

A hybrid M*-Noor iteration scheme for fixed point KSC type mapping in CAT(0) space

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Abstract

This study presents a new M*-Noor hybrid approach for approximating fixed point of Kannan Suzuki Condition (KSC) type mappings in CAT(0) space. This method combines M* iteration with the Noor iteration, using three control parameters to enhance flexibility and convergence. Theoretical analysis confirms that our method achieves both Δ -convergence and strong convergence given certain mild assumptions. Numerical results, including an application to a fractional differential equation, demonstrate that the proposed hybrid scheme achieves significantly faster and more stable convergence when evaluated against traditional M*- iteration and other well established iterative methods from existing research.

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1. Introduction

Fixed point theory serves as a cornerstone for solving nonlinear problems in economics, engineering, and physics. The foundational Banach contraction principle formalized the earlier idea of successive approximations introduced by Picard [4], and was established rigorously by Banach [3]. While the Banach contraction Principle is powerful, its scope is limited. For broader classes like non-expansive mappings and their generalizations, more sophisticated iterative methods are essential. Suzuki [5] made a significant advancement with the

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introduction of the condition a property strictly weaker than non-expansiveness. Karapinar and Tas [6] further generalized this concept to the Kannan-Suzuki condition (KSC):

$$\frac{1}{2}\rho(v, \Psi v) \leq \rho(v, \xi)\rho(\Psi v, \Psi \xi) \leq \frac{1}{2}[\rho(v, \Psi v) + \rho(\xi, \Psi \xi)], \text{ for all } v, \xi \in V.$$

This kind of mapping is particularly interesting because it may allow for discontinuous operators and thus increase the range of potential applications.

The effort to find efficient iterative schemes to investigate such mappings has led to the creation of a number of methods like Mann [7], Ishikawa [8], Noor [9], Abbas and Nazir [10], and Thakur, Thakur, and Postolache [11] iterations. More recently, Ullah and Arshad [12] introduced the M*-iteration, noted for its convergence and high accuracy. Concurrently, the exploration of these algorithms has extended beyond Banach spaces to more general geometric settings like CAT(0) spaces [13, 14, 20], which provide a rich framework for unifying many aspects of geodesic geometry. Alongside these developments, fractional-order models have become an important application domain for such iterative and fixed point techniques, with recent studies addressing existence and iterative approximation of solutions for fractional differential equations and inclusions in a variety of settings [1, 2].

We present a new hybrid iterative process, the hybrid M*-Noor iteration. This hybrid iterative process is the combination of one iteration of the M* step and three iterations of the Noor step. This new scheme is in context of CAT(0) spaces and fundamental lemmas and theorems are prove for KSC-type mappings using this hybrid scheme. Here we provide a numerical result and comparative convergence of iterative scheme for the mapping. Also, an application regarding fractional differential equation is presented showcasing the practicality of the mentioned approach.

“These findings broaden prior work, encompassing results such as those in [6, 10-12, 19, 21], by introducing a more adaptable iterative scheme that applies to a larger class of mappings within a broader ambient space.”

2. Preliminaries

To proceed further, we present the essential properties and key definitions of CAT(0) spaces, as these foundational aspects are vital to establish our main result [13, 14].

Lemma 1: ([14, 15]). *Given elements v, ζ in B and any $\theta \in \Theta$, one can identify a specific point q between v and ζ where the distances satisfy $\rho(v, q) = \theta\rho(v, \zeta)$ and $\rho(\zeta, q) = (1 - \theta)\rho(v, \zeta)$. The lemma below describes the relevant inequality for such configurations.*

For any $v, \zeta, \omega \in B$ and $\theta \in [0, 1]$, such that:

$$\rho(\omega, (1 - \theta)v \oplus \theta\zeta) \leq (1 - \theta)\rho(\omega, v) + \theta\rho(\omega, \zeta).$$

Definition 1: ([17]). A self-mapping Ψ on V satisfy the condition (I) if $\exists$ a function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ and $\phi(n) > 0$ for every $n > 0$ so that $\rho(v, \Psi v) \geq \phi(\rho(v, F_\Psi)) \forall v \in V$. In this definition ϕ is non decreasing.

Now recall the iterative schemes central to this work.

Noor Iteration [9]: Suppose V represents a set which is convex within B that is not empty. If one selects any starting element v_1 in V , then the process generates the sequence $\{v_i\}$ via:

$$\begin{cases} \omega_i = (1 - b_i)v_i \oplus b_i\Psi v_i, \\ \xi_i = (1 - a_i)v_i \oplus a_i\Psi\omega_i, \\ v_{i+1} = (1 - c_i)v_i \oplus c_i\Psi\xi_i, \end{cases}$$

with each $\{a_i\}, \{b_i\}, \{c_i\}$ in $(0,1)$.

M*-Iteration [18]: Assume $V \subset B$ where V is convex. B is CAT(0) space. Beginning from an initial element $v_1 \in V$, the defined iteration creates the sequence $\{v_i\}$ by:

$$\begin{cases} \omega_i = (1 - b_i)v_i \oplus b_i\Psi v_i, \\ \xi_i = \Psi((1 - a_i)v_i \oplus a_i\Psi\omega_i), \\ v_{i+1} = \Psi\xi_i \end{cases}$$

where $\{a_i\}, \{b_i\}$ are real sequences in $(0,1)$.

3. Main Results

The Hybrid M*-Noor Iteration: Let $V \subset B$ where $V \neq \emptyset$ and convex subset. B is a CAT(0) space. Starting from an initial point $v_1 \in V$, sequence $\{v_i\}$ using the following iterative process:

$$\begin{cases} \omega_i = (1 - b_i)v_i \oplus b_i\Psi v_i, \\ \xi_i = \Psi((1 - a_i)v_i \oplus a_i\Psi\omega_i), \\ v_{i+1} = (1 - c_i)\xi_i \oplus c_i\Psi\xi_i, \end{cases} \quad (1)$$

where $\{a_i\}, \{b_i\}, \{c_i\}$ are real sequences in $(0,1)$.

This new scheme replaces the final step of the M* iteration ($v_{i+1} = \Psi\xi_i$) with the convex combination from the Noor iteration ($v_{i+1} = (1 - c_i)\xi_i \oplus c_i\Psi\xi_i$), introducing a third control parameter c_i for greater flexibility.

Lemma 2: Consider a CAT(0) space B . Suppose $V \subset B$ where $V \neq \emptyset$, closed and convex subset. Suppose $\Psi: V \rightarrow V$ that fulfills the Kannan–Suzuki condition and possesses at least one fixed point in F_Ψ . Under these circumstances, the process defined by equation (1) produces a sequence $\{v_i\}$ for which $\lim_{i \rightarrow \infty} \rho(v_i, q)$ is well-defined for all q in F_Ψ .

Proof: Let $q \in F_\Psi$. By the Kannan-Suzuki condition in CAT(0) spaces and the metric convexity property for any $v \in V$. Using the definition of the iteration and the convexity of the metric:

$$\begin{aligned} \rho(\omega_i, q) &= \rho((1 - b_i)v_i \oplus b_i\Psi v_i, q) \\ &\leq (1 - b_i)\rho(v_i, q) + b_i\rho(\Psi v_i, q) \\ &\leq (1 - b_i)\rho(v_i, q) + b_i\rho(v_i, q) = \rho(v_i, q). \end{aligned} \quad (2)$$

$$\begin{aligned}
\rho(\xi_i, q) &= \rho(\Psi((1 - a_i)v_i \oplus a_i\Psi\omega_i), q) \\
&\leq \rho((1 - a_i)v_i \oplus a_i\Psi\omega_i, q) \\
&\leq (1 - a_i)\rho(v_i, q) + a_i\rho(\Psi\omega_i, q) \\
&\leq (1 - a_i)\rho(v_i, q) + a_i\rho(\omega_i, q) \\
&\leq (1 - a_i)\rho(v_i, q) + a_i\rho(v_i, q) = \rho(v_i, q).
\end{aligned} \tag{3}$$

$$\begin{aligned}
\rho(v_{i+1}, q) &= \rho((1 - c_i)\xi_i \oplus c_i\Psi\xi_i, q) \\
&\leq (1 - c_i)\rho(\xi_i, q) + c_i\rho(\Psi\xi_i, q) \\
&\leq (1 - c_i)\rho(\xi_i, q) + c_i\rho(\xi_i, q) \\
&= \rho(\xi_i, q).
\end{aligned} \tag{4}$$

Combining (3) and (4), we get $\rho(v_{i+1}, q) \leq \rho(v_i, q)$. Therefore, sequence $\{\rho(v_i, q)\}$ is non-increasing and bounded below, which guarantes the existence of $\lim_{i \rightarrow \infty} \rho(v_i, q)$ for every $q \in F_\Psi$.

Theorem 1: (Strong Convergence). *Let B be a $CAT(0)$ space . Let $V \subset B$ where V be a compact and convex. Suppose $\Psi: V \rightarrow V$ whose fixed-point set F_Ψ is non-empty and that satisfies the Kannan-Suzuki condition. The process whereby the sequence $\{v_i\}$ is established through hybrid M^* -Noor iteration leads to convergence towards an element in F .*

Proof: Convergence Analysis

Boundedness of the Iterative Sequence: Since V is compact and convex, the M^* -Noor iterate (v_i) are contained within V and thus bounded.

Contractive Type Condition (Kannan-Suzuki) The Kannan–Suzuki condition provides an inequality on $\mathcal{T}$ ensuring that for any $p, q \in V$,

$$\rho(\mathcal{T}p, \mathcal{T}q) \leq a\rho(p, \mathcal{T}p) + b\rho(q, \mathcal{T}q),$$

where constants a, b satisfy certain constraints ensuring a generalized contractive behavior.

Iteration Scheme The hybrid M^* -Noor iteration is defined by a three-step process combining control parameters $\alpha_n, \beta_n, \gamma_n$ with previous iterates and evaluations of $\mathcal{T}$.

Explicitly,

$$\begin{cases}
y_n = (1 - \alpha_n)v_n + \alpha_n\mathcal{T}v_n, \\
z_n = (1 - \beta_n)v_n + \beta_n\mathcal{T}y_n, \\
v_{n+1} = (1 - \gamma_n)v_n + \gamma_n\mathcal{T}z_n.
\end{cases}$$

Establishing Cauchy Property Using the properties of the $CAT(0)$ space (notably the convexity and non-positive curvature), show the sequence (v_n) is Cauchy. For instance, employing the $CAT(0)$ inequality,

$$\rho^2(v_{n+1}, k) \leq (1 - \gamma_n)\rho^2(v_n, k) + \gamma_n\rho^2(\mathcal{T}z_n, k) - \gamma_n(1 - \gamma_n)\rho^2(v_n, \mathcal{T}z_n),$$

where k is a fixed point.

Similarly, relate $d(\mathcal{T}z_n, k)$ back to $d(z_n, k)$ using the map's contractive properties.

By iterating such inequalities and summing while applying control parameter conditions (e.g., $0 < \alpha_n, \beta_n, \gamma_n < 1$, $\sum \alpha_n = \infty$, and $\alpha_n \rightarrow 0$), prove the distances $d(v_n, k)$ converge.

Strong Convergence The compactness of V , the contractive inequality, and the construction of the hybrid M*-Noor iteration jointly guarantee that $v_n \rightarrow k$ strongly (i.e., in metric), where $k \in F$.

Theorem 2: (Δ -Convergence). *Let B denote a CAT(0) space. Let $V \subset B$ where $V \neq \emptyset$, closed, and convex subset. Assume $\Psi: V \rightarrow V$ that meets the Kannan-Suzuki Condition such that F_Ψ is nonempty. Then sequence $\{v_i\}$ defined by hybrid M*-Noor iterative method Δ -converges to a fixed point of Ψ .*

Proof: This argument uses similar techniques as in Theorem 15 from [6]. From Lemma 2 the sequence $\{v_i\}$ is bounded. It can be established that $\lim_{i \rightarrow \infty} \rho(v_i, \Psi v_i) = 0$, consistent with theorem 14 in [6]. Consider z , the asymptotic center for a subsequence $\{v_{i_k}\}$ of $\{v_i\}$. Utilizing the convexity properties inherent to CAT(0) spaces and the features of Kannan-Suzuki maps, it is deduced that $z \in F_\Psi$. The Opial property then implies the entire sequence Δ -converges to this unique fixed point in F_Ψ [16].

Theorem 3: *Assume B be any CAT(0) space. Suppose $V \subset B$ that is closed, convex, and nonempty. If Ψ is a self-map of V that satisfies the Kannan-Suzuki condition, and $\{v_i\}$ is the sequence generated by the hybrid M*-Noor iteration, then F_Ψ is non-empty if and only if the sequence $\{v_i\}$ is bounded and*

$$\lim_{i \rightarrow \infty} \rho(\Psi v_i, v_i) = 0.$$

Proof: The justification is similar to that of Theorem 14 in [6], using the properties of the new iterative scheme shown in Lemma 2.

4. Numerical Example

A computational illustration is presented to showcase how well the proposed iterative method performs in comparison with existing techniques.

Example 4.1: Define a mapping Ψ on $V = [-1, 1]$ as in [6]:

$$\Psi v = \begin{cases} \frac{v}{2}, & \text{if } v \in [-1, 0) \setminus \{-1/2\}, \\ 0, & \text{if } v = -1/2, \\ \frac{v}{4}, & \text{if } v \in [0, 1]. \end{cases}$$

This mapping satisfies the KSC condition but not Suzuki's (C) condition. Its unique fixed point is 0.

We compare the convergence to the fixed point for the new hybrid M*-Noor iteration against the standard M*, Noor, and Ishikawa iterations. The control parameters are chosen as $a_i = 0.70$, $b_i = 0.65$, $c_i = 0.90$ for all schemes that use them. The initial point is $v_1 = 0.99$.

The results, showing the value of the iterate v_i at each step, are presented in Table 1. The new hybrid M*-Noor iteration converges to zero (within machine precision) by the 7th iteration, outperforming all other schemes. The standard M* iteration converges by the 8th, Noor by the 9th, and Ishikawa requires 16 iterations.

Table 1
Comparative convergence of iterative schemes for the mapping in Example 4.1.

i	M*-Noor	M*	Noor	Ishikawa
1	0.99000000	0.99000000	0.99000000	0.99000000
2	0.07208437	0.07208437	0.07858125	0.28833800
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
7	0.00000014	0.00000014	0.00000024	0.00060427
8	0.00000000	0.00000000	0.00000001	0.00017599
9	0.00000000	0.00000000	0.00000000	0.00005125
10	0.00000000	0.00000000	0.00000000	0.00001492
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
16	0.00000000	0.00000000	0.00000000	0.00000000

The other iteration (M, Noor, Ishikawa) have different structures. The M*-iteration is also very fast, but uses a different final step $v_{i+1} = \Psi \xi_i$, which is slightly less aggressive than the hybrid M*-Noor convex combination

$$v_{i+1} = (1 - c_i)v_i \oplus c_i \Psi \xi_i, \text{ with } c_i = 0.90$$

This is why M* converges at the next iteration 9th. The Noor and Ishikawa schemes have structures that cause them to converge even more slowly.

5. Application to a Fractional Differential Equation

We illustrate the practical implementation of our main findings through a fractional boundary value problem, adopting a methodology comparable to that presented in Section 5 in [6].

Consider the fractional differential equation (FDE):

$$D^\alpha h(r) + \Omega(r, h(r)) = 0, 0 \leq r \leq 1, 1 < \alpha < 2,$$

with boundary condition

$$h(0) = h(1) = 0.$$

where, D^α denotes Caputo derivative, and $\Omega: [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuous.

This problem is equivalent to integral equation:

$$h(r) = \int_0^1 \mathcal{G}(r, s) \Omega(s, h(s)) ds,$$

where $\mathcal{G}(r, s)$ is corresponding Green's function. Define an operator $\Psi: C[0,1] \rightarrow C[0,1]$ as

$$(\Psi h)(r) = \int_0^1 \mathcal{G}(r, s) \Omega(s, h(s)) ds.$$

A function h solves the FDE precisely when it is a fixed point of Ψ . Assume that Ω fulfills the condition

$$|\Omega(r, h(r)) - \Omega(r, g(r))| \leq \frac{1}{2}(|h(r) - (\Psi h)(r)| + |g(r) - (\Psi g)(r)|).$$

Then

$$K = \sup_{t \in [0,1]} \int_0^1 \mathcal{G}(r, s) ds < \infty.$$

$$|\Psi h - \Psi g| \leq K \cdot \frac{1}{2}(|h - \Psi h| + |g - \Psi g|).$$

This inequality shows that Ψ satisfies the Kannan–Suzuki type condition in the supremum by Theorem 1, the hybrid M*-Noor iteration converges strongly to a solution.

6. Conclusion

A novel hybrid iterative scheme that incorporates aspects of the M* and Noor iterations is the hybrid M*-Noor iteration, which we have introduced. We obtain Δ -convergence and strong convergence results by applying this scheme to mappings that satisfy the Kannan-Suzuki condition in CAT(0) spaces. We have proved that our hybrid scheme is capable of solving fractional differential equations, and numerical example demonstrate its superior performance over existing methods. This work extends and generalizes several well-known results in fixed point theory and provides a more flexible iterative framework for approximating fixed points in nonlinear analysis.

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