mathfrak{w} \right)^{\frac{1}{q}} \right],
\end{aligned}$$

where $\mathcal{U} \in I, \mathcal{A} \in [0,1], q > 1$ and $\phi(\mathcal{A})$ is normalization function.

Proof: By using Lemma 33, we can write

$$\begin{aligned}
& \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}}\phi(\mathcal{A})\Gamma(\mathcal{A})} [\mathcal{J}(\mathcal{V}_1) + \mathcal{J}(\mathcal{V}_2)] \right. \\
& \quad \left. - \frac{2(-\mathcal{A}+1)\mathcal{J}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right)}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2}\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\int_0^1 ((-\mathfrak{w}+1)^{\mathcal{A}+1} - 1) \left| \mathcal{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1)\mathcal{V}_1 \right) \right| d\mathfrak{w} \right. \\
& \quad \left. + \int_0^1 (\mathfrak{w}^{\mathcal{A}+1} - 1) \left| \mathcal{U}'' \left(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| d\mathfrak{w} \right] \\
& = \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2}\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})} \left[\int_0^1 (-\mathfrak{w}+1)^{\mathcal{A}+1} \left| \mathcal{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1)\mathcal{V}_1 \right) \right| d\mathfrak{w} \right. \\
& \quad - \int_0^1 \left| \mathcal{U}'' \left(\mathfrak{w} \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-\mathfrak{w}+1)\mathcal{V}_1 \right) \right| d\mathfrak{w} \\
& \quad \left. + \int_0^1 \mathfrak{w}^{\mathcal{A}+1} \left| \mathcal{U}'' \left(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| d\mathfrak{w} - \int_0^1 \left| \mathcal{U}'' \left(\mathfrak{w}\mathcal{V}_2 + (-\mathfrak{w}+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right| d\mathfrak{w} \right].
\end{aligned}$$

By applying power mean inequality, we get

$$\begin{aligned}
& \left| \frac{AB}{\mathcal{V}_1} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + \frac{AB}{\mathcal{V}_2} I_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{A}} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}}\phi(\mathcal{A})\Gamma(\mathcal{A})} [\mathcal{J}(\mathcal{V}_1) + \mathcal{J}(\mathcal{V}_2)] \right. \\
& \quad \left. - \frac{2(-\mathcal{A}+1)\mathcal{J}\left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2}\right)}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2}\Gamma(\mathcal{A})\phi(\mathcal{A})(1+\mathcal{A})}
\end{aligned}$$

$$\begin{aligned}
& \times \left[\left(\int_0^1 (-w+1)^{\mathcal{E}+1} - dw \right)^{1-\frac{1}{q}} \left(\int_0^1 (-w+1)^{\mathcal{E}+1} \left| \mathcal{U}'' \left(w \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-w+1)\mathcal{V}_1 \right) \right|^q dw \right)^{\frac{1}{q}} \right. \\
& - \left(\int_0^1 (w^0) dt \right)^{1-\frac{1}{p}} \left(\int_0^1 \left| \mathcal{U}'' \left(w \frac{\mathcal{V}_2+\mathcal{V}_1}{2} + (-w+1)\mathcal{V}_1 \right) \right|^q dw \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 w^{\mathcal{E}+1} dw \right)^{1-\frac{1}{q}} \left(\int_0^1 w^{\mathcal{E}+1} \left| \mathcal{U}'' \left(w \mathcal{V}_2 + (-w+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q dw \right)^{\frac{1}{q}} \\
& \left. + \left(\int_0^1 (w^0) dw \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| \mathcal{U}'' \left(w \mathcal{V}_2 + (-w+1) \frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q dw \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Utilizing convexity of $|J''|^q$ and doing simple calculation, we get the acquired theorem.

Next, we present some results which can be easily proved using Theorem 49.

I. If $h(w) = w$, then we arrive to the following consequence:

Corollary 50: *If $|J''|^q$ is MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^AB \mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} J \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + {}^AB \mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} J \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \Phi(\mathcal{E}) \Gamma(\mathcal{E})} [J(\mathcal{V}_1) + J(\mathcal{V}_2)] \right. \\
& \left. - \frac{2(-\mathcal{E}+1)J \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\Phi(\mathcal{E})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \Phi(\mathcal{E}) (1+\mathcal{E})} + \left[\left(\frac{1}{(\mathcal{E}+2)} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{(\mathcal{E}+2)(\mathcal{E}+3)} + \frac{|\mathcal{U}''(\mathcal{V}_1)|^q}{(\mathcal{E}+3)} \right)^{\frac{1}{q}} \right. \\
& + \left(\frac{1}{(\mathcal{E}+2)} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{(\mathcal{E}+3)} + \frac{|\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{(\mathcal{E}+2)(\mathcal{E}+3)} \right)^{\frac{1}{q}} \\
& \left. \times \left[\left(\frac{|\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{2} \right)^{\frac{1}{q}} \right] \right].
\end{aligned}$$

II. If $h(W) = W^s$, then one have the following result:

Corollary 51: *If $|\mathcal{U}''|^q$ is Breckner type of MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^AB \mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} J \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + {}^AB \mathcal{I}_{\frac{\mathcal{V}_2+\mathcal{V}_1}{2}}^{\mathcal{E}} J \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}}}{2^{\mathcal{E}} \Phi(\mathcal{E}) \Gamma(\mathcal{E})} [J(\mathcal{V}_1) + J(\mathcal{V}_2)] \right. \\
& \left. - \frac{2(-\mathcal{E}+1)J \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\Phi(\mathcal{E})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{E}+2}}{2^{\mathcal{E}+2} \Gamma(\mathcal{E}) \Phi(\mathcal{E}) (1+\mathcal{E})}
\end{aligned}$$

$$\begin{aligned}
& + \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\phi(s+1, \mathcal{A}+2) \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q + \frac{|\mathcal{U}''(\mathcal{V}_1)|^q}{(\mathcal{A}+s+2)} \right)^{\frac{1}{q}} \right. \\
& + \left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(+ \frac{|\mathcal{U}''(\mathcal{V}_2)|^q}{(\mathcal{A}+s+2)} + \phi(s+1, \mathcal{A}+2) \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q \right)^{\frac{1}{q}} \\
& \left. \times \left[\left(\frac{|\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{s+1} \right)^{\frac{1}{q}} \right] \right].
\end{aligned}$$

III. If $h(\mathfrak{w}) = 1$, then one arrive to the consequence as follows:

Corollary 52: *If $|\mathcal{U}''|^q$ is P -MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^AB I_{\mathcal{V}_2+\mathcal{V}_1}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + {}^AB I_{\mathcal{V}_2+\mathcal{V}_1}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}} \phi(\mathcal{A}) \Gamma(\mathcal{A})} [\mathcal{J}(\mathcal{V}_1) + \mathcal{J}(\mathcal{V}_2)] \right. \\
& \left. - \frac{2(-\mathcal{A}+1) \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \phi(\mathcal{A}) (1+\mathcal{A})} + \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{(\mathcal{A}+2)} \right)^{\frac{1}{q}} \right. \\
& + \left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{(\mathcal{A}+2)} \right)^{\frac{1}{q}} \\
& \left. \times \left[\left(\left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q + |\mathcal{U}''(\mathcal{V}_1)|^q \right)^{\frac{1}{q}} + \left(|\mathcal{U}''(\mathcal{V}_2)|^q + \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q \right)^{\frac{1}{q}} \right] \right].
\end{aligned}$$

IV. If $h(\mathfrak{w}) = \mathfrak{w}(-\mathfrak{w}+1)$, then we arrive to the following consequence:

Corollary 53: *If $|\mathcal{U}''|^q$ is tgs -MN-convex, then*

$$\begin{aligned}
& \left| \frac{{}^AB I_{\mathcal{V}_2+\mathcal{V}_1}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) + {}^AB I_{\mathcal{V}_2+\mathcal{V}_1}^{\mathcal{A}} \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) - \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}}}{2^{\mathcal{A}} \phi(\mathcal{A}) \Gamma(\mathcal{A})} [\mathcal{J}(\mathcal{V}_1) + \mathcal{J}(\mathcal{V}_2)] \right. \\
& \left. - \frac{2(-\mathcal{A}+1) \mathcal{J} \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right)}{\phi(\mathcal{A})} \right| \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \phi(\mathcal{A}) (1+\mathcal{A})} \\
& \leq \frac{(-\mathcal{V}_1+\mathcal{V}_2)^{\mathcal{A}+2}}{2^{\mathcal{A}+2} \Gamma(\mathcal{A}) \phi(\mathcal{A}) (1+\mathcal{A})} + \left[\left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q + |\mathcal{U}''(\mathcal{V}_1)|^q}{(\mathcal{A}+3)(\mathcal{A}+4)} \right)^{\frac{1}{q}} \right. \\
& + \left(\frac{1}{\mathcal{A}+2} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{U}''(\mathcal{V}_2)|^q + |\mathcal{U}''(\frac{\mathcal{V}_2+\mathcal{V}_1}{2})|^q}{(\mathcal{A}+3)(\mathcal{A}+4)} \right)^{\frac{1}{q}} \\
& \left. \times \left[\left(\left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q + |\mathcal{U}''(\mathcal{V}_1)|^q \right)^{\frac{1}{q}} + \left(|\mathcal{U}''(\mathcal{V}_2)|^q + \left| \mathcal{U}'' \left(\frac{\mathcal{V}_2+\mathcal{V}_1}{2} \right) \right|^q \right)^{\frac{1}{q}} \right] \right].
\end{aligned}$$

$$- \left[\left(\frac{|v''(\frac{v_2+v_1}{2})|^q + |v''(v_1)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{|v''(v_2)|^q + |v''(\frac{v_2+v_1}{2})|^q}{6} \right)^{\frac{1}{q}} \right].$$

Competing interests

We confirm that authors have no competing interests.

Funding

No funding

Authors' contributions

Ibrahim Slimane: Suggested and initiated this work, Conceptualization, Methodology, Formal analysis, Investigation, Writing original draft, review and editing.

Muhammad Uzair Awan: Conceptualization, Methodology, Formal analysis, Investigation, Writing original draft.

Zarnab Aslam: Formal analysis, Investigation, review and editing.

Received

- [1] T. Abdeljawad and D. Baleanu, "Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler non-singular kernel," *J. Nonlinear Sci. Appl.*, vol. 10, pp. 1098–1107 (2017).
- [2] A. Atangana and D. Baleanu, "New fractional derivatives with non-local and nonsingular kernel: Theory and application to heat transfer model," *Therm. Sci.*, vol. 20, no. 2, pp. 763–769 (2016).
- [3] M. Alomari, M. Darus, and S. S. Dragomir, "New inequalities of Hermite–Hadamard type for functions whose second derivatives absolute value are quasi-convex," *RGMIA Res. Rep. Collect.*, vol. 12, Art. 17.
- [4] M. U. Awan, M. A. Noor, T. Du, and K. I. Noor, "On M-convex function," *AIMS Math.*, vol. 5, no. 3, pp. 2376–2387 (2020).
- [5] S. Belarbi and Z. Dahmani, "On some new fractional integral inequalities," *J. Inequal. Pure Appl. Math.*, vol. 10, no. 3, Art. 86, pp. 1–12 (2009).
- [6] H. Budak, M. A. Ali, N. Alp, and H. M. Awais, "Some new q-Hermite–Hadamard type inequalities for the product of convex functions," *J. Interdiscip. Math.*, vol. 25, no. 8, pp. 2141–2166, 2022.2.
- [7] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Prog. Fract. Differ. Appl.*, vol. 1, no. 2, pp. 73–85 (2015).

- [8] G. Cristescu and L. Lupsa, *Non-connected Convexities and Applications*. Dordrecht, The Netherlands: Kluwer Academic (2002).
- [9] S. S. Dragomir, "Inequalities of Hermite–Hadamard type for h -convex functions on linear spaces," *Proyecciones*, vol. 34, no. 4, pp. 323–341 (2015).
- [10] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Appl. Math. Lett.*, vol. 11, no. 5, pp. 91–95 (1998).
- [11] S. S. Dragomir and B. Mond, "Integral inequalities of Hadamard's type for logconvex functions," *Demonstr. Math.*, vol. 2, pp. 354–364 (1998).
- [12] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite–Hadamard Inequality and Applications*. Melbourne, Australia: Victoria Univ. (2000).
- [13] Gulalai, S. Ahmad, F. A. Rihan, A. Ullah, Q. M. Al-Mdallal, and A. Akgül, "Nonlinear analysis of a nonlinear modified KdV equation under Atangana–Baleanu Caputo," *AIMS Math.*, vol. 7, no. 5, pp. 7847–7865 (2022).
- [14] M. Gürbüz, A. O. Akdemir, and S. Rashid, "Hermite–Hadamard inequality for fractional integrals of Caputo–Fabrizio type and related inequalities," *J. Inequal. Appl.*, vol. 2020, Art. 172 (2020).
- [15] D. A. Ion, "Some estimates on the Hermite–Hadamard inequality through quasiconvex functions," *An. Univ. Craiova Math. Comput. Sci. Ser.*, vol. 34, pp. 82–87 (2007).
- [16] M. Khaled and P. Agarwal, "New Hermite–Hadamard type integral inequalities for convex functions and their applications," *J. Comput. Appl. Math.*, vol. 350, pp. 274–285 (2019).
- [17] M. A. Khan, Y. Khurshid, T.-S. Du, and Y.-M. Chu, "Generalization of Hermite–Hadamard type inequalities via conformable fractional integrals," *J. Funct. Spaces*, vol. 2018, Art. ID 5357463 (2018).
- [18] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Amsterdam, The Netherlands: Elsevier (2006).
- [19] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, "Fractional Hermite–Hadamard inequalities for some new classes of Godunova–Levin functions," *Appl. Math. Inf. Sci.*, vol. 8, no. 6, pp. 2865–2872 (2014).
- [20] M. E. Özdemir, "On Iyengar-type inequalities via quasi-convexity and quasiconcavity," *Miskolc Math. Notes*, vol. 15, no. 1, pp. 171–181 (2014).

- [21] F. Rahman, A. Ali, and S. Saifullah, "Analysis of time-fractional ϕ -equation with singular and non-singular kernels," *Int. J. Appl. Comput. Math.*, vol. 7, Art. 192 (2021).
- [22] M. Z. Sarikaya, E. Set, H. Yaldiz, et al., "Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities," *Math. Comput. Model.*, vol. 57, pp. 2403–2407 (2013).
- [23] S. Saifullah, A. Ali, M. Irfan, and K. Shah, "Time-fractional Klein–Gordon equation with solitary/shock wave solutions," *Math. Probl. Eng.*, vol. 2021, Art. ID (2021).
- [24] E. Set, S. I. Butt, A. O. Akdemir, A. Karaoglan, and T. Abdeljawad, "New integral inequalities for differentiable convex functions via Atangana–Baleanu fractional integral operators," *Chaos Solitons Fractals*, vol. 143, Art. 110554 (2021).
- [25] E. Set, J. Choi, and B. Celik, "Certain Hermite–Hadamard type inequalities involving generalized fractional integral operators," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, vol. 112, no. 4, pp. 1539–1547 (2018).
- [26] I. Slimane and Z. Dahmani, "A continuous and fractional derivative dependence of random differential equations with nonlocal conditions," *J. Interdiscip. Math.*, vol. 24, no. 6, pp. 1457–1470 (2021).
- [27] S. Varosanec, "On h -convexity," *J. Math. Anal. Appl.*, vol. 326, pp. 303–311 (2007).
- [28] B. Y. Xi, S.-P. Bai, and F. Qi, "On integral inequalities of the Hermite–Hadamard type for coordinated (α, m_1) – (s, m_2) -convex functions," *J. Interdiscip. Math.*, vol. 21, no. 7–8, pp. 1505–1518 (2018).
- [29] Y. Sabir, M. Misiran, and Z. Omar, "Hermite–Hadamard inequality for product of (h_1, h_2, s) -convex and m -harmonically convex function," *J. Interdiscip. Math.*, vol. 26, no. 4, pp. 675–689 (2023).
- [30] Y. Sabir, Z. Omar, and M. Misiran, "Hermite–Hadamard and Simpson's type inequalities for first and second ordered derivatives using product of (h_1, h_2, s) -convex and m -convex function," *J. Interdiscip. Math.*, vol. 26, no. 8, pp. 1733–1744 (2023).
- [31] Y. Zhang, T. S. Du, and H. Wang, "Some new k -fractional integral inequalities containing multiple parameters via generalized (s, m) -preinvexity," *Italian J. Pure Appl. Math.*, vol. 40, pp. 510–527, 2018.

Received November, 2023