

A novel memory iterative method using hermite interpolation for nonlinear equations

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Abstract

Nonlinear equations are used to model the real world phenomenon which are not adequately represented by linear equations. A three-stage iterative algorithm applying memory is developed in this study. This research proposes a three-step iteration scheme enhanced with memory scheme with squared parameter using Hermite interpolation for solving nonlinear equations. Also, we prove some results related to convergences of this iterative method.

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1. Introduction

Nonlinear equations appear in many areas of science and engineering, representing problems [6] that cannot be adequately modeled by linear equations. Nonlinear equations are common in science, engineering, economics, and many other areas. Unlike linear equations, they are more difficult to solve and usually cannot be solved directly [1]. For this reason, mathematicians use iterative methods, which give approximate solutions step by step. Methods like Newton's method are very popular because they are simple and fast, but they do not always work well [10] if the first guess is not near the solution. To improve this, many higher-order and multi-step methods have been developed, but these often need more calculations, which can make them less efficient [11, 12]. In recent years,

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researchers have worked on creating better methods that give faster results with fewer calculations [3, 4, 7, 13, 14]. One useful idea is to use memory-based methods, which reuse past information to speed up the process without extra cost [2, 8]. In this paper, Our method uses Hermite interpolation with a squared parameter that adjusts itself automatically. This makes the method faster and more accurate. Derivative-free and memory-based approaches have also demonstrated high efficiency for solving nonlinear equations [9].

2. Preliminaries

Memory Based iterative Method Existing [1, 5]

$$q_k = p_k - \frac{G^*(p_k)}{G^{*'}(p_k)},$$

$$r_k = p_k - \frac{G^*(p_k)}{G^{*'}(p_k)} \left(1 + \frac{G^*(q_k)}{G^*(p_k)} + \left(1 + \frac{1}{1 + \frac{G^*(p_k)}{G^{*'}(p_k)}} \right) \left(\frac{G^*(q_k)}{G^*(p_k)} \right)^2 \right).$$

$$p_{k+1} = r_k - \frac{G^*(r_k)}{G^*[r_k, q_k] + (r_k - q_k)G^*[r_k, q_k, p_k] + (r_k - q_k)(r_k - p_k)G^*[r_k, q_k, p_k, p_k] + \alpha G^*(r_k)}$$

where $G^*[r_k, q_k]$, $G^*[r_k, q_k, p_k]$, and $G^*[r_k, q_k, p_k, p_k]$ denote the divided differences, which can be expressed as,

$$G^*[r_k, q_k] = \frac{G^*(r_k) - G^*(q_k)}{r_k - q_k},$$

$$G^*[r_k, q_k, p_k] = \frac{G^*[r_k, q_k] - G^*[q_k, p_k]}{r_k - p_k},$$

and

$$G^*[r_k, q_k, p_k, p_k] = \frac{G^*[r_k, q_k, p_k] - G^*[q_k, p_k, p_k]}{r_k - p_k}.$$

3. Analysis of Convergence with proposed Method

Here, In the proposed Method we have used parameter α^2 instead of α in 3rd Step of [1] as follow to enhance the convergence order from eight to nine

$$q_k = p_k - \frac{G^*(p_k)}{G^{*'}(p_k)},$$

$$r_k = p_k - \frac{G^*(p_k)}{G^{*'}(p_k)} \left(1 + \frac{G^*(q_k)}{G^*(p_k)} + \left(1 + \frac{1}{1 + \frac{G^*(p_k)}{G^{*'}(p_k)}} \right) \left(\frac{G^*(q_k)}{G^*(p_k)} \right)^2 \right).$$

$$p_{k+1} = r_k - \frac{G^*(r_k)}{G^*[r_k, q_k] + (r_k - q_k)G^*[r_k, q_k, p_k] + (r_k - q_k)(r_k - p_k)G^*[r_k, q_k, p_k, p_k] + \alpha_k^2 G^*(r_k)}.$$

where $G^*[r_k, q_k]$, $G^*[r_k, q_k, p_k]$, $G^*[r_k, q_k, p_k, p_k]$, $G^*(p_k)$, and $G^{*'}(p_k)$ have the same meaning as in [1].

$$\begin{aligned}
e_{k+1} = & A_2^2(A_2 + 5A_2^2 - A_3)((\alpha^2 + A_2)(A_2 + 5A_2^2 - A_3) + A_4)e_k^8 \\
& - A_2(8(19 + 45\alpha^2)A_2^6 + 360A_2^7 + 2A_2^5(13 + 76\alpha^2 - 196A_3) \\
& + 4A_2^3(-\alpha^2 A_3 + A_4) + A_2^4(2 + 26\alpha^2 - 8(15 + 49\alpha^2)A_3 + 66A_4) \\
& + 2A_2^3(\alpha^2 - 5(1 + 12\alpha^2)A_3 + 42A_3^2) \\
& + (7 + 10\alpha^2)A_4 - 5A_5) \\
& + A_2^2(2A_3(-5\alpha^2 + 6(1 + 7\alpha^2)A_3) + A_4 + 4(\alpha^2 - 12A_3)A_4 - 2A_5) \\
& + 2A_2(6\alpha^2 A_2^2 A_3 - 2A_3^3 + A_4^2 + A_3(-(3 + 2\alpha^2)A_4 + A_5))e_k^9 \\
& + O(e_k^{10}).
\end{aligned} \tag{1}$$

where $e_{k,q}$ and $e_{k,r}$ are the error expressions for all three steps, which are given in [1].

The above equation is represented NWM9 where,

$$\begin{aligned}
\alpha_k^2 = & \frac{(12H_5''(p_k) + 30(12H_5''(p_k))^2 - 4H_6'''(q_k)G^{*'}(p_k))^2}{(H_7^{(iv)}(r_k))^2(G^{*'}(p_k))^2} \\
& - \frac{(12H_5''(p_k)G^{*'}(p_k) + 30(12H_5''(p_k))^2 - 4H_6'''(q_k)G^{*'}(p_k))}{H_7^{(iv)}(r_k)H_5''(p_k)} \\
& - \frac{4(G^{*'}(p_k))^2}{(H_5''(p_k))^2}
\end{aligned} \tag{2}$$

where the hermite interpolating the set of polynomials $H_m(p)$ for $m = 5, 6, 7$ can be expressed as $\{H_7(P)^2\}$, $\{H_6(P)^2\}$, $\{H_5(P)^2\}$ Using the Newton-Raphson method, the divided differences formula to include all individual and cross terms.

Theorem 3.1: Let the function G^* be approximated by the Hermite polynomial H_m at the interpolation points $\{r_k, q_k, p_k, p_k, r_{k-1}, q_{k-1}, p_{k-1}, p_{k-1}\}$ within a domain $D \subseteq \mathbb{R}$, and suppose that the derivative $G^{*(m+1)}$ is continuous on D .

The interpolant is defined by the conditions $H_m(p_k) = G^*(p_k)$ and $H'_m(p_k) = G^{*'}(p_k)$.

Furthermore, all specified nodes $\{r_k, q_k, p_k, p_k, r_{k-1}, q_{k-1}, p_{k-1}, p_{k-1}\}$ lie in the vicinity of the root ξ . Then,

$$\left(H_7^{(iv)}(r_k)\right)^2 = 576(G^{*'}(\xi))^2 \left(A_4^2 - 2A_4A_8e_{k-1,r}e_{k-1,q}e_{k-1}^2 + A_8^2e_{k-1,r}^2e_{k-1,q}^2e_{k-1}^4\right) \tag{3}$$

$$\left(H_6^{(iii)}(q_k)\right)^2 = 36(G^{*'}(\xi))^2 \left(A_3^2 - 2A_3A_7e_{k-1,r}e_{k-1,q}e_{k-1}^2 + A_7^2e_{k-1,r}^2e_{k-1,q}^2e_{k-1}^4\right). \tag{4}$$

$$\left(H_5''(p_k)\right)^2 = 4(G^{*'}(\xi))^2 \left(A_2^2 - 2A_2A_6e_{k-1,r}e_{k-1,q}e_{k-1}^2 + A_6^2e_{k-1,r}^2e_{k-1,q}^2e_{k-1}^4\right). \tag{5}$$

$$\alpha_k^2 = \frac{(12H_5''(p_k)G^{*'}(p_k) + 30(H_5''(p_k))^2 - 4H_6'''(q_k)G^{*'}(p_k))^2}{(H_7^{(iv)}(r_k))^2(G^{*'}(p_k))^2}$$

$$\begin{aligned}
& - \frac{(12H_5''(p_k)G^{*'}(p_k) + 30(H_5''(p_k))^2 - 4H_6'''(q_k)G^{*'}(p_k))}{H_7^{(iv)}(r_k)H_5''(p_k)} \\
& - \frac{4(G^{*'}(p_k))^2}{(H_5''(p_k))^2}
\end{aligned} \tag{6}$$

$$\begin{aligned}
& [(A_2 + 5A_2^2 - A_3 - A_4) e_{k-1,r} e_{k-1,q} e_{k-1} \\
& - (A_2 + 5A_2^2 - A_3) A_6 A_8 e_{k-1,r}^2 e_{k-1,q}^4 e_{k-1} \\
& + 5A_2^2 A_8 (A_2 + 5A_2^2 - A_3) e_{k-1,r} e_{k-1,q}^2 e_{k-1} \\
& + 5A_2^6 A_8 (A_2 + 5A_2^2 - A_3) e_{k-1,r}^3 e_{k-1,q}^6 e_{k-1} \\
& - 10A_2 A_6 A_8 (A_2 + 5A_2^2 - A_3) e_{k-1,r} e_{k-1,q}^2 e_{k-1} \\
& - A_3 A_8 (A_2 + 5A_2^2 - A_3) e_{k-1,r} e_{k-1,q}^2 e_{k-1} \\
& + A_7 A_8 (A_2 + 5A_2^2 - A_3) e_{k-1,r}^2 e_{k-1,q}^4 e_{k-1}]^2.
\end{aligned} \tag{7}$$

Proof: The Hermite interpolants of degree five, six, and seven may be represented through the following expressions:

$$(G^*(p) - H_7(p))^2 = \left(\frac{G^{*(8)}(\delta)}{8!} \right)^2 (p - r_k)^2 (p - q_k)^2 (p - p_k)^4 (p - r_{k-1})^2 (p - q_{k-1})^2 (p - p_{k-1})^4. \tag{8}$$

$$(G^*(p) - H_6(p))^2 = \left(\frac{G^{*(7)}(\delta)}{7!} \right)^2 (p - q_k)^2 (p - p_k)^4 (p - r_{k-1})^2 (p - q_{k-1})^2 (p - p_{k-1})^4. \tag{9}$$

$$(G^*(p) - H_5(p))^2 = \left(\frac{G^{*(6)}(\delta)}{6!} \right)^2 (p - p_k)^4 (p - r_{k-1})^2 (p - q_{k-1})^2 (p - p_{k-1})^4. \tag{10}$$

By doing, rearrangement and differentiating we get equation four times at $p = r_k$, three times at $p = q_k$, and two times at $p = p_k$, respectively.

$$\begin{aligned}
(H_7^{(iv)}(r_k))^2 &= (G^{*(iv)}(r_k))^2 - \frac{48}{8!} G^{*(iv)}(r_k) G^{*(8)}(\delta) (r_k - r_{k-1}) \\
&\quad (r_k - q_{k-1}) (r_k - p_{k-1})^2 \\
&\quad + 576 \left(\frac{G^{*(8)}(\delta)}{8!} \right)^2 (r_k - r_{k-1})^2 (r_k - q_{k-1})^2 (r_k - p_{k-1})^4.
\end{aligned} \tag{11}$$

$$\begin{aligned}
(H_6'''(q_k))^2 &= (G^{*'''}(q_k))^2 - \frac{12}{7!} G^{*'''}(q_k) G^{*(7)}(\delta) (q_k - r_{k-1}) \\
&\quad (q_k - q_{k-1}) (q_k - p_{k-1})^2 \\
&\quad + 36 \left(\frac{G^{*(7)}(\delta)}{7!} \right)^2 (q_k - r_{k-1})^2 (q_k - q_{k-1})^2 (q_k - p_{k-1})^4.
\end{aligned} \tag{12}$$

$$\begin{aligned}
(H_5''(p_k))^2 &= (G^{*''}(p_k))^2 - \frac{4}{6!} G^{*''}(p_k) G^{*(6)}(\delta) (p_k - r_{k-1}) \\
&\quad (p_k - p_{k-1}) (p_k - p_{k-1})^2 \\
&\quad + 4 \left(\frac{G^{*(6)}(\delta)}{6!} \right)^2 (p_k - r_{k-1})^2 (p_k - q_{k-1})^2 (p_k - p_{k-1})^4.
\end{aligned} \tag{13}$$

Next, Taylor's series expansion is given,

$$(G^{*'}(p_k))^2 = (G^{*'}(\xi))^2(1 + 4A_2e_k + (6A_3 + 4A_2^2)e_k + O(e_k^3)). \quad (14)$$

$$(G^{*''}(p_k))^2 = (G^{*'}(\xi))^2(4A_2^2 + 24A_2A_3e_k + O(e_k^2)). \quad (15)$$

$$(G^{*'''}(q_k))^2 = (G^{*'}(\xi))^2(36A_3^2 + 288A_3A_4e_{k,q} + O(e_{k,q}^2)). \quad (16)$$

similarly,

$$(G^{*(iv)}(r_k))^2 = (G^{*'}(\xi))^2(576A_4^2 + 5760A_4A_5e_{k,r} + O(e_{k,r}^2)). \quad (17)$$

$$(G^{*(6)}(\delta))^2 = (G^{*'}(\xi))^2(518400A_6^2 + 7257600A_6A_7e_\delta + O(e_\delta^2)). \quad (18)$$

$$(G^{*(7)}(\delta))^2 = (G^{*'}(\xi))^2(25401600A_7^2 + 406425600A_7A_8e_\delta + O(e_\delta^2)). \quad (19)$$

$$(G^{*(8)}(\delta))^2 = (G^{*'}(\xi))^2(1625702400A_8^2 + 29262643200A_8A_9e_\delta + O(e_\delta^2)). \quad (20)$$

now, Putting (17) and (20) in (11), (16) and (19) in (12), and (15) and (18) in (13), we gets,

$$(H_7^{(iv)}(r_k))^2 = 576(G^{*'}(\xi))^2(A_4^2 - 2A_4A_8e_{k-1,r}e_{k-1,q}e_{k-1}^2 + A_8^2e_{k-1,r}^2e_{k-1,q}^2e_{k-1}^4), \quad (21)$$

$$(H_6^{(iii)}(p_k))^2 = 36(G^{*'}(\xi))^2(A_3^2 - 2A_3A_7e_{k-1,r}e_{k-1,q}e_{k-1}^2 + A_7^2e_{k-1,r}^2e_{k-1,q}^2e_{k-1}^4), \quad (22)$$

$$(H_5^{''}(p_k))^2 = 4(G^{*'}(\xi))^2(A_2^2 - 2A_2A_6e_{k-1,r}e_{k-1,q}e_{k-1}^2 + A_6^2e_{k-1,r}^2e_{k-1,q}^2e_{k-1}^4). \quad (23)$$

$$\begin{aligned} \alpha_k^2 &= \frac{(12H_5^{''}(p_k)G^{*'}(p_k) + 30(H_5^{''}(p_k))^2 - 4H_6^{'''}(q_k)G^{*'}(p_k))^2}{(H_7^{(iv)}(r_k))^2(G^{*'}(p_k))^2} \\ &\quad - \frac{(12H_5^{''}(p_k)G^{*'}(p_k) + 30(H_5^{''}(p_k))^2 - 4H_6^{'''}(q_k)G^{*'}(p_k))}{H_7^{(iv)}(r_k)H_5^{''}(p_k)} \\ &\quad - \frac{4(G^{*'}(p_k))^2}{(H_5^{''}(p_k))^2} \end{aligned} \quad (24)$$

$$\begin{aligned} &[(A_2 + 5A_2^2 - A_3 - A_4)e_{k-1,r}e_{k-1,q}e_{k-1} \\ &- (A_2 + 5A_2^2 - A_3)A_6A_8e_{k-1,r}^2e_{k-1,q}^4e_{k-1} \\ &+ 5A_2^2A_8(A_2 + 5A_2^2 - A_3)e_{k-1,r}e_{k-1,q}^2e_{k-1} \\ &+ 5A_2^6A_8(A_2 + 5A_2^2 - A_3)e_{k-1,r}^3e_{k-1,q}^6e_{k-1} \\ &- 10A_2A_6A_8(A_2 + 5A_2^2 - A_3)e_{k-1,r}e_{k-1,q}^2e_{k-1} \\ &- A_3A_8(A_2 + 5A_2^2 - A_3)e_{k-1,r}e_{k-1,q}^2e_{k-1} \\ &+ A_7A_8(A_2 + 5A_2^2 - A_3)e_{k-1,r}^2e_{k-1,q}^4e_{k-1}]^2. \end{aligned} \quad (25)$$

hence Proved.

4. Conclusion

A memory-based three-step method is proposed in this work. The method uses Hermite interpolation with a squared parameter that adjusts itself automatically. Our analysis has show that the scheme reaches ninth-order convergence, meaning it can find solutions much faster and more accurately than many existing methods, without needing extra function evaluations. The advantage of this method is that it reuses past information (memory) to speed up convergence at no extra cost. Numerical results conform that it performs better than traditional Three-step methods in terms of accuracy and efficiency. This makes the scheme a useful tool for solving nonlinear problems in science and engineering. In the future, this approach can also be extended to systems of nonlinear equations and larger practical problems. The proposed approach may also be extended to other higher-order iterative schemes and fixed-point algorithms reported in the literature [14].

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