

A nonlinear fractional differential system : Existence and UHML-stability results

Mohamed Bezziou *

Department of Mathematics

Djilali Bounaama University

Khemis Miliana

Ain Defla 44000

Algeria

Zoubir Dahmani [†]

Laboratory of Discrete and Applied Mathematics and Operations Research LAMDA-RO

Faculty of Sciences

University of Blida 1

Blida 09000

Algeria

Abstract

In this paper, we study the existence, uniqueness and stability in the sense of Ulam-Hyers Mittag-Leffler approach for a particular class of nonlinear coupled systems of Caputo fractional differential equations equipped with six homogeneous and nonhomogeneous integral boundary conditions. The uniqueness result for the given problem is based on the Banach fixed point theorem, while the existence of its solutions is accomplished by applying the Schaefer fixed point theorem. In addition, the results obtained are supported with illustrative examples.

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* ORCID-ID: 0009-0009-2911-7003

[†] ORCID-ID: 0000-0003-4659-0723

* E-mail: m.bezziou@univ-dbk.m.dz (Corresponding Author)

[†] E-mail: zzdahmani@yahoo.fr

1. Introduction

The study of fractional calculus has seen a significant increase in these last years, while fractional differential equations has been attractive to many researchers, involves rational numbers in the order of derivation operator properties, one can see [13,23]. Many fractional differential and integral operators have been defined and used by mathematicians for their specific purposes, often employing various fixed point theorems. For details on the theory and application, see [6, 7, 14, 15, 19, 21]. Recently, significant attention has been focused on the study of the existence, uniqueness and Mittag-Leffler stability of solutions for boundary value problems with nonlocal integral boundary conditions. For more information on this topic, we refer the reader to [2, 4, 12, 16, 20, 26, 27, 29].

Gyulov, Morosanu and Tersian, [18] have worked on

$$x^{(6)} + ax^{(4)} + bx^{(2)} + cx(s) = f(s, x(s)), s \in (0, M),$$

with

$$x^{(k)}(0) = x^{(k)}(M) = 0, k \in \{0, 2, 4\}.$$

Danet in [11] has worked on the case:

$$\begin{cases} y^{(vi)}(\tau) + ay^{(iv)}(\tau) + by^{(ii)}(\tau) = c(\tau)y(\tau) - f(\tau, y(\tau)), & \text{in } \Xi = (0, L), \\ y^{(j)} = 0, \quad j \in \{0, 2, 4\}, & \text{on } \partial\Xi. \end{cases}$$

for $a, b \in \mathbb{R}$ and $c(\tau)$ is a given function.

In [5], the authors proved the existence, uniqueness for the problem:

$$\begin{cases} \prod_{k=1}^4 D^{\alpha_k} u(\tau) = H_1(\tau, u(\tau), v(\tau)) + a_1 f_1(u(\tau)) + b_1 g_1(\prod_{k=1}^2 D^{\alpha_k} u(\tau)), \\ \tau \in J = [0, 1], \\ \prod_{k=1}^4 D^{\beta_k} v(\tau) = H_2(\tau, u(\tau), v(\tau)) + a_2 f_2(v(\tau)) + b_2 g_2(\prod_{k=1}^2 D^{\beta_k} v(\tau)), \\ \tau \in J = [0, 1], \\ u(0) = v(1) = \prod_{k=1}^2 D^{\beta_k} v(1) = D^{\alpha_4} u(0) = 0, \\ v(0) = u(1) = \prod_{k=1}^2 D^{\alpha_k} u(1) = D^{\beta_4} v(0) = 0, \end{cases}$$

For further results on this topic, we refer the reader to [1, 3, 8, 9, 10, 24, 25, 28].

In this work, we discuss the existence, uniqueness and Ulam-Hyers-Mittag-Leffler (UHML, for short) stability of solutions for the nonlinear coupled system:

$$\begin{cases} {}^C D^{\alpha_1} x_1(\tau) = \lambda_1 F_1(\tau, x_1^{(4)}(\tau), x_2^{(4)}(\tau)) + \delta_1 G_1(\tau, x_1^{(2)}(\tau), x_2^{(2)}(\tau)) \\ \quad + \gamma_1 H_1(\tau, x_1(\tau), x_2(\tau)), \quad 0 \leq \tau \leq 1, \\ {}^C D^{\alpha_2} x_2(\tau) = \lambda_2 F_2(\tau, x_1^{(4)}(\tau), x_2^{(4)}(\tau)) + \delta_2 G_2(\tau, x_1^{(2)}(\tau), x_2^{(2)}(\tau)) \\ \quad + \gamma_2 H_2(\tau, x_1(\tau), x_2(\tau)), \quad 0 \leq \tau \leq 1, \end{cases} \quad (1)$$

subject to the following six non-separated boundary conditions.

$$\begin{cases} x_j^{(m)}(0) = 0, m \in \{0, 1, 3, 5\}, \\ \sigma_j \int_0^1 x_j(\tau) d\tau = x_j^{(2)}(0), \quad \rho_j \int_0^\xi x_j(\tau) d\tau = x_j^{(4)}(1), \\ 0 < \xi < 1, \quad j = 1, 2, \end{cases} \quad (2)$$

where ${}^C D^{\alpha_1}, {}^C D^{\alpha_2}$ denote the Caputo fractional derivatives of order α_1, α_2 such that $5 < \alpha_1, \alpha_2 \leq 6$, respectively, $\lambda_i, \delta_i, \gamma_i, \sigma_i, \rho_i \in \mathbb{R}, i = 1, 2$ and $F_i, G_i, H_i, i = 1, 2$, are nonlinear functions satisfying some assumptions in $C(J \times \mathbb{R}^2, \mathbb{R})$.

2. Preliminaries

We introduce Caputo and RL approaches (see [17, 22]).

Definition 1: The RL integrations is defined as

$$J_a^\gamma \varphi(x) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_a^x (x - \tau)^{\gamma-1} \varphi(\tau) d\tau, & \gamma > 0, \quad a < x \leq b, \\ \varphi(x), & \gamma = 0, \end{cases}$$

where $\Gamma(\gamma) = \int_0^\infty e^{-u} u^{\gamma-1} du$.

Note that for $\gamma > 0, \delta > 0$, we have:

$$J_a^\gamma J_a^\delta \varphi(x) = J_a^{\gamma+\delta} \varphi(x).$$

Definition 2: The Caputo differentiation is given by:

$$D^\gamma \varphi(x) = \frac{1}{\Gamma(m-\gamma)} \int_a^x (x - \tau)^{m-\gamma-1} \varphi^{(m)}(\tau) d\tau,$$

The Following lemmas give some properties of fractional calculus theory (see [22, 23])

Lemma 3: We have for $D^\gamma y(\tau) = 0$ what follows:

$$y(\tau) = a_0 + a_1 \tau + a_2 \tau^2 + \cdots + a_{m-1} \tau^{m-1},$$

where $a_k \in \mathbb{R}, k = 0, 1, \dots, m-1, m = [\gamma] + 1$.

Lemma 4: For $\gamma > 0, m \in \mathbb{N}^*$

$$J^\gamma [D^\gamma y(\tau)] = y(\tau) + a_0 + a_1 \tau + a_2 \tau^2 + \cdots + a_{m-1} \tau^{m-1},$$

for some $a_k \in \mathbb{R}, k = 0, 1, \dots, m-1, m = [\gamma] + 1$.

Lemma 5: Let $q > p > 0$ and $y \in L^1([a, b])$. Then

$$D^p J^q [y(\tau)] = J^{q-p} [y(\tau)], \quad \tau \in [a, b].$$

We need also the following result:

Lemma 6: *The problem:*

$$\begin{cases} {}^C D^{\alpha_i} x_i(\tau) = \varphi_i(t), & t \in [0, 1], \\ x_i^{(k)}(0) = 0, & k \in \{0, 1, 3, 5\} \\ \sigma_i \int_0^1 x_i(t) dt = x_i^{(2)}(0), & \rho_i \int_0^\xi x_i(t) dt = x_i^{(4)}(1), \quad 0 < \xi < 1, \\ 6 \geq \alpha_i > 5, & \sigma_i \in \mathbb{R}, \quad \rho_i \in \mathbb{R}, \quad i \in \{1, 2\} \end{cases} \quad (3)$$

can admit:

$$\begin{aligned} & x_i(t) \\ &= \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} \varphi_i(s) ds + (\Lambda_{1i} t^2 + \Lambda_{4i} t^4) \frac{1}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} \varphi_i(s) ds \\ &+ (\Lambda_{2i} t^2 + \Lambda_{5i} t^4) \frac{1}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} \varphi_i(s) ds \\ &+ (\Lambda_{3i} t^2 - \Lambda_{6i} t^4) \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} \varphi_i(s) ds, \end{aligned} \quad (4)$$

$$i = 1, 2.$$

where,

$$\begin{aligned} \Lambda_{1i} &= \frac{\frac{1}{2} \sigma_i [6(120 - \rho_i \xi^5 - 20\sigma_i) + \xi^3 \rho_i \sigma_i]}{(6 - \sigma_i)(120 - \rho_i \xi^5 - 20\sigma_i)} \\ \Lambda_{2i} &= \frac{\rho_i \sigma_i}{2(120 - \rho_i \xi^5 - 20\sigma_i)} \\ \Lambda_{3i} &= \frac{\sigma_i [36(120 - \rho_i \xi^5 - 20\sigma_i) + \rho_i \times (6 - \sigma_i) \xi^3]}{1440(6 - \sigma_i)(120 - \rho_i \xi^5 - 20\sigma_i)} \\ \Lambda_{4i} &= \frac{\frac{1}{6} \frac{5\rho_i \xi^3 \sigma_i}{(120 - \rho_i \xi^5 - 20\sigma_i)}}{6(120 - \rho_i \xi^5 - 20\sigma_i)} \\ \Lambda_{5i} &= \frac{\frac{1}{6} \frac{5\rho_i (6 - \sigma_i)}{(120 - \rho_i \xi^5 - 20\sigma_i)}}{6(120 - \rho_i \xi^5 - 20\sigma_i)} \\ \Lambda_{6i} &= \frac{\frac{\rho_i (6 - \sigma_i) \xi^3}{864(120 - \rho_i \xi^5 - 20\sigma_i)}}{24} + \frac{1}{24}, \end{aligned}$$

such that, $(6 - \sigma_i)(120 - \rho_i \xi^5 - 20\sigma_i) \neq 0$, $i \in \{1, 2\}$.

Proof: We have:

$${}^C D^{\alpha_i} x_i(t) = \varphi_i(t), \quad i = 1, 2,$$

implies that:

$$x_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} \varphi_i(s) ds + \sum_{i=0}^5 c_i t^i, \quad (5)$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \dots, 5$.

Applying the boundary conditions,

$$x_i(0) = x_i'(0) = x_i^{(3)}(0) = x_i^{(5)}(0) = 0,$$

allow us to write

$$c_i = 0, i \in \{0, 1, 3, 5\}.$$

By substituting these values into the equation (5), we find

$$x_i(t) = \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \varphi_i(s) ds + c_2 t^2 + c_4 t^4, \quad i = \{1, 2\}, \quad (6)$$

then, from

$$\begin{cases} \sigma_i \int_0^1 x_i(t) dt = x_i^{(2)}(1), \\ \rho_i \int_0^\xi x_i(t) dt = x_i^{(4)}(1), \quad i = \{1, 2\}, \end{cases}$$

we get

$$\begin{cases} c_2 = \frac{\sigma_i}{2} \int_0^1 x_i(t) dt, \\ c_4 = \frac{1}{24} \left(\rho_i \int_0^\xi x_i(t) dt - \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} \varphi_i(s) ds \right), \quad i = 1, 2. \end{cases}$$

By replacing c_2 and c_4 in the formula (6), we obtain.

$$\begin{aligned} x_i(t) &= \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} \varphi_i(s) ds + \frac{\sigma_i t^2}{2} \int_0^1 x_i(t) dt \\ &\quad + \rho_i t^4 \int_0^\xi \frac{x_i(t)}{24} dt - \frac{t^4}{24\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} \varphi_i(s) ds. \end{aligned} \quad (7)$$

From (7), we have

$$\begin{aligned} \int_0^\xi x(t) dt &= \frac{20(6-\sigma_i)}{(120-\rho_i \xi^5-20\sigma_i)\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} \varphi_i(s) ds \\ &\quad + \frac{20\sigma_i \xi^3}{120-\rho_i \xi^5-20\sigma_i} \frac{1}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} \varphi_i(s) ds \\ &\quad - \frac{(6-\sigma_i)\xi^3}{36(120-\rho_i \xi^5-20\sigma_i)} \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} \varphi_i(s) ds, \end{aligned} \quad (8)$$

and

$$\begin{aligned} \int_0^1 x(t) dt &= \frac{6(-\rho_i \xi^5+120-20\sigma_i)+\rho_i \sigma_i \xi^3}{(6-\sigma_i)(120-\rho_i \xi^5-20\sigma_i)} \frac{1}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} \varphi_i(s) ds \\ &\quad + \frac{\rho_i}{120-\rho_i \xi^5-20\sigma_i} \frac{1}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} \varphi_i(s) ds \\ &\quad + \frac{36(120-\rho_i \xi^5-20\sigma_i)+(6-\sigma_i)\rho_i \xi^3}{720(6-\sigma_i)(-\rho_i \xi^5+120-20\sigma_i)} \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} \varphi_i(s) ds. \end{aligned} \quad (9)$$

Now, by inserting (8) and (9) into (7), we deduce (4).

3. Main results:

We will use the spaces:

$$E = \left\{ x_i \mid x_i, x_i^{(2)}, x_i^{(4)} \in C(J, \mathbb{R}) \right\}, \quad i = 1, 2,$$

and

$$\|x_i^{(k)}\|_\infty = \sup_{t \in [0,1]} |x_i^{(k)}(t)|, \quad k = 0, 2, 4, \quad i = 1, 2, \quad (10)$$

where

$$\|x_i\|_E = \|x_i\|_\infty + \|x_i^{(2)}\|_\infty + \|x_i^{(4)}\|_\infty, \quad i \in \{1,2\}. \quad (11)$$

And also $(E \times E, \|\cdot\|_{E \times E})$ with

$$\|(x_1, x_2)\|_{E \times E} = \|x_1\|_E + \|x_2\|_E. \quad (12)$$

will be utilized. We work with the application $\mathcal{T}: E \times E \rightarrow E \times E$:

$$\mathcal{T}(x_1(t), x_2(t)) = \begin{pmatrix} \mathcal{T}_1(x_1(t), x_2(t)) \\ \mathcal{T}_2(x_1(t), x_2(t)) \end{pmatrix},$$

where, for any $t \in J$, we have for $i = 1, 2$

$$\begin{aligned} & T_i(x_1, x_2)(t) \\ = & \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i-1} [\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) \\ & + \gamma_i H_i(s, x_1(s), x_2(s))] ds \\ & + (\Lambda_{1i} t^2 + \Lambda_{4i} t^4) \frac{1}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} [\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\ & + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))] ds \\ & + (\Lambda_{2i} t^2 + \Lambda_{5i} t^4) \frac{1}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} [\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\ & + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))] ds \\ & + (\Lambda_{3i} t^2 - \Lambda_{6i} t^4) \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\ & + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))] ds, \quad i = 1, 2. \end{aligned}$$

Lemma 7: Assume that $F_i, G_i, H_i \in C(J \times \mathbb{R}^2, \mathbb{R})$. Then $(x_1, x_2) \in E^2$ is a solution of (1)-(2) if (x_1, x_2) is a solution for $\begin{pmatrix} T_1(x_1, x_2) \\ T_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Proof: Trivial.

For convenience, put:

(H1): The functions F_i, G_i and $H_i, i = 1, 2$, are continuous over $J \times \mathbb{R}^2$.

(H2): One has $P_i, Q_i, R_i, i \in \{1, 2\}$, with:

$$\begin{aligned} |(F_i(t, x_1, x_2) - F_i(t, y_1, y_2))| & \leq P_i \sum_{j=1}^2 |x_j - y_j|, \\ |(G_i(t, x_1, x_2) - G_i(t, y_1, y_2))| & \leq Q_i \sum_{j=1}^2 |x_j - y_j|, \\ |(H_i(t, x_1, x_2) - H_i(t, y_1, y_2))| & \leq R_i \sum_{j=1}^2 |x_j - y_j|, \end{aligned}$$

where, $i \in \{1, 2\}$.

(H3) One has $L_{1i}, L_{2i}, L_{3i}, i = 1, 2$, such that we have:

$$|F_i(t, y_1, y_2)| \leq L_{1i}, \quad |G_i(t, y_1, y_2)| \leq L_{2i}, \quad |H_i(t, y_1, y_2)| \leq L_{3i}.$$

Then, we may put:

$$\begin{aligned}
\Omega_i &= |\lambda_i|P_i + |\delta_i|Q_i + |\gamma_i|R_i, \\
M_i &= \frac{1}{\Gamma(\alpha_i+1)} + \frac{|\Lambda_{1i}+\Lambda_{4i}|}{\Gamma(\alpha_i+2)} + \frac{|\Lambda_{2i}+\Lambda_{5i}|\xi^{\alpha_i+1}}{\Gamma(2+\alpha_i)} + \frac{|\Lambda_{3i}-\Lambda_{6i}|}{\Gamma(\alpha_i-3)}, \\
N_i &= \frac{1}{\Gamma(\alpha_i-1)} + \frac{2|\Lambda_{1i}+6\Lambda_{4i}|}{\Gamma(2+\alpha_i)} + \frac{2|\Lambda_{2i}+6\Lambda_{5i}|\xi^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + \frac{2|\Lambda_{3i}-6\Lambda_{6i}|}{\Gamma(\alpha_i-3)}, \\
V_i &= \frac{1}{\Gamma(\alpha_i-3)} + \frac{24|\Lambda_{4i}|}{\Gamma(\alpha_i+2)} + \frac{24|\Lambda_{5i}|\xi^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + \frac{24|\Lambda_{6i}|}{\Gamma(\alpha_i-3)}, \\
W_i &= M_i + N_i + V_i.
\end{aligned}$$

Theorem 8: If (H1) and (H2) are satisfied and suppose also that:

$$\sum_{i=1}^2 \Omega_i W_i < 1. \quad (13)$$

Then, (1)-(2) can admit one solution.

Proof: We can use Banach principal to prove this result. For $(x_1, x_2), (y_1, y_2) \in E \times E$, we can write

$$\begin{aligned}
& |T_i(x_1(t), x_2(t)) - T_i(y_1(t), y_2(t))| \\
& \leq \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i-1} [|\lambda_i(F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) - F_i(s, y_1^{(4)}(s), y_2^{(4)}(s))) \\
& \quad + \delta_i(G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) - G_i(s, y_1^{(2)}(s), y_2^{(2)}(s))) \\
& \quad + \gamma_i(H_i(s, x_1(s), x_2(s)) - H_i(s, y_1(s), y_2(s)))|] ds \\
& \quad + \frac{|\Lambda_{1i}+\Lambda_{4i}|}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} [|\lambda_i(F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad - F_i(s, y_1^{(4)}(s), y_2^{(4)}(s))) + \delta_i(G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) - G_i(s, y_1^{(2)}(s), y_2^{(2)}(s))) \\
& \quad + \gamma_i(H_i(s, x_1(s), x_2(s)) - H_i(s, y_1(s), y_2(s)))|] ds \\
& \quad + \frac{|\Lambda_{2i}+\Lambda_{5i}|}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} [|\lambda_i(F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad - F_i(s, y_1^{(4)}(s), y_2^{(4)}(s))) + \delta_i(G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) - G_i(s, y_1^{(2)}(s), y_2^{(2)}(s))) \\
& \quad + \gamma_i(H_i(s, x_1(s), x_2(s)) - H_i(s, y_1(s), y_2(s)))|] ds \\
& \quad + \frac{|\Lambda_{3i}-\Lambda_{6i}|}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [|\lambda_i(F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad - F_i(s, y_1^{(4)}(s), y_2^{(4)}(s))) + \delta_i(G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) - G_i(s, y_1^{(2)}(s), y_2^{(2)}(s))) \\
& \quad + \gamma_i(H_i(s, x_1(s), x_2(s)) - H_i(s, y_1(s), y_2(s)))|] ds \}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|T_i(x_1, x_2) - T_i(y_1, y_2)\|_\infty \\
& \leq |\lambda_i|P_i \left(\frac{1}{\Gamma(1+\alpha_i)} + \frac{1}{\Gamma(2+\alpha_i)} |\Lambda_{1i} + \Lambda_{4i}| + \frac{|\Lambda_{2i}+\Lambda_{5i}|\xi^{\alpha_i+1}}{\Gamma(2+\alpha_i)} \right. \\
& \quad \left. + \frac{|\Lambda_{3i}-\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) (\|x_1^{(4)} - y_1^{(4)}\|_\infty + \|x_2^{(4)} - y_2^{(4)}\|_\infty)
\end{aligned}$$

$$\begin{aligned}
& + |\delta_i| Q_i \left(\frac{1}{\Gamma(1+\alpha_i)} + \frac{|\Lambda_{1i} + \Lambda_{4i}|}{\Gamma(2+\alpha_i)} + \frac{|\Lambda_{2i} + \Lambda_{5i}| \xi^{\alpha_i+1}}{\Gamma(2+\alpha_i)} \right. \\
& + \left. \frac{|\Lambda_{3i} - \Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) \left(\|x_1^{(2)} - y_1^{(2)}\|_\infty + \|x_2^{(2)} - y_2^{(2)}\|_\infty \right) \\
& + |\gamma_i| R_i \left(\frac{1}{\Gamma(1+\alpha_i)} + \frac{1}{\Gamma(2+\alpha_i)} |\Lambda_{1i} + \Lambda_{4i}| + \frac{|\Lambda_{2i} + \Lambda_{5i}| \xi^{\alpha_i+1}}{\Gamma(\alpha_i+2)} \right. \\
& + \left. \frac{|\Lambda_{3i} - \Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) (\|x_1 - y_1\|_\infty + \|x_2 - y_2\|_\infty) \\
& \leq \Omega_i M_i (\|x_1 - y_1\|_E + \|x_2 - y_2\|_E), i = 1, 2.
\end{aligned} \tag{14}$$

Also, the reader can observe that

$$\begin{aligned}
& |D^2 T_i(x_1, x_2)(t) - D^2 T_i(y_1, y_2)(t)| \\
& \leq \frac{1}{\Gamma(\alpha_i-2)} \int_0^1 (1-s)^{\alpha_i-3} [|\lambda_i(F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) - F_i(s, y_1^{(4)}(s), y_2^{(4)}(s))) \\
& + \delta_i(G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) - G_i(s, y_1^{(2)}(s), y_2^{(2)}(s))) \\
& + \gamma_i(H_i(s, x_1(s), x_2(s)) - H_i(s, y_1(s), y_2(s)))|] ds \\
& + \frac{2|\Lambda_{1i} + 6\Lambda_{4i}|}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} [|\lambda_i(F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& - F_i(s, y_1^{(4)}(s), y_2^{(4)}(s))) + \delta_i(G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) - G_i(s, y_1^{(2)}(s), y_2^{(2)}(s))) \\
& + \gamma_i(H_i(s, x_1(s), x_2(s)) - H_i(s, y_1(s), y_2(s)))|] ds \\
& + \frac{2|\Lambda_{2i} + 6\Lambda_{5i}|}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} [|\lambda_i(F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& - F_i(s, y_1^{(4)}(s), y_2^{(4)}(s))) + \delta_i(G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) - G_i(s, y_1^{(2)}(s), y_2^{(2)}(s))) \\
& + \gamma_i(H_i(s, x_1(s), x_2(s)) - H_i(s, y_1(s), y_2(s)))|] ds \\
& + \frac{2|\Lambda_{3i} - 6\Lambda_{6i}|}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [|\lambda_i(F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& - F_i(s, y_1^{(4)}(s), y_2^{(4)}(s))) + \delta_i(G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) - G_i(s, y_1^{(2)}(s), y_2^{(2)}(s))) \\
& + \gamma_i(H_i(s, x_1(s), x_2(s)) - H_i(s, y_1(s), y_2(s)))|] ds\}.
\end{aligned}$$

Implies that

$$\begin{aligned}
& \|D^2 T_i(x_1, x_2) - D^2 T_i(y_1, y_2)\|_\infty \\
& \leq |\lambda_i| P_i \left(\frac{1}{\Gamma(\alpha_i-1)} + \frac{2(|\Lambda_{1i} + 6\Lambda_{4i}| + |\Lambda_{2i} + 6\Lambda_{5i}| \xi^{\alpha_i+1})}{\Gamma(\alpha_i+2)} \right. \\
& + \left. \frac{2|\Lambda_{3i} - 12\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) (\|x_1^{(4)} - y_1^{(4)}\|_\infty + \|x_2^{(4)} - y_2^{(4)}\|_\infty) \\
& + |\delta_i| Q_i \left(\frac{1}{\Gamma(\alpha_i-1)} + \frac{2(|\Lambda_{1i} + 6\Lambda_{4i}| + |\Lambda_{2i} + 6\Lambda_{5i}| \xi^{\alpha_i+1})}{\Gamma(2+\alpha_i)} \right. \\
& + \left. \frac{2|\Lambda_{3i} - 12\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) (\|x_1^{(2)} - y_1^{(2)}\|_\infty + \|x_2^{(2)} - y_2^{(2)}\|_\infty) \\
& + |\gamma_i| R_i \left(\frac{1}{\Gamma(\alpha_i-1)} + \frac{2(|\Lambda_{1i} + 6\Lambda_{4i}| + |\Lambda_{2i} + 6\Lambda_{5i}| \xi^{\alpha_i+1})}{\Gamma(2+\alpha_i)} \right. \\
& + \left. \frac{2|\Lambda_{3i} - 6\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) (\|x_1 - y_1\|_\infty + \|x_2 - y_2\|_\infty) \\
& \leq \Omega_i N_i (\|x_1 - y_1\|_E + \|x_2 - y_2\|_E), i = 1, 2.
\end{aligned} \tag{15}$$

Also, we can state that

$$\begin{aligned}
& \|D^4 T_i(x_1, x_2) - D^4 T_i(y_1, y_2)\|_\infty \\
& \leq |\lambda_i| P_i \left(\frac{1}{\Gamma(\alpha_i-3)} + \frac{24(|\Lambda_{4i}| + |\Lambda_{5i}| \xi^{\alpha_i+1})}{\Gamma(2+\alpha_i)} + \frac{24|\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) \\
& \quad \times \left(\|x_1^{(4)} - y_1^{(4)}\|_\infty + \|x_2^{(4)} - y_2^{(4)}\|_\infty \right) \\
& \quad + |\delta_i| Q_i \left(\frac{1}{\Gamma(\alpha_i-3)} + \frac{24|\Lambda_{4i}|}{\Gamma(\alpha_i+2)} + \frac{24|\Lambda_{5i}| \xi^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + \frac{24|\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) \\
& \quad \times \left(\|x_1^{(2)} - y_1^{(2)}\|_\infty + \|x_2^{(2)} - y_2^{(2)}\|_\infty \right) \\
& \quad + |\gamma_i| R_i \left(\frac{1}{\Gamma(\alpha_i-3)} + \frac{24(|\Lambda_{4i}| + |\Lambda_{5i}| \xi^{\alpha_i+1})}{\Gamma(\alpha_i+2)} + \frac{24|\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right) \\
& \quad \times (\|x_1 - y_1\|_\infty + \|x_2 - y_2\|_\infty) \\
& \leq \Omega_i V_i (\|(x_1 - y_1)\|_E + \|(x_2 - y_2)\|_E), i = 1, 2.
\end{aligned} \tag{16}$$

Thanks to (14), (15), (16) and (11), we find

$$\begin{aligned}
& \|T_i(x_1, x_2) - T_i(y_1, y_2)\|_E \\
& = \|T_i(x_1, x_2) - T_i(y_1, y_2)\|_\infty + \|D^2 T_i(x_1, x_2) - D^2 T_i(y_1, y_2)\|_\infty \\
& \quad + \|D^4 T_i(x_1, x_2) - D^4 T_i(y_1, y_2)\|_\infty \\
& \leq \Omega_i \left[\frac{1}{\Gamma(\alpha_i+1)} + \frac{|\Lambda_{1i} + \Lambda_{4i}| + |\Lambda_{2i} + \Lambda_{5i}| \xi^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + \frac{|\Lambda_{3i} - \Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right. \\
& \quad + \frac{1}{\Gamma(\alpha_i-1)} + \frac{2(|\Lambda_{1i} + 6\Lambda_{4i}| + |\Lambda_{2i} + 6\Lambda_{5i}| \xi^{\alpha_i+1})}{\Gamma(2+\alpha_i)} + \frac{2|\Lambda_{3i} - 6\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \\
& \quad \left. + \frac{1}{\Gamma(\alpha_i-3)} + \frac{24(|\Lambda_{4i}| + |\Lambda_{5i}| \xi^{\alpha_i+1})}{\Gamma(2+\alpha_i)} + \frac{24|\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right] (\|(x_1 - y_1)\|_E + \|(x_2 - y_2)\|_E) \\
& \leq \Omega_i (M_i + N_i + V_i) (\|(x_1 - y_1)\|_E + \|(x_2 - y_2)\|_E), i = 1, 2.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& \|T_1(x_1, x_2) - T_1(y_1, y_2)\|_E \\
& \leq \Omega_1 (M_1 + N_1 + V_1) (\|(x_1 - y_1)\|_E + \|(x_2 - y_2)\|_E) \\
& \leq \Omega_1 W_1 (\|(x_1 - y_1)\|_E + \|(x_2 - y_2)\|_E),
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
& \|T_2(x_1, x_2) - T_2(y_1, y_2)\|_E \\
& \leq \Omega_2 (M_2 + N_2 + V_2) (\|(x_1 - y_1)\|_E + \|(x_2 - y_2)\|_E) \\
& \leq \Omega_2 W_2 (\|(x_1 - y_1)\|_E + \|(x_2 - y_2)\|_E).
\end{aligned} \tag{18}$$

Therefore, from (17), (18) and (12), we have

$$\|T(x_1, x_2) - T(y_1, y_2)\|_{E \times E} \leq (\Omega_1 W_1 + \Omega_2 W_2) \|(x_1, x_2) - (y_1, y_2)\|_{E \times E}.$$

The proof ends.

Now, we prove the existence solution by applying following Lemma of Schaefer alternative fixed point.

Theorem 9: When (H1)-(H3) are valid, (1)-(2) has a solution.

Proof: **Step 1:** Continuity is very trivial.

Step 2: We define the set $B_r = \{(x_1, x_2) \in E \times E: \|(x_1, x_2)\| \leq r\}$.

For all $(x_1, x_2) \in B_r$ and for each $t \in J$, by (H3) we have

$$\begin{aligned}
& \|T_i(x_1, x_2)\|_\infty \\
& \leq \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i-1} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \quad + \frac{|\Lambda_{1i} + \Lambda_{4i}|}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} [|\lambda_i (F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \quad + \frac{|\Lambda_{2i} + \Lambda_{5i}|}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \quad + \frac{|\Lambda_{3i} - \Lambda_{6i}|}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [|\lambda_i (F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds\} \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i| + L_{3i}|\gamma_i|) \left[\frac{1}{\Gamma(\alpha_i+1)} + \frac{|\Lambda_{1i} + \Lambda_{4i}|}{\Gamma(2+\alpha_i)} \right. \\
& \quad \left. + \frac{\xi^{\alpha_i+1} |\Lambda_{2i} + \Lambda_{5i}|}{\Gamma(2+\alpha_i)} + \frac{|\Lambda_{3i} - \Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right] \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i| + L_{3i}|\gamma_i|) M_i < +\infty, \quad i = 1, 2,
\end{aligned}$$

then

$$\begin{aligned}
& \|D^2 T_i(x_1, x_2)\|_\infty \\
& \leq \frac{1}{\Gamma(\alpha_i-2)} \int_0^1 (1-s)^{\alpha_i-3} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \quad + 2|\Lambda_{1i} + 6\Lambda_{4i}| \frac{1}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \quad + 2|\Lambda_{2i} + 6\Lambda_{5i}| \frac{1}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \quad + 2|\Lambda_{3i} - 6\Lambda_{6i}| \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds\} \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i| + L_{3i}|\gamma_i|) \left[\frac{1}{\Gamma(\alpha_i-1)} + \frac{2|\Lambda_{1i} + 6\Lambda_{4i}|}{\Gamma(\alpha_i+2)} \right. \\
& \quad \left. + \frac{2|\Lambda_{2i} + 6\Lambda_{5i}| \xi^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + \frac{2|\Lambda_{3i} - 6\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right] \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i| + L_{3i}|\gamma_i|) N_i < +\infty, \quad i = 1, 2
\end{aligned}$$

and

$$\begin{aligned}
& \|D^4 T_i(x_1, x_2)\|_\infty \\
& \leq \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \quad + \frac{24\Lambda_{4i}}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \quad + \frac{24\Lambda_{5i}}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds ds \\
& \quad + \frac{24\Lambda_{6i}}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [|\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
& \quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))|] ds \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i| + L_{3i}|\gamma_i|) \left[\frac{1}{\Gamma(\alpha_i-1)} + \frac{24|\Lambda_{4i}|}{\Gamma(\alpha_i+2)} \right. \\
& \quad \left. + \frac{24|\Lambda_{2i}|\xi^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + \frac{24|\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right] \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i| + L_{3i}|\gamma_i|) V_i < +\infty, \quad i = 1, 2.
\end{aligned}$$

Hence, for any $(x_1, x_2) \in B_r$, we obtain $\|T_i(x_1, x_2)\|_{E \times E} < +\infty, i \in \{1, 2\}$.

Step 3: Boundedness

Let $t_1, t_2 \in [0, 1]$, so, we have

$$\begin{aligned}
& |T(x_1, x_2)(t_2) - T(x_1, x_2)(t_1)| \\
& \leq \frac{1}{\Gamma(\alpha_i)} \left| \int_0^{t_1} ((t_2-s)^{\alpha_i-1} - (t_1-s)^{\alpha_i-1}) [\lambda_i F_i(t, x_1^{(4)}(t), x_2^{(4)}(t)) \right. \\
& \quad + \delta_i G_i(t, x_1^{(2)}(t), x_2^{(2)}(t)) + \gamma_i H_i(t, x_1(t), x_2(t))] ds \Big| \\
& \quad + \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_2-s)^{(\alpha-1)} [\lambda_i F_i(t, x_1^{(4)}(t), x_2^{(4)}(t)) \right. \\
& \quad + \delta_i G_i(t, x_1^{(2)}(t), x_2^{(2)}(t)) + \gamma_i H_i(t, x_1(t), x_2(t))] ds \Big| \\
& \quad + \left| (\Lambda_{1i}(t_2^2 - t_1^2) + \Lambda_{4i}(t_2^4 - t_1^4)) \frac{1}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} [\lambda_i F_i(t, x_1^{(4)}(t), x_2^{(4)}(t)) \right. \\
& \quad + \delta_i G_i(t, x_1^{(2)}(t), x_2^{(2)}(t)) + \gamma_i H_i(t, x_1(t), x_2(t))] ds \Big| \\
& \quad + \left| (\Lambda_{2i}(t_2^2 - t_1^2) + \Lambda_{5i}(t_2^4 - t_1^4)) \frac{1}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} [\lambda_i F_i(t, x_1^{(4)}(t), x_2^{(4)}(t)) \right. \\
& \quad + \delta_i G_i(t, x_1^{(2)}(t), x_2^{(2)}(t)) + \gamma_i H_i(t, x_1(t), x_2(t))] ds \Big| \\
& \quad + \left| (\Lambda_{3i}(t_2^2 - t_1^2) - \Lambda_{6i}(t_2^4 - t_1^4)) \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [\lambda_i F_i(t, x_1^{(4)}(t), x_2^{(4)}(t)) \right. \\
& \quad + \delta_i G_i(t, x_1^{(2)}(t), x_2^{(2)}(t)) + \gamma_i H_i(t, x_1(t), x_2(t))] ds \Big|.
\end{aligned}$$

Hence

$$\begin{aligned}
& |T(x_1, x_2)(t_2) - T(x_1, x_2)(t_1)| \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|) \times \left\{ \frac{[2|t_1 - t_2|^\alpha + |t_1^\alpha - t_2^\alpha|]}{\Gamma(\alpha_i + 1)} \right. \\
& \quad + \frac{(|\Lambda_{1i}|(t_2^2 - t_1^2) + |\Lambda_{4i}|(t_2^4 - t_1^4))}{\Gamma(\alpha_i + 2)} \\
& \quad \left. + \frac{(|\Lambda_{2i}|(t_2^2 - t_1^2) + |\Lambda_{5i}|(t_2^4 - t_1^4))\xi^{\alpha_i + 1}}{\Gamma(\alpha_i + 2)} + \frac{(|\Lambda_{3i}|(t_2^2 - t_1^2) + |\Lambda_{6i}|(t_2^4 - t_1^4))}{\Gamma(\alpha_i - 3)} \right\}
\end{aligned} \tag{19}$$

As before, we have

$$\begin{aligned}
& |D^2T(x_1, x_2)(t_2) - D^2T(x_1, x_2)(t_1)| \\
& \leq \frac{[2(t_2 - t_1)^{\alpha-1} + |t_2^{\alpha-1} - t_1^{\alpha-1}|]}{\Gamma(\alpha_i - 1)} (L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|) \\
& \quad + (L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|) \left[\frac{(12|\Lambda_{4i}|(t_2^2 - t_1^2))}{\Gamma(\alpha_i + 2)} \right. \\
& \quad \left. + \frac{(12|\Lambda_{5i}|(t_2^2 - t_1^2))\xi^{\alpha_i + 1}}{\Gamma(\alpha_i + 2)} + \frac{(12|\Lambda_{6i}|(t_2^2 - t_1^2))}{\Gamma(\alpha_i - 3)} \right],
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
& |D^4T(x_1, x_2)(t_2) - D^4T(x_1, x_2)(t_1)| \\
& \leq \frac{[2(t_2 - t_1)^{\alpha-3} + |t_2^{\alpha-3} - t_1^{\alpha-3}|]}{\Gamma(\alpha_i - 3)} (L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|).
\end{aligned} \tag{21}$$

Therefore, T is completely continuous.

Step 4: The subset

$$F = \{(x_1, x_2) \in E \times E : (x_1, x_2) = \mu T(x_1, x_2), 0 < \mu < 1\},$$

satisfies the boundedness.

So, we can write

$$\begin{aligned}
& \|x_i\|_\infty = \sup_{t \in [0,1]} |x_i(t)| \\
& \leq \mu(L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|) \left[\frac{1}{\Gamma(\alpha_i + 1)} + \frac{|\Lambda_{1i} + \Lambda_{4i}|}{\Gamma(\alpha_i + 2)} \right. \\
& \quad \left. + \frac{\xi^{\alpha_i + 1}|\Lambda_{2i} + \Lambda_{5i}|}{\Gamma(2 + \alpha_i)} + \frac{|\Lambda_{3i} + \Lambda_{6i}|}{\Gamma(\alpha_i - 3)} \right] \\
& \leq \mu(L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|)M_i \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|)M_i \leq +\infty, i = 1, 2,
\end{aligned} \tag{22}$$

also, we have

$$\begin{aligned}
& \|x_i^{(2)}\|_\infty = \sup_{t \in [0,1]} |x_i^{(2)}(t)| \\
& \leq \mu(L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|) \left[\frac{1}{\Gamma(\alpha_i - 1)} + \frac{2|\Lambda_{1i} + 6\Lambda_{4i}|}{\Gamma(\alpha_i + 2)} \right. \\
& \quad \left. + \frac{2|\Lambda_{2i} + 6\Lambda_{5i}|\xi^{\alpha_i + 1}}{\Gamma(\alpha_i + 2)} + \frac{2|\Lambda_{3i} - 6\Lambda_{6i}|}{\Gamma(\alpha_i - 3)} \right] \\
& \leq \mu(L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|)N_i \\
& \leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|)N_i < +\infty, i = 1, 2,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\|x_i^{(4)}\|_\infty &= \sup_{t \in [0,1]} |x_i^{(4)}(t)| \\
&\leq \mu(L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|) \left[\frac{1}{\Gamma(\alpha_i-1)} + \frac{24|\Lambda_{4i}|}{\Gamma(\alpha_i+2)} \right. \\
&\quad \left. + \frac{24|\Lambda_{2i}|\xi^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + \frac{24|\Lambda_{6i}|}{\Gamma(\alpha_i-3)} \right] \\
&\leq \mu(L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|)V_i \\
&\leq (L_{1i}|\lambda_i| + L_{2i}|\delta_i|L_{2i} + L_{3i}|\gamma_i|)V_i < +\infty, i = 1,2,
\end{aligned} \tag{24}$$

From (22), (23) and (24), we see that $\|(x_1, x_2)\|_{E \times E} < +\infty$. Consequently, F is thus bounded.

4. UHML stability

Definition 10: The System (1)-(2) has UHML stability if one can get $C_{\alpha_i} > 0, i = 1,2$ such that for each $\varepsilon_1, \varepsilon_2 > 0, t \in J$ and for each solution $(x_1, x_2) \in E \times E$ of

$$\begin{cases} |D^{\alpha_1}x_1(t) - \lambda_1 F_1(t, x_1^{(4)}(t), x_2^{(4)}(t)) - \delta_1 G_1(t, x_1^{(2)}(t), x_2^{(2)}(t)) \\ - \gamma_1 H_1(t, x_1, x_2)| \leq \varepsilon_1 E_{\alpha_1}(t^{\alpha_1}), \\ |D^{\alpha_2}x_2(t) - \lambda_2 F_2(t, x_1^{(4)}(t), x_2^{(4)}(t)) - \delta_2 G_2(t, x_1^{(2)}(t), x_2^{(2)}(t)) \\ - \gamma_2 H_2(t, x_1, x_2)| \leq \varepsilon_2 E_{\alpha_2}(t^{\alpha_2}), \end{cases} \tag{25}$$

under the conditions:

$$\begin{cases} x_i(0) = x_i'(0) = x_i^{(3)}(0) = x_i^{(5)}(0) = 0, \\ x_i^{(2)}(0) = \sigma_i \int_0^1 x_i(t)dt, \quad x_i^{(4)}(1) = \rho_i \int_0^\xi x_i(t)dt, \quad 0 < \xi < 1, \\ \lambda_i, \delta_i, \gamma_i, \sigma_i, \rho_i \in \mathbb{R}, \quad 5 < \alpha_i \leq 6, \quad i = 1,2, \end{cases} \tag{26}$$

there exists a solution $(y_1, y_2) \in E \times E$ of the problem (1)-(2) with

$$\|(x_1, x_2) - (y_1, y_2)\|_{E \times E} \leq C_\alpha \varepsilon E_\alpha(t^\alpha), \varepsilon > 0, t \in [0,1]. \tag{27}$$

where $E_\alpha(t^\alpha) = \max\{E_{\alpha_1}(t^{\alpha_1}), E_{\alpha_2}(t^{\alpha_2})\}$, $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and $E_{\alpha_i}(\cdot)$ represent the Mittag-Leffler function defined by:

$$E_{\alpha_i}(\cdot) = \sum_{j=0}^{\infty} \frac{(\cdot)^j}{\Gamma(1+j\alpha_i)}, \quad 0 < \operatorname{Re}(\alpha_i), \quad (\cdot) \in \mathbb{C}, i \in \{1,2\}.$$

We need the assumptions:

(H4): There exists ψ , $|\psi| \leq \varepsilon E_\alpha$.

Theorem 11: Assume that the hypotheses of Theorem 8 are valid and (H4) holds. Then, (1)-(2) can have UHML stability.

Proof: If one consider $(y_1, y_2) \in E \times E$ as the unique solution of (1)-(2). So, suppose that

$$\begin{aligned}
& y_i(t) \\
&= \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i-1} [\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) \\
&\quad + \gamma_i H_i(s, x_1(s), x_2(s))] ds \\
&\quad + (\Lambda_{1i} t^2 + \Lambda_{4i} t^4) \frac{1}{\Gamma(\alpha_i+1)} \int_0^1 (1-s)^{\alpha_i} [\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
&\quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))] ds \\
&\quad + (\Lambda_{2i} t^2 + \Lambda_{5i} t^4) \frac{1}{\Gamma(\alpha_i+1)} \int_0^\xi (\xi-s)^{\alpha_i} [\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
&\quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))] ds \\
&\quad + (\Lambda_{3i} t^2 - \Lambda_{6i} t^4) \frac{1}{\Gamma(\alpha_i-4)} \int_0^1 (1-s)^{\alpha_i-5} [\lambda_i F_i(s, x_1^{(4)}(s), x_2^{(4)}(s)) \\
&\quad + \delta_i G_i(s, x_1^{(2)}(s), x_2^{(2)}(s)) + \gamma_i H_i(s, x_1(s), x_2(s))] ds \} \\
&= I_i, i = 1, 2.
\end{aligned}$$

Thanks to (H4), we get

$$|y_i(t) - I_i| \leq C_{\alpha_i} \varepsilon_i E_{\alpha_i}(t^{\alpha_i}), C_{\alpha_i} > 0, i = 1, 2. \quad (28)$$

By theorem 8, the problem (1)-(2) has a unique solution x_i , $i = 1, 2$. Then, using (28) and (H4)), we get

$$\begin{aligned}
\|y_i - x_i\|_\infty &\leq M_i [(|\lambda_i|P_i + |\delta_i|Q_i + |\gamma|R_i) \|y_i - x_i\|_E + \varepsilon_i E_{\alpha_i}(t^{\alpha_i})] \\
&\leq M_i (\Omega_i \|y_i - x_i\|_E + \varepsilon_i E_{\alpha_i}(t^{\alpha_i})), \quad i = 1, 2,
\end{aligned}$$

and

$$\begin{aligned}
& \|y_i^{(2)} - x_i^{(2)}\|_\infty \\
&\leq N_i [(|\lambda_i|P_i + |\delta_i|Q_i + |\gamma|R_i) \|y_i - x_i\|_E + \varepsilon_i E_{\alpha_i}(t^{\alpha_i})] \\
&\leq N_i (\Omega_i \|y_i - x_i\|_E + \varepsilon_i E_{\alpha_i}(t^{\alpha_i})), \quad i = 1, 2.
\end{aligned}$$

also, we have

$$\begin{aligned}
& \|y_i^{(4)} - x_i^{(4)}\|_\infty \\
&\leq V_i [(|\lambda_i|P_i + |\delta_i|Q_i + |\gamma|R_i) \|y_i - x_i\|_E + \varepsilon_i E_{\alpha_i}(t^{\alpha_i})] \\
&\leq V_i (\Omega_i \|y_i - x_i\|_E + \varepsilon_i E_{\alpha_i}(t^{\alpha_i})), \quad i = 1, 2.
\end{aligned}$$

implies that

$$\begin{aligned}
\|y_i - x_i\|_E &= \|y_i - x_i\|_\infty + \|y_i^{(2)} - x_i^{(2)}\|_\infty + \|y_i^{(4)} - x_i^{(4)}\|_\infty \\
&\leq W_i \Omega_i \|y_i - x_i\|_E + W_i \varepsilon_i E_{\alpha_i}(t^{\alpha_i}), \quad i = 1, 2,
\end{aligned}$$

thus

$$\begin{aligned}
& \|(y_1 - x_1, y_2 - x_2)\|_{E \times E} \\
&= \sum_{i=1}^2 \|y_i - x_i\|_E \leq \left(\frac{W_1 + W_2}{1 - (W_1 \Omega_1 + W_2 \Omega_2)} \right) \varepsilon E_\alpha(t^\alpha).
\end{aligned}$$

Therefore

$$C_\alpha = \frac{W_1 + W_2}{1 - (W_1 \Omega_1 + W_2 \Omega_2)}.$$

Thus, problem (1)-(2) is Ulam-Hyers Mittag-Leffler stable.

Example 12: We consider the following problem:

$$\begin{cases} D^{5.6}x(t) = \frac{1}{10^8} \frac{\sin(t)(x^{(4)}(t)+|y^{(4)}(t)|)}{2e^{t+1}} + 10^{-9}t^3y^{(2)}(t) + \frac{e^{t^2}}{6 \times 10^8}x(t), & 0 \leq t \leq 1, \\ D^{5.5}x(t) = \frac{1}{10^9} \arctan\left(\frac{y^{(4)}(t)}{1+|y^{(4)}(t)|}\right) + 10^{-8} \frac{\cos(\pi t^2)x^{(2)}(t)}{\ln(e^{t+5})} + \frac{\sqrt{e^{t+1}}}{3 \times 10^8}y(t), & 0 \leq t \leq 1, \\ x_i^{(k)}(0) = 0, k \in \{0,1,3,5\} \\ \sigma_i \int_0^1 x_i(t)dt = x_i^{(2)}(0), \quad \rho_i \int_0^\xi x_i(t)dt = x_i^{(4)}(1), i = 1,2, \\ \sigma_1 = 10^{-2}, \sigma_2 = 10^{-3}, \rho_1 = 10^{-4}, \rho_2 = 10^{-5}, \quad \xi = \frac{1}{2}. \end{cases}$$

For this example, we have

$$\begin{aligned} \lambda_1 F_1(t, x^{(4)}(t), y^{(4)}(t)) &= \frac{1}{10^8} \frac{\sin(t)(x^{(4)}(t)+|y^{(4)}(t)|)}{2e^{t+1}}, \\ \lambda_2 F_2(t, x^{(4)}, y^{(4)}) &= \frac{1}{10^9} \arctan\left(\frac{y^{(4)}(t)}{1+|y^{(4)}(t)|}\right), \delta_1 G_1(t, x^{(2)}(t), y^{(2)}(t)) = \\ 10^{-9}t^3y^{(2)}(t), \quad \delta_2 G_2(t, x^{(2)}(t), y^{(2)}(t)) &= 10^{-8} \frac{\cos(\pi t^2)x^{(2)}(t)}{\ln(e^{t+5})}, \gamma_1 H_1(t, x, y) = \\ \frac{e^{t^2}}{6 \times 10^8}x(t), \\ \gamma_2 H_2(t, x, y) &= \frac{\sqrt{e^{t+1}}}{3 \times 10^8}y(t), \text{ where } P_1 = \frac{1}{2e}, P_2 = 1, Q_1 = 1, Q_2 = \frac{1}{5}, R_1 = \\ e, R_2 = \sqrt{e+1} \text{ and } \Omega_1 &= \frac{15+8e}{3e \times 10^9}, \Omega_2 = \frac{9+\sqrt{e+1}}{3 \times 10^9}, \text{ also we have:} \\ \Lambda_{11} &= 5.0083 \times 10^{-3}, \Lambda_{12} = 5.0008 \times 10^{-4}, \Lambda_{21} = 4.1736 \times 10^{-9}, \\ \Lambda_{22} &= 4.1674 \times 10^{-11}, \Lambda_{31} = 1.0333 \times 10^6, \Lambda_{32} = 4.1674 \times 10^{-6}, \\ \Lambda_{41} &= 8.695 \times 10^{-10}, \Lambda_{42} = 8.682 \times 10^{-12}, \Lambda_{51} = 4.1667 \times 10^{-6}, \\ \Lambda_{52} &= 4.1667 \times 10^{-7}, \Lambda_{61} = 4.1667 \times 10^{-2}, \Lambda_{62} = 4.1667 \times 10^{-2} \text{ and} \\ M_1 &= 7.2279 \times 10^5, M_2 = 3.4816 \times 10^{-2}, N_1 = 1.4456 \times 10^6, \\ N_2 &= 0.46210, V_1 = 1.3990, V_2 = 1.5045, \text{ where} \\ W_1 &= 2.1684 \times 10^6, W_2 = 2.0014. \end{aligned}$$

Consequently, the following inequality hold: $\sum_{i=1}^2 W_i \Omega_i = 9.7710 \times 10^{-3} < 1$. The reader remarks that the conditions of Theorem 8 are satisfied. Hence, the problem above has a unique solution on $[0,1]$.

Remark 13: If we consider the case $C_\alpha = \frac{W_1+W_2}{1-(W_1\Omega_1+W_2\Omega_2)}$ in the above example, the solution $(x(t), y(t))$ is Ulam Hyers Mittag-Leffler stable.

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