

Advanced higher-order degree irregularity measures in Pythagorean fuzzy graph under threshold conditions

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Abstract

This paper introduces advanced irregularity measures for Pythagorean fuzzy graphs under threshold conditions. Two types of indices are proposed. The first type focuses on edge-based irregularity, including a higher-order edge degree irregularity index and a correlation-based index that captures relationships between the degrees of adjacent vertices. The second type focuses on degree-based irregularity, introducing an entropy-based index that measures uncertainty in the degree distribution. For each measure, we establish fundamental properties, monotonicity behavior under increasing thresholds, and sharp upper and lower bounds. Extremal cases are also characterized to identify graphs that attain equality in these bounds. Numerical examples are provided to illustrate the validity and effectiveness of the proposed measures. The results offer a refined perspective on degree irregularity and extend the theoretical framework of Pythagorean fuzzy graphs under threshold conditions.

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1. Introduction:

Pythagorean fuzzy sets, introduced as an extension of intuitionistic fuzzy sets, provide greater flexibility for modeling uncertainty by allowing the squared sum of membership and non-membership degrees to be at most one. Owing to their ability to represent uncertain and imprecise information more effectively, Pythagorean fuzzy sets have found extensive applications in decision-making,

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pattern recognition, clustering, and optimization. Several similarity measures and distance-based approaches have also been proposed to enhance their practical applicability in multi-criteria decision-making problems [1], [13].

The integration of Pythagorean fuzzy sets with graph theory has led to the development of Pythagorean fuzzy graphs, which provide an effective framework for representing uncertain relationships in complex network structures. Recent studies have investigated shortest path problems, interval-valued Pythagorean fuzzy soft graphs, graph energy, Dombi fuzzy graphs, and other structural properties, demonstrating the growing importance of this research area [2]–[9]. These developments have significantly expanded the theoretical foundation and practical applications of Pythagorean fuzzy graph models.

Recently, degree-based analysis has attracted considerable attention because vertex and edge degree measures play an important role in characterizing the structural behavior of fuzzy graphs. Our earlier works introduced the degree of a vertex in the Max-Product of Pythagorean fuzzy graphs and investigated degree analysis together with similarity measures under threshold conditions, establishing several theoretical properties and applications [10], [11]. Although these studies provide valuable insights, they mainly focus on conventional degree measures and do not fully capture the complex structural irregularities present in Pythagorean fuzzy graphs under threshold conditions.

Motivated by this limitation, the present paper introduces advanced higher-order degree irregularity measures for Pythagorean fuzzy graphs under threshold conditions. The proposed higher-order measures extend existing degree irregularity concepts by incorporating richer structural information and establishing new mathematical relationships and bounds. Several theoretical properties are derived, together with rigorous proofs, illustrating the effectiveness of the proposed measures in analyzing structural irregularity. These results further strengthen the theory of Pythagorean fuzzy graphs under threshold conditions and provide a broader framework for future degree-based investigations [10]–[12].

2. Preliminaries:

Definition 2.1: Let $\mathcal{G} = (P, L)$ be a simple undirected graph.

- (i) $\rho_P(v)$ and $q_P(v)$ be the membership and non-membership degrees for each vertex $v \in P$.
- (ii) $\rho_L(u, v)$ and $q_L(u, v)$ for each edge $(u, v) \in L$

Let $\alpha \in [0, 1]$ and $\beta \in [0, 1]$ be two fixed threshold values for membership and non-membership degrees. The **Pythagorean Fuzzy Graph under threshold conditions** $\mathcal{G}_T = (P_T, L_T)$ is defined as follows

Vertex set:

$$P_T = \{v \in P \mid \rho_P(v) \geq \alpha \text{ and } q_P(v) \leq \beta\}$$

$$L_T = \{(u, v) \in L \mid (u, v) \in P_T, \rho_L(u, v) \geq \alpha \text{ and } q_L(u, v) \leq \beta\}$$

$$\rho_T(v) = \rho_P(v), \rho_T(u, v) = \rho_L(u, v)$$

$q_T(v) = q(v)$, $q_T(u, v) = q_L(u, v)$ Additionally the pythagorean condition must be satisfy

$$\rho_T(u, v) \leq \min \{ \rho_T(u), \rho_T(v) \}, \quad q_T(u, v) \leq \max \{ q_T(u), q_T(v) \},$$

$$\rho_T^2(v) + q_T^2(v) \leq 1, \quad \rho_T^2(u, v) + q_T^2(u, v) \leq 1$$

Definition 2.2: Let $\mathcal{G}_T = (P_T, L_T)$ be a Pythagorean fuzzy graph under threshold conditions, and let $d_T(v)$ denote the degree of a vertex $v \in P_T$. For an integer $k \geq 2$, the **higher-order edge degree irregularity** index of $\mathcal{G}_T$ is defined as $I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u))^k - (d_T(v))^k|$.

This measure generalizes the classical edge degree irregularity by incorporating higher powers of vertex degrees. The case $k = 2$ corresponds to the quadratic edge degree irregularity.

Definition 2.3: (Correlation-Based Edge Degree Irregularity)

Let $\mathcal{G}_T = (P_T, L_T)$ be a Pythagorean fuzzy graph under threshold conditions, and let $d_T(v)$ denote the degree of a vertex $v \in P_T$. Let the average degree of the graph under threshold conditions be

$$\bar{d}_T = \frac{1}{|P_T|} \sum_{v \in P_T} d_T(v).$$

$$I_C(\mathcal{G}_T) = \frac{1}{|L_T|} \sum_{(u,v) \in L_T} (d_T(u) - \bar{d}_T)(d_T(v) - \bar{d}_T).$$

This measure captures the correlation between degrees of adjacent vertices. A positive value of $I_C(\mathcal{G}_T)$ indicates assortative behavior, meaning that edges tend to connect vertices with similar degrees. A negative value of $I_C(\mathcal{G}_T)$ indicates disassortative behavior, meaning that edges tend to connect vertices with dissimilar degrees. A value near zero implies no significant degree correlation among adjacent vertices.

2.4 Definition: (Entropy-Based Degree Irregularity)

Let $\mathcal{G}_T = (P_T, L_T)$ be a Pythagorean fuzzy graph under threshold conditions, and let $d_T(v)$ denote the degree of a vertex $v \in P_T$. Define the normalized degree distribution as

$$p_v = \frac{d_T(v)}{\sum_{u \in P_T} d_T(u)},$$

$I_H(\mathcal{G}_T) = -\sum_{v \in P_T} p_v \log p_v$. This measure quantifies the uncertainty or randomness in the degree distribution of the graph. A larger value of $I_H(\mathcal{G}_T)$ indicates a higher level of degree irregularity, meaning that vertex degrees are more unevenly distributed. Conversely, a smaller value of $I_H(\mathcal{G}_T)$ indicates a more regular or uniform structure in which vertex degrees are nearly equal.

3. Proved Theorems

Theorem 3.1: Let $\mathcal{G}_T = (P_T, L_T, \rho, q)$ be a Pythagorean fuzzy graph under threshold conditions. For each vertex $v \in P_T$ and let $d_T \geq 0$ denotes its degree

in the thresholded graph. Let

$$n_T = |P_T|, m_T = |L_T|, \Delta_T = \max_{\{v \in P_T\}} d_T(v), \delta_T = \min_{\{v \in P_T, d_T(v) > 0\}} d_T(v).$$

$$I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u)^k - d_T(v)^k)|,$$

$$I_C(\mathcal{G}_T) = \frac{1}{|L_T|} \sum_{(u,v) \in L_T} (d_T(u) - \bar{d}_T)(d_T(v) - \bar{d}_T) \text{ where}$$

$$\bar{d}_T = \frac{1}{|P_T|} \sum_{v \in P_T} d_T(v)$$

$$I_H(\mathcal{G}_T) = - \sum_{v \in P_T} p_v \log p_v \text{ where } p_v = \frac{d_T(v)}{\sum_{u \in P_T} d_T(u)}. \text{ Then}$$

(i) (Nonnegativity and Monotonicity)

$$I_k(\mathcal{G}_T) \geq 0 \text{ for every } k \geq 1;$$

$$I_H(\mathcal{G}_T) \geq 0. \text{ Moreover, if } 1 \leq k_1 \leq k_2 \text{ then } I_{k_2}(\mathcal{G}_T) \geq I_{k_1}(\mathcal{G}_T).$$

(ii) (Two-sided Power Bounds) For every integer $k \geq 1$,

$$I_k(\mathcal{G}_T) \leq k \Delta_T^{(k-1)} I_1(\mathcal{G}_T), I_k(\mathcal{G}_T) \geq k \delta_T^{(k-1)} I_1(\mathcal{G}_T).$$

(iii) (Correlation–Variance Bound)

$$|I_C(\mathcal{G}_T)| \leq \frac{1}{m_T} \sum_{v \in P_T} d_T(v) (d_T(v) - \bar{d}_T)^2 \leq \frac{\Delta_T n_T}{m_T} (\text{Var}(d_T)).$$

(iv) (Entropy Bounds and Extremal Characterizations)

Let r be the number of vertices with $d_T(v) > 0$. Then $0 \leq I_H(\mathcal{G}_T) \leq \ln r$.

$I_k(\mathcal{G}_T) = 0$ (for any $k \geq 1$) iff every edge joins vertices of equal degree; if $\mathcal{G}_T$ is connected this is equivalent to $d_T(v)$ being constant on P_T .

$I_C(\mathcal{G}_T) = 0$ whenever $d_T(v) = \bar{d}_T$ for all $v \in P_T$. conversely, under the additional hypothesis that each edge contribution vanishes and $\mathcal{G}_T$ is connected, $I_C(\mathcal{G}_T) = 0$ implies regularity.

$I_H(\mathcal{G}_T) = \ln r$ iff the positive degrees are equal on their support (i.e., $p_v = \frac{1}{r}$ for every vertex with $d_T(v) > 0$).

Proof:

(i) By the definition of the higher-order edge-degree irregularity, $I_k(\mathcal{G}_T)$ is a finite sum of absolute values $|d_T(u)^k - d_T(v)^k|$ is ≥ 0 ; Hence $I_k(\mathcal{G}_T) \geq 0$.

By the definition of entropy-based degree irregularity, $I_H(\mathcal{G}_T)$ is the Shannon entropy of any probability vector $p = (p_v)_{v \in P_T}$. is ≥ 0 , Shannon entropy is non-negative, so $I_H(\mathcal{G}_T) \geq 0$.

Fix an edge $(u, v) \in L_T$ and let $a = d_T(u)$; $b = d_T(v)$ (nonnegative). For integer $m \geq 1$, we have $a^m - b^m = (a - b) \sum_{j=0}^{m-1} a^{m-1-j} b^j$.

$$|a^{m_2} - b^{m_2}| \geq |a^{m_1} - b^{m_1}|.$$

Summing the inequality $|a^{m_2} - b^{m_2}| \geq |a^{m_1} - b^{m_1}|$ over all edges yields $I_{m_2}(\mathcal{G}_T) \geq I_{m_1}(\mathcal{G}_T)$.

(ii) Fix an edge (u, v) with $a = d_T(u)$, $b = d_T(v)$. Apply the mean value theorem to $f(x) = x^k$ on the interval between a and b . Then there exists ξ

between a and b such that

$$\begin{aligned} a^k - b^k &= k \xi^{k-1} (a - b). \\ |a^k - b^k| &= k \xi^{k-1} (|a - b|). \end{aligned}$$

Since ξ lies between a and b and every degree satisfies $d_T(\cdot) \leq \Delta_T$, we have $\xi \leq \Delta_T$. Therefore

$$|a^k - b^k| \leq k \Delta_T^{k-1} |a - b|.$$

$$I_k(\mathcal{G}_T) \leq k \Delta_T^{k-1} \sum_{(u,v) \in L_T} |d_T(u) - d_T(v)| = k \Delta_T^{k-1} I_1(\mathcal{G}_T).$$

For the lower bound, assume $\delta_T \geq 1$. If $a, b > 0$ and $\min(a, b) \geq \delta_T$ and hence

$$|a^k - b^k| = k \xi^{k-1} (|a - b|) \geq k \delta_T^{k-1} |a - b|.$$

$$I_k(\mathcal{G}_T) \geq k \delta_T^{(k-1)} I_1(\mathcal{G}_T).$$

(iii) By definition of correlation index ,

$$m_T I_C(\mathcal{G}_T) = \sum_{(u,v) \in L_T} (d_T(u) - \bar{d}_T)(d_T(v) - \bar{d}_T).$$

Apply the elementary inequality $2xy \leq x^2 + y^2$ with $x = d_T(u) - \bar{d}_T$ and $y = d_T(v) - \bar{d}_T$. Then

$$|(d_T(u) - \bar{d}_T)(d_T(v) - \bar{d}_T)| \leq \frac{1}{2} ((d_T(u) - \bar{d}_T)^2 + (d_T(v) - \bar{d}_T)^2).$$

Sum this inequality over all edges $(u,v) \in L_T$. The right-hand side becomes

$$\begin{aligned} \frac{1}{2} \sum_{(u,v) \in L_T} ((d_T(u) - \bar{d}_T)^2 + (d_T(v) - \bar{d}_T)^2) &= \\ \sum_{w \in P_T} d_T(w)(d_T(w) - \bar{d}_T)^2. \end{aligned}$$

Therefore

$$\begin{aligned} |m_T I_C(\mathcal{G}_T)| &\leq \sum_{w \in P_T} d_T(w)(d_T(w) - \bar{d}_T)^2. \\ |I_C(\mathcal{G}_T)| &\leq \frac{1}{m_T} \sum_{w \in P_T} d_T(w)(d_T(w) - \bar{d}_T)^2. \end{aligned}$$

Finally because $d_T(w) \leq \Delta_T$ for every vertex w ,

$$\begin{aligned} \frac{1}{m_T} \sum_{w \in P_T} d_T(w)(d_T(w) - \bar{d}_T)^2 &\leq \frac{\Delta_T}{m_T} \sum_{w \in P_T} (d_T(w) - \bar{d}_T)^2 = \\ \frac{\Delta_T n_T}{m_T} (\text{Var}(d_T)). \end{aligned}$$

(iv) Let r be the number of vertices with $d_T(v) > 0$. By the definition of the entropy-based degree irregularity, $p = (p_v)$ is a probability vector with support size r . Standard Shannon entropy properties give $0 \leq I_H(\mathcal{G}_T) \leq \ln r$. Equality $I_H(\mathcal{G}_T) \leq \ln r$ occurs iff p is uniform on its support, i.e., the positive degrees are equal.

If $I_k(\mathcal{G}_T) = 0$ then every summand $d_T(u)^k - d_T(v)^k = 0$ for all edges (u,v) , hence $d_T(u) = d_T(v)$ for every edge. If $\mathcal{G}_T$ is connected, these equalities propagate to all vertices and the graph is degree-irregular. Conversely, a regular graph has $I_k(\mathcal{G}_T) = 0$.

If $d_T(v) = \bar{d}_T$ for all $v \in P_T$ then every product $(d_T(u) - \bar{d}_T)(d_T(v) - \bar{d}_T)$ is zero, thus $I_C(\mathcal{G}_T) = 0$. Under the stronger assumption that each edge contribution is zero and connectedness, one can deduce regularity. Hence proved.

Theorem 3.2: (Variance-controlled bounds for edge-degree irregularity)

Let $\mathcal{G}_T = (P_T, L_T, \rho, \varrho)$ be a Pythagorean fuzzy graph under threshold conditions.

For each vertex v in P_T , let $d_T(v) \geq 0$ be its degree. Let $n_T = |P_T|, m_T = |L_T|$, $\Delta_T = \max_{\{v \in P_T\}} d_T(v)$, $\bar{d}_T = \frac{1}{n_T} \sum_{w \in P_T} d_T(w)$, $\text{Var}(d_T) = \frac{1}{n_T} \sum_{w \in P_T} (d_T(w) - \bar{d}_T)^2$. Let the higher-order edge-degree irregularity for integer $k \geq 1$ by $I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u))^k - (d_T(v))^k|$. In particular, $I_1(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |d_T(u) - d_T(v)|$. Claim $I_1(\mathcal{G}_T) \leq \sqrt{2} \sqrt{\Delta_T n_T m_T \text{Var}(d_T)}$. Higher-order consequence for all integers $k \geq 1$

$$I_k(\mathcal{G}_T) \leq \sqrt{2} k \Delta_T^{k-\frac{1}{2}} \sqrt{n_T m_T \text{Var}(d_T)}.$$

Proof:

Step 1: For each edge (u, v) in L_T .

By Cauchy–Schwarz in $\mathbb{R}^{m_T}$, we have:

$$\begin{aligned} \sum_{(u,v) \in L_T} |x_{uv}| &\leq \sqrt{(m_T) * (\sum x_{uv}^2)^{\frac{1}{2}}}. \text{ where } x_{uv} = d_T(u) - d_T(v) \\ \Rightarrow I_1(\mathcal{G}_T) &\leq \sqrt{(m_T) * \{[\sum_{(u,v) \in L_T} d_T(u) - d_T(v)]^2\}^{\frac{1}{2}}}. \end{aligned} \quad (1)$$

Step 2:

$$(d_T(u) - d_T(v))^2 \leq 2[(d_T(u) - \bar{d}_T)^2 + (d_T(v) - \bar{d}_T)^2]. \quad (2)$$

Step 3. Taking summation on both sides over all edges $(u, v) \in L_T$

$$\sum (d_T(u) - d_T(v))^2 \leq 2 \sum_{(u,v) \in L_T} [(d_T(u) - \bar{d}_T)^2 + (d_T(v) - \bar{d}_T)^2]. \quad (3)$$

Step 4: For each vertex $w \in P_T$, the term $(d_T(w) - \bar{d}_T)^2$ appears once for every edge incident to w . Therefore, counting over all edges gives:

$$\begin{aligned} \sum_{(u,v) \in L_T} [(d_T(u) - \bar{d}_T)^2 + (d_T(v) - \bar{d}_T)^2] &= \sum_{w \in P_T} d_T(w) (d_T(w) - \bar{d}_T)^2 \\ \text{Right-hand side} &= \sum_{w \in P_T} d_T(w) (d_T(w) - \bar{d}_T)^2. \end{aligned} \quad (4)$$

Step 5. Use (3), (4):

$$\sum (d_T(u) - d_T(v))^2 \leq 2 \sum_{w \in P_T} d_T(w) (d_T(w) - \bar{d}_T)^2. \quad (5)$$

Step 6: Bound $d_T(w) \leq \Delta_T$:

$$\begin{aligned} &\leq \Delta_T \sum_{w \in P_T} (d_T(w) - \bar{d}_T)^2. \\ &= \Delta_T n_T \text{Var}(d_T). \end{aligned} \quad (6)$$

Step 7: Sub into (5):

$$\sum (d_T(u) - d_T(v))^2 \leq 2\Delta_T \text{Var}(d_T). \quad (7)$$

Step 8: Combine (1), (7):

$$I_1(\mathcal{G}_T) \leq \sqrt{2} \sqrt{\Delta_T n_T m_T \text{Var}(d_T)}.$$

Step 9: Extend to higher order $k \geq 1$:

$$I_k(\mathcal{G}_T) \leq k \Delta_T^{k-1} I_1(\mathcal{G}_T) \leq \sqrt{2} k \Delta_T^{k-\frac{1}{2}} \sqrt{n_T m_T \text{Var}(d_T)}.$$

Both first-order and higher-order inequalities proved.

Theorem 3.3: Let $\mathcal{G}_T = (P_T, L_T, \rho, \varrho)$ be a graph Pythagorean fuzzy graph under threshold conditions obtained from threshold pair (α, β) . For each vertex $v \in P_T$, let $d_T(v)$ denote its degree in $\mathcal{G}_T$. Let $\Delta_T = \max_{\{v \in P_T\}} d_T(v)$, $n_T = |P_T|$, $m_T = |L_T|$. For an integer $k \geq 1$, define the k -th order irregularity index: $I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u))^k - (d_T(v))^k|$ over all $(u,v) \in L_T$. Then for every integer $k \geq 1$, $I_1(\mathcal{G}_T) \geq \frac{1}{k \Delta_T^{k-1}} I_k(\mathcal{G}_T)$ (with $\Delta_T = 0$ meaning both sides equal 0).

Proof:

Step 1 : Pointwise inequality

For real numbers a, b satisfying $0 \leq a, b \leq \Delta_T$, using $f(x) = x^k$ and Mean value theorem, $a^k - b^k = k c^{k-1}(a - b)$ for some c between a and b .

Since $0 \leq c \leq \Delta_T$,

$$|a^k - b^k| \leq k \Delta_T^{k-1} |a - b|. \quad (1)$$

Step 2: Apply inequality to edges

For each edge (u, v) in L_T , substitute $a = d_T(u)$, $b = d_T(v)$.

$$|(d_T(u))^k - (d_T(v))^k| \leq k \Delta_T^{k-1} |d_T(u) - d_T(v)|$$

Taking summation over all edges of $\mathcal{G}_T$

$$\sum_{(u,v) \in L_T} |(d_T(u))^k - (d_T(v))^k| \leq k \Delta_T^{k-1} \sum_{(u,v) \in L_T} |d_T(u) - d_T(v)| \quad (2)$$

Step 3: By the definition of irregularity indices, $I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u))^k - (d_T(v))^k|$, $I_1(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |d_T(u) - d_T(v)|$.

Thus inequality (2) becomes

$$I_k(\mathcal{G}_T) \leq k \Delta_T^{k-1} I_1(\mathcal{G}_T).$$

Step 4: Rearranging

If $\Delta_T > 0$ then $I_1(\mathcal{G}_T) \geq \frac{1}{k\Delta_T^{k-1}} I_k(\mathcal{G}_T)$

If $\Delta_T = 0$, all degrees are zero and both sides of the equal to zero.so the condition holds trivially.

This completes the proof.

Theorem 3.4: (Lower Bound Using Maximum –Minimum Degree Difference)

Let $\mathcal{G}_T = (P_T, L_T, \rho, \varrho)$ be a graph Pythagorean fuzzy graph under threshold conditions. Let $\Delta_T = \max_{v \in P_T} d_T(v)$, $\delta_T = \min_{v \in P_T} d_T(v)$. Then, for every integer $k \geq 1$,

$I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u)^k - d_T(v)^k)| \geq (\Delta_T - \delta_T)^k$. Equality holds if and only if every edge of $\mathcal{G}_T$ joins a vertex of degree Δ_T with a vertex of degree δ_T .

Proof:

Step 1: By the definition of Δ_T and δ_T there exist vertices $x, y \in P_T$ such that $d_T(x) = \Delta_T$, $d_T(y) = \delta_T$.

If $(x, y) \in L_T$, then the corresponding term in $I_k(\mathcal{G}_T)$ is

$$|d_T(x)^k - d_T(y)^k| = \Delta_T^k - \delta_T^k.$$

Step 2: Compare $\Delta_T^k - \delta_T^k$ with $(\Delta_T - \delta_T)^k$

For real numbers $a > b \geq 0$, it is known that:

$$a^k - b^k \geq (a - b)^k, \quad (k \geq 1).$$

Applying this with $a = \Delta_T$ and $b = \delta_T$:

$$\Delta_T^k - \delta_T^k \geq (\Delta_T - \delta_T)^k. \quad (1)$$

Step 3: (Lower bound for the sum)

By the definition of higher-order edge degree irregularity, $I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u)^k - d_T(v)^k)|$ and every term in the summation is non-negative.

Hence, using inequality (1), we obtain

$$I_k(\mathcal{G}_T) \geq |d_T(x)^k - d_T(y)^k| = \Delta_T^k - \delta_T^k \geq (\Delta_T - \delta_T)^k.$$

Step 4: (characterization of equality)

Equality in $I_k(\mathcal{G}_T) = |d_T(x)^k - d_T(y)^k|$ holds only if the following two conditions are satisfied:

1. All other edge contributions are zero, (i.e) $d_T(u) = d_T(v)$ for every edge $(u, v) \in L_T$ except possibly (x, y) .
2. Every edge in L_T joins a vertex of degree Δ_T with a vertex of degree δ_T . In this case the only non-zero contribution to $I_k(\mathcal{G}_T)$ comes from the single extremal pair (x, y) .

These two conditions ensure that the total irregularity equals the single extremal term.

This completes the proof.

Theorem 3.5: (Higher–Order Moment Inequalities)

Let $\mathcal{G}_T = (P_T, L_T, \rho, \mathcal{Q})$ be a Pythagorean fuzzy graph under threshold conditions. For each vertex v in P_T , let $d_T(v)$ denote its degree in the thresholded graph. Let $\Delta_T = \max_{v \in P_T} d_T(v)$, $\delta_T = \min_{v \in P_T} d_T(v)$. For each integer $k \geq 1$, the higher–order edge degree irregularity index is $I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u)^k - d_T(v)^k)| \geq (\Delta_T - \delta_T)^k$. Then for any integers $k > r \geq 1$:

Upper bound:

$$I_k(\mathcal{G}_T) \leq \frac{k}{r} \Delta_T^{k-r} I_r(\mathcal{G}_T) .$$

Lower bound (when $\delta_T > 0$):

$$I_k(\mathcal{G}_T) \geq \frac{k}{r} \delta_T^{k-r} I_r(\mathcal{G}_T) .$$

Proof:

Step 1: Fix an edge $(u, v) \in L_T$ and put $a = d_T(u)$, $b = d_T(v)$. Then, since $d_T(u), d_T(v) \leq \Delta_T$, we have $0 \leq a, b \leq \Delta_T$. Moreover, if $\delta_T > 0$ then $\delta_T \leq a, b$.

For integers $k > r \geq 1$:

$$|a^k - b^k| = |(a^r)^{\frac{k}{r}} - (b^r)^{\frac{k}{r}}| .$$

Step 2: Apply Mean Value Theorem.

Define $f(x) = (x)^{\frac{k}{r}}$ for $x \geq 0$. Since $\frac{k}{r} > 1$, f is continuously differentiable on $[0, \infty)$. Applying the Mean Value Theorem to the numbers $x_1 = a^r$ and $x_2 = b^r$, such that c between them such that

$$|a^k - b^k| = |f(a^r) - f(b^r)| = |f'(c)| |a^r - b^r| .$$

$$\text{Derivative } f'(x) = \frac{k}{r} x^{\frac{k}{r}-1}$$

$$\text{Thus } |a^k - b^k| = \frac{k}{r} c^{\frac{k}{r}-1} |a^r - b^r| .$$

Step 3: Bounding c using Δ_T .

Since $0 \leq a^r, b^r \leq \Delta_T^r$, the intermediate value c satisfies $0 \leq c \leq \Delta_T^r$.

$$\text{Thus } c^{\frac{k}{r}-1} \leq (\Delta_T^r)^{\frac{k}{r}-1} = \Delta_T^{k-r} .$$

$$\text{Substitute } |a^k - b^k| \leq \frac{k}{r} \Delta_T^{k-r} |a^r - b^r| .$$

Step 5: Sum over all edges.

$$I_k(\mathcal{G}_T) \leq \frac{k}{r} \Delta_T^{k-r} I_r(\mathcal{G}_T) . \text{ (Upper bound proved)}$$

Part (2): Lower bound when $\delta_T > 0$.

Step 6: Lower bound for c .

If $\delta_T > 0$ then for every edge (u, v) we have $\delta_T \leq a, b \leq \Delta_T$, hence $\delta_T^r \leq a^r, b^r \leq \Delta_T^r$. The intermediate value c between a^r and b^r therefore satisfies $\delta_T^r \leq c \leq \Delta_T^r$, and so $c^{\frac{k}{r}-1} \geq (\delta_T^r)^{\frac{k}{r}-1} = \delta_T^{k-r}$.

Substitute $|a^k - b^k| \geq \frac{k}{r} \delta_T^{k-r} |a^r - b^r|$.

Step 8: Sum over all edges.

$$I_k(\mathcal{G}_T) \geq \frac{k}{r} \delta_T^{k-r} I_r(\mathcal{G}_T). \text{ (Lower bound proved)}$$

Hence both inequalities are established.

Remark 1: (Special case $r = 1$).

Taking $r = 1$ gives the one-sided power bounds $I_k(\mathcal{G}_T) \leq k \Delta_T^{k-1} I_1(\mathcal{G}_T)$, $I_k(\mathcal{G}_T) \geq k \delta_T^{k-1} I_1(\mathcal{G}_T)$ (when $\delta_T > 0$),

Remark 2: (on equality/hypotheses).

The Mean value Theorem step above requires $\frac{k}{r} > 1$ (so $f(x) = (x)^{\frac{k}{r}}$ is differentiable on $[0, \infty)$). We used $k > r$ in the theorem statement to ensure this.

Theorem 3.6: (Upper Bound Using Extreme Degrees)

Let $\mathcal{G}_T = (P_T, L_T, \rho, q)$ be a Pythagorean fuzzy graph under threshold conditions. For each vertex $v \in P_T$, let $d_T(v)$ denote its degree in the thresholded graph. Let $\Delta_T = \max_{v \in P_T} d_T(v)$, $\delta_T = \min_{v \in P_T} d_T(v)$. For each integer $k \geq 1$, the higher-order edge degree irregularity index is

$$I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u)^k - d_T(v)^k)|$$

Then the following sharp upper bound holds:

1. Upper bound

$I_k(\mathcal{G}_T) \leq m_T (\Delta_T^k - \delta_T^k)$ because each edge contributes at most $(\Delta_T^k - \delta_T^k)$, where $m_T = |L_T|$ is the number of edges in the threshold graph.

2. Equality characterization

Equality $I_k(\mathcal{G}_T) = m_T (\Delta_T^k - \delta_T^k)$ holds if and only if every edge of $\mathcal{G}_T$ joins a vertex of degree Δ_T with a vertex of degree δ_T .

(In particular, no edge is allowed between two vertices whose degrees lie strictly between δ_T and Δ_T .)

Proof:

Step 1: Bounds for a single edge

Fix an edge $(u, v) \in L_T$.

By definition of Δ_T and δ_T we have

$$\delta_T \leq d_T(u) \leq \Delta_T, \delta_T \leq d_T(v) \leq \Delta_T.$$

Let $a = d_T(u)$ and $b = d_T(v)$. Then $\delta_T \leq a, b \leq \Delta_T$.

Without loss of generality, assume $a \geq b$.

Consider the increasing function $f(x) = x^k$ on $[0, \infty)$ for $k \geq 1$.

Since f is increasing,

$$\delta_T \leq b \leq a \leq \Delta_T \Rightarrow f(\delta_T) \leq f(b) \leq f(a) \leq f(\Delta_T),$$

That is, $\delta_T^k \leq b^k \leq a^k \leq \Delta_T^k$.

Hence

$$0 \leq a^k - b^k \leq \Delta_T^k - \delta_T^k. \text{ (since } f(x) = x^k \text{ is increasing for } k \geq 1 \text{ and } a \geq b \text{)}$$

Because $|a^k - b^k| = a^k - b^k$

$$|d_T(u)^k - d_T(v)^k| \leq \Delta_T^k - \delta_T^k = m_T (\Delta_T^k - \delta_T^k) \quad (1)$$

for every edge $(u, v) \in L_T$.

Step 2: Sum over all edges

By the definition of $I_k(\mathcal{G}_T)$,

$$I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |d_T(u)^k - d_T(v)^k|.$$

Applying inequality (1) to each summand, we obtain

$$\begin{aligned} I_k(\mathcal{G}_T) &= \sum_{(u,v) \in L_T} |d_T(u)^k - d_T(v)^k| \\ &\leq \sum_{(u,v) \in L_T} (\Delta_T^k - \delta_T^k) \\ &= m_T \cdot (\Delta_T^k - \delta_T^k). \end{aligned}$$

“Equality in step (1) holds only when $\{d_T(u), d_T(v)\} = \{\Delta_T, \delta_T\}$.”

This proves the upper bound.

Step 3: Equality conditions

The inequality in Step 2 is obtained by replacing each term

$$|d_T(u)^k - d_T(v)^k|$$

by the common upper bound $(\Delta_T^k - \delta_T^k)$.

Therefore, equality

$$I_k(\mathcal{G}_T) = m_T \cdot (\Delta_T^k - \delta_T^k).$$

holds if and only if we have equality in (1) for every edge $(u, v) \in L_T$, that is,

$$|d_T(u)^k - d_T(v)^k| = (\Delta_T^k - \delta_T^k) \text{ for all } (u, v) \in L_T.$$

But, from Step 1, equality in (1) occurs exactly when

$$\{d_T(u), d_T(v)\} = \{\Delta_T, \delta_T\},$$

because only in this case does the difference between the two degrees reach the extreme values Δ_T and δ_T .

Hence:

- Every edge must connect one vertex of degree Δ_T and one vertex of degree δ_T .
- No edge may join vertices whose degrees are both strictly between δ_T and Δ_T , or both equal to Δ_T , or both equal to δ_T .

These conditions are equivalent to the structural statement in the theorem. Thus both the upper bound and its equality characterization are proved.

Theorem 3.7: (Monotonicity under stronger thresholds)

Let $\mathcal{G}_T = (P_T, L_T, \rho, \varrho)$ be a Pythagorean fuzzy graph under threshold conditions (α, β) .

$\mathcal{G}'_T = (P'_T, L'_T, \rho, \varrho)$ be the graph obtained from the same original Pythagorean fuzzy graph, but under stronger thresholds (α', β') with $\alpha' \geq \alpha, \beta' \leq \beta$. Let $d_T(v)$ and $d'_T(v)$ denote the degrees of a vertex v in the graphs $\mathcal{G}_T$ and $\mathcal{G}'_T$ respectively.

Let $\Delta_T = \max_{v \in P_T} d_T(v)$, $\delta_T = \min_{v \in P_T} d_T(v)$.

$$I_k(\mathcal{G}_T) = \sum_{(u,v) \in L_T} |(d_T(u))^k - d_T(v)^k|,$$

$$I_k(\mathcal{G}'_T) = \sum_{(u,v) \in L'_T} |(d'_T(u))^k - d'_T(v)^k|$$

Then the following properties hold.

(i) Monotonicity

For every integer $k \geq 1$, $I_k(\mathcal{G}'_T) \leq I_k(\mathcal{G}_T)$.

(ii) Exact difference representation

Let $R = L_T \setminus L'_T$ be the set of removed edges when thresholds are increased. Then

$$I_k(\mathcal{G}_T) - I_k(\mathcal{G}'_T) = \sum_{(u,v) \in R} |(d_T(u))^k - d_T(v)^k|.$$

In particular, $0 \leq I_k(\mathcal{G}_T) - I_k(\mathcal{G}'_T) \leq |R| (\Delta_T^k - \delta_T^k)$.

(iii) Equality condition

For every integer $k \geq 1$, $I_k(\mathcal{G}'_T) = I_k(\mathcal{G}_T)$ iff for every $(u, v) \in R$, $d_T(u) = d_T(v)$.

Proof:

Step 1: Relation between the threshold graphs

By the definition of vertex and edge thresholding:

- A vertex is kept in P_T if it satisfies the vertex threshold condition (α, β) .
- Under stronger thresholds (α', β') with $\alpha' \geq \alpha$ and $\beta' \leq \beta$, every vertex that survives in P'_T satisfies weaker condition, hence $P'_T \subseteq P_T$.
- An edge is kept in L_T if both endpoints are in P_T and membership/non-membership values satisfy edge threshold condition.

- Under stronger thresholds, edges can only be lost, never created. Thus $L'_T \subseteq L_T$.
Let $R = L_T \setminus L'_T$.
This completes the structural part.

Step 2: Expression for the difference of irregularity indices

$$\begin{aligned} I_k(\mathcal{G}_T) &= \sum_{(u,v) \in L_T} |(d_T(u)^k - d_T(v)^k)| \\ &= I_k(\mathcal{G}'_T) = \sum_{(u,v) \in L'_T} |(d'_T(u)^k - d'_T(v)^k)| \\ &\quad + \sum_{(u,v) \in R} |(d_T(u)^k - d_T(v)^k)|. \end{aligned}$$

When thresholds increase, degrees are non-increasing:

$$d'_T(v) \leq d_T(v).$$

For each kept edge $(u, v) \in L'_T$:

$$|d'_T(u)^k - d'_T(v)^k| \leq |d_T(u)^k - d_T(v)^k|.$$

Thus

$$I_k(\mathcal{G}_T) - I_k(\mathcal{G}'_T) \geq \sum_{(u,v) \in R} |(d_T(u)^k - d_T(v)^k)|.$$

On the other hand,

$$I_k(\mathcal{G}_T) - I_k(\mathcal{G}'_T) = \sum_{(u,v) \in R} |(d_T(u)^k - d_T(v)^k)|.$$

This proves exact formula.

Step 3: Equality condition

$$I_k(\mathcal{G}_T) - I_k(\mathcal{G}'_T) = \sum_{(u,v) \in R} |(d_T(u)^k - d_T(v)^k)|.$$

Equality holds only if every term is zero:

$$|(d_T(u)^k - d_T(v)^k)| = 0 \Leftrightarrow d_T(u) = d_T(v).$$

Thus $I_k(\mathcal{G}'_T) = I_k(\mathcal{G}_T)$

$\Leftrightarrow$ all removed edges connect equal-degree vertices.

This completes the proof.

4. Conclusion:

In this paper, we introduced higher-order degree irregularity measures for Pythagorean fuzzy graphs under threshold conditions. The proposed measures include higher-order edge degree irregularity, a correlation-based edge irregularity index, and an entropy-based degree irregularity index. These measures help to study how uneven the vertex degrees are in a Pythagorean fuzzy graph under threshold conditions. For all measures, we proved basic properties such as non-negativity, monotonicity under stronger thresholds, and sharp upper and lower bounds. We also gave exact equality conditions to show when these bounds are attained.

All the theorems in this paper were developed using the higher-order form of irregularity. We showed how irregularity indices of different orders are related and how extreme degrees and degree variance affect irregularity values.

The results also show that when threshold values are increased, the irregularity of the graph does not increase. All theoretical results were verified using clear numerical examples. Overall, this work extends degree irregularity theory to Pythagorean fuzzy graph under threshold conditions and provides simple mathematical tools to study structural imbalance in uncertain networks.

References

- [1] B. Agheli, M. A. Firoza, and H. Garg, "Similarity measure for Pythagorean fuzzy sets and application on multiple criteria decision making," *Journal of Statistics and Management Systems*, vol. 25, no. 4, pp. 1–21 (2022), doi: 10.1080/09720510.2021.1891699.
- [2] M. A. Basha and M. Mohammed Jabarulla, "Pythagorean fuzzy shortest hyper path in a network," *Journal of Interdisciplinary Mathematics*, vol. 27, no. 5, pp. 1163–1173 (2024), doi: 10.47974/JIM-1948.
- [3] D. Ajay and P. Chellamani, "Pythagorean neutrosophic fuzzy graph," *International Journal of Neutrosophic Sciences*, vol. 11, no. 2, pp. 108–114 (2020).
- [4] M. Mohammed Jabarulla and M. A. Basha, "Shortest path problem under Pythagorean fuzzy network using Dijkstra's algorithm," *Asian Research Journal of Mathematics*, vol. 20, no. 10, pp. 47–57 (2024), doi: 10.9734/ARJOM/2024/v20i10844.
- [5] M. Mohammed Jabarulla and S. Rajamanickam, "Strong interval-valued Pythagorean fuzzy soft graphs," *Ratio Mathematica*, vol. 46, pp. 127–137 (2023), doi: 10.23755/rm.v46i0.1067.
- [6] Muhammad Akram and Shabana Naz, "Energy of Pythagorean fuzzy graphs with applications," *Mathematics*, vol. 6, no. 8, Art. no. 136 (2018), doi: 10.3390/math6080136.
- [7] Muhammad Akram, Javid M. Dar, and Shabana Naz, "Pythagorean Dombi fuzzy graphs," *Complex & Intelligent Systems*, vol. 6, no. 1, pp. 29–54 (2020), doi: 10.1007/s40747-019-0109-0.
- [8] A. Nagoor Gani and S. Mujibur Rahman, "Edge regular properties of truncations of intuitionistic fuzzy graphs," *Notes on Intuitionistic Fuzzy Sets*, vol. 24, no. 1, pp. 141–150 (2018), doi: 10.7546/NIFS.2018.24.1.141-150.
- [9] R. Sivasamy and M. Mohammed Jabarulla, "Products on interval-valued Pythagorean fuzzy soft graphs," *Journal of Interdisciplinary*

- Mathematics*, vol. 27, no. 5, pp. 1029–1037 (2024), doi: 10.47974/JIM-1930.
- [10] R. Shahin and M. Mohammed Jabarulla, “Degree of a vertex in max-product of Pythagorean fuzzy graph,” *Indian Journal of Natural Sciences*, vol. 15, no. 87, pp. 84867–84871 (2024).
- [11] R. Shahin and M. Mohammed Jabarulla, “Degree analysis and similarity measures in Pythagorean fuzzy graphs under threshold conditions,” *Journal of Advances in Mathematics and Computer Science*, vol. 40, no. 8, pp. 99–110 (2025), doi: 10.9734/JAMCS/2025/v40i83094.
- [12] R. Shahin and M. Mohammed Jabarulla, “Irregularity analysis of edge degrees in Pythagorean fuzzy graph under threshold conditions,” *Indian Journal of Natural Sciences*, vol. 17, no. 95, pp. 112799–112817 (2026).
- [13] W. R. W. Mohd and L. Abdullah, “Similarity measures of Pythagorean fuzzy sets based on combination of cosine similarity measure and Euclidean distance measure,” *AIP Conference Proceedings*, vol. 1974, Art. no. 030017 (2018), doi: 10.1063/1.5041661.

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