

PARTICULAR FUZZY NUMBERS AND THEIR APPLICATION TO REGULAR 2X2 FUZZY MARKOV CHAINS: UNCERTAIN PROBABILITIES

SAED F. MALLAK

DEPARTMENT OF APPLIED MATHEMATICS
PALESTINE TECHNICAL UNIVERSITY -KADOORIE

E-MAILS: S.MALLAK@PTUK.EDU.PS AND SAEDMALLAK@YAHOO.COM

ABSTRACT. We introduce particular fuzzy numbers and discuss some of their properties. We also discuss the limit of the 2x2 regular fuzzy Markov chains (using a restricted fuzzy matrix multiplication) when some of their entries consist of these fuzzy numbers.

AMS Classification: 15B15

Keywords: Fuzzy sets, Fuzzy Numbers, Parabolic and Trapezoidal-Parabolic Fuzzy Number, Finite Fuzzy Markov Chains, A restricted Fuzzy Matrix Multiplication.

1. INTRODUCTION

Fuzzy Sets, Fuzzy Numbers, Fuzzy Markov chains and Probability Measure of Fuzzy Events have been interesting areas of research in the last few decades for many authors and many articles were published (for examples all the references except [4, 7, 9, 16]) since introducing the concepts of Fuzzy sets and Probability Measure of Fuzzy Events by Zadeh [22, 23], respectively. In particular, Fuzzy Markov Chains have attracted the interest of many authors, see for example [1, 2, 3], [5, 6], [8], [10]-[15], [19], and [21]. This paper originates from [5] and [15]. Using a restricted fuzzy

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 9, NUMBER 2 (2011) 137-150.
©TARU PUBLICATIONS

matrix multiplication, Buckley and Eslami in [5] examined the properties of regular and absorbing fuzzy Markov chains. It was mentioned there, in example 2, that the limit of the regular 2x2 fuzzy Markov chain consists of triangular fuzzy numbers when the entries are triangular fuzzy numbers. From the first look, it seems so in general. Indeed, it is not so. In [15], we gave an example where the entries are triangular fuzzy numbers where the limit consists of triangular shaped fuzzy numbers. Besides, we gave an example where the entries are trapezoidal fuzzy numbers where the limit consists of trapezoidal shaped fuzzy numbers. Moreover, we gave criteria that guarantee a limit consisting of triangular fuzzy numbers when the entries are triangular fuzzy numbers and another one that guarantee a limit consisting of trapezoidal fuzzy numbers when the entries are trapezoidal fuzzy numbers. In section 2, we give a quick review of fuzzy sets and numbers. A restricted fuzzy matrix multiplication is discussed in section 3 and regular fuzzy Markov chains are discussed in section 4. In section 5, we introduce particular fuzzy numbers, namely Trapezoidal-Parabolic Fuzzy Numbers and Trapezoidal-Parabolic shaped Fuzzy Numbers, while Parabolic Fuzzy Numbers and Parabolic shaped Fuzzy Numbers are introduced in section 6. In all cases the fuzzy numbers are determined by particular points; we also mention some of their properties. In section 7, we apply the introduced fuzzy numbers to regular 2x2 fuzzy Markov chains. We give criteria that guarantee a limit consisting of trapezoidal-parabolic fuzzy numbers when the entries are trapezoidal-parabolic fuzzy numbers and another one that guarantee a limit consisting of parabolic fuzzy numbers when the entries are parabolic fuzzy numbers in the regular 2x2 fuzzy Markov chains using a restricted fuzzy matrix multiplication.

2. FUZZY SETS AND NUMBERS

We will present some preliminary needed definitions.

Definition 1: A fuzzy subset A of some set Ω is defined by its membership function written $A(x)$ which produces values in $[0, 1]$ for all x in Ω . That is $A(x)$ is a function mapping Ω into $[0, 1]$. We place a bar over a letter to denote a fuzzy set, that is $\bar{A}$. The term crisp means not fuzzy. A crisp set is a regular set.

Definition 2: Let $\Omega = R$. An α -cut of $\bar{A}$, written $\bar{A}[\alpha]$, is defined as $\{x : \bar{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$. $\bar{A}[0]$, the support of $\bar{A}$ is defined as the closure of the union of all the $\bar{A}[\alpha]$, for $0 < \alpha \leq 1$.

Definition 3: A fuzzy number $\bar{\bar{N}}$ is a fuzzy subset of the real numbers satisfying

- (1) $\exists x \in R : \bar{\bar{N}}(x) = 1$
- (2) $\bar{\bar{N}}[\alpha]$ is a closed and bounded interval for $0 \leq \alpha \leq 1$.

A special type of fuzzy numbers $\bar{M}$ is called a triangular fuzzy number. $\bar{M}$ is defined by three numbers $a_1 < a_2 < a_3$ where

- (1) $\bar{M}(x) = 1$ at $x = a_2$.
- (2) The graph of $\bar{M}(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_2, a_3]$ the graph is a straight line from $(a_2, 1)$ to $(a_3, 0)$
- (3) $\bar{M}(x) = 0$ for $x \leq a_1$ or $x \geq a_3$.

We write $\bar{M} = (a_1/a_2/a_3)$ for triangular fuzzy number $\bar{M}$. If at least one of the graphs described above is not a straight line (curve), then $\bar{\bar{M}}$ is called triangular shaped fuzzy number and we write $\bar{\bar{M}} \approx (a_1/a_2/a_3)$.

Another special type of fuzzy numbers $\bar{M}$ is called a trapezoidal fuzzy number. $\bar{M}$ is defined by four numbers $a_1 < a_2 < a_3 < a_4$ where

- (1) $\bar{M}(x) = 1$ on $[a_2, a_3]$
- (2) The graph of $\bar{M}(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_3, a_4]$ the graph is a straight line from $(a_3, 1)$ to $(a_4, 0)$
- (3) $\bar{M}(x) = 0$ for $x \leq a_1$ or $x \geq a_4$.

We write $\bar{M} = (a_1, a_2, a_3, a_4)$ for trapezoidal fuzzy number $\bar{M}$. If at least one of the graphs described above is not a straight line (curve), then $\bar{M}$ is called trapezoidal shaped fuzzy number and we write $\bar{M} \approx (a_1, a_2, a_3, a_4)$.

More details, properties and operations, can be found in [6] and [24].

3. RESTRICTED FUZZY MATRIX MULTIPLICATION ([5] AND [6])

Let $P = [p_{ij}]$ be an $r \times r$ transition matrix of a Markov chain. If $p_{ij} = 0$ or $p_{ij} = 1$ then there is no uncertainty, and so we assume there is uncertainty in the transition probability p_{ij} if $0 < p_{ij} < 1$. In the latter case we replace each of p_{ij} by a fuzzy number $\bar{\bar{p}}_{ij}$ where $0 < p_{ij} < 1$, with the restriction that there are $p_{ij} \in \bar{\bar{p}}_{ij}[1]$ such that $P = [p_{ij}]$ is a transition matrix, and we define the fuzzy transition matrix $\bar{\bar{P}} = [\bar{\bar{p}}_{ij}]$, with the understanding that $\bar{\bar{p}}_{ij} = 0$ when $p_{ij} = 0$ and $\bar{\bar{p}}_{ij} = 1$ when

$p_{ij} = 1$. The restriction that there are $p_{ij} \in \bar{p}_{ij}[1]$ such that $P = [p_{ij}]$ is a transition matrix, guarantees that $p_{ij} \in \bar{p}_{ij}[\alpha]$ for all $0 \leq \alpha \leq 1$.

Now we define restricted fuzzy matrix multiplication because we need to compute $\bar{P}^n$ for $n = 2, 3, \dots$

Let

$$S = \left\{ x = (x_1, \dots, x_r) : x_i \geq 0, \sum_{i=1}^r x_i = 1 \right\},$$

and then define (Using Dom for "domain")

$$Dom_i[\alpha] = \left(\prod_{j=1}^r \bar{p}_{ij}[\alpha] \right) \cap S = \left\{ (p_{i1}, \dots, p_{ir}) : p_{i1}, \dots, p_{ir} \geq 0, \sum_{j=1}^r p_{ij} = 1 \right\},$$

for $0 \leq \alpha \leq 1$ and $1 \leq i \leq r$. Then

$$Dom[\alpha] = \prod_{j=1}^r Dom_j[\alpha] = \left\{ (p_{11}, \dots, p_{rr}) : p_{ij} \geq 0, \sum_{j=1}^r p_{ij} = 1, i, j = 1, \dots, r \right\}.$$

Next set $\bar{P}^n = [\bar{p}_{ij}^{(n)}]$ where we will define $\bar{p}_{ij}^{(n)}$ and show that they are fuzzy numbers.

Consider a crisp transition matrix P and $P^n = [p_{ij}^{(n)}]$. We know that

$$p_{ij}^{(n)} = f_{ij}^{(n)}(p_{11}, \dots, p_{rr}),$$

for some function $f_{ij}^{(n)}$. That is the elements of P^n are just some function of the elements of P . Now consider $f_{ij}^{(n)}$ a function of

$$p = (p_{11}, \dots, p_{rr}) \in Dom[\alpha].$$

Let

$$\Gamma_{ij}^{(n)}[\alpha] = f_{ij}^{(n)}(Dom[\alpha])$$

.Now $f_{ij}^{(n)}$ is continuous and $Dom[\alpha]$ is connected, closed and bounded (compact), which implies that $\Gamma_{ij}^{(n)}[\alpha]$ is a closed and bounded interval for all α, i, j & n . We set

$$\bar{p}_{ij}^{(n)}[\alpha] = \Gamma_{ij}^{(n)}[\alpha],$$

giving the α -cuts of the $\bar{p}_{ij}^{(n)}$ in $\bar{P}^n$. The resulting $\bar{p}_{ij}^{(n)}$ is a fuzzy number because its α -cuts are closed, bounded, intervals and surely it is normalized. Since $\Gamma_{ij}^{(n)}[\alpha]$ is a closed and bounded interval so it is determined by computing its endpoints.

Therefore,

$$\overline{\overline{p_{ij}}}^{(n)}[\alpha] = \left[p_{ij1}^{(n)}(\alpha), p_{ij2}^{(n)}(\alpha) \right], \quad \forall \alpha,$$

where

$$p_{ij1}^{(n)}(\alpha) = \min \left\{ f_{ij}^{(n)}(p) : p \in \text{Dom}[\alpha] \right\}$$

and

$$p_{ij2}^{(n)}(\alpha) = \max \left\{ f_{ij}^{(n)}(p) : p \in \text{Dom}[\alpha] \right\}.$$

4. REGULAR FUZZY MARKOV CHAINS

Let P be an rxr crisp transition matrix for a Markov chain. We say that the Markov chain is regular if $P^k > 0$ for some $k \in \mathbb{N}$ that is $p_{ij}^k > 0, \forall i, j$. If P is a (crisp) transition matrix for a regular Markov chain, then $\lim_{n \rightarrow \infty} P^n = \Pi$ where each row in Π is

$$w = (w_1, \dots, w_r), w_i > 0$$

and

$$\sum_{i=1}^r w_i = 1.$$

Here w is the solution of $wP = w$ satisfying $w_i > 0$ and

$$\sum_{i=1}^r w_i = 1,$$

more can be found in [4, 7, 16].

If P is an rxr crisp transition matrix for a regular Markov chain, then consider $\overline{\overline{P}} = [\overline{\overline{p_{ij}}}]$ where $\overline{\overline{p_{ij}}}$ gives the uncertainty (if any) in p_{ij} .

If $(p_{11}, \dots, p_{rr}) \in \text{Dom}[\alpha]$, then $P = [p_{ij}]$ is also a regular Markov chain.

Let $\overline{\overline{P}} \rightarrow \overline{\overline{\Pi}}$ where each row in $\overline{\overline{\Pi}}$ is $\overline{\overline{\pi}} = (\overline{\overline{\pi}}_1, \dots, \overline{\overline{\pi}}_r)$ and

$$\overline{\overline{\pi}}_j[\alpha] = [\pi_{j1}(\alpha), \pi_{j2}(\alpha)], \quad 1 \leq j \leq r.$$

We now show how to compute the α -cuts of the $\overline{\overline{\pi}}_j$. For each $(p_{11}, \dots, p_{rr}) \in \text{Dom}[\alpha]$ set $P = [p_{ij}]$ and we get $P^n \rightarrow \Pi$. Let

$$\Gamma(\alpha) = \{w : w \text{ is a row in } \Pi, (p_{11}, \dots, p_{rr}) \in \text{Dom}[\alpha]\}.$$

Then

$$\pi_{j1}(\alpha) = \min\{w_j : w \in \Gamma(\alpha)\}$$

and

$$\pi_{j2}(\alpha) = \max\{w_j : w \in \Gamma(\alpha)\}$$

where w_j is the j^{th} component in the vector w ([5, 6]) .

As an illustration of the previous discussion , let P be a 2×2 transition matrix of a regular Markov chain, then consider

$$\bar{\bar{P}} = \begin{bmatrix} \bar{\bar{p}}_{11} & \bar{\bar{p}}_{12} \\ \bar{\bar{p}}_{21} & \bar{\bar{p}}_{22} \end{bmatrix}$$

where $\bar{\bar{p}}_{ij}$ gives the uncertainty (if any) in p_{ij} for $i = 1, 2, j = 1, 2$.

If $(p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]$, then

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

is a regular transition matrix and so P^n is convergent. We solve

$$[w_1, w_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [w_1, w_2]$$

where $w_1, w_2 > 0$ and $w_1 + w_2 = 1$. We have

$$p_{11}w_1 + p_{21}w_2 = w_1 \Rightarrow p_{21}w_2 = (1 - p_{11})w_1 \Rightarrow w_2 = \frac{p_{12}}{p_{21}}w_1$$

$$p_{12}w_1 + p_{22}w_2 = w_2 \Rightarrow p_{12}w_1 = (1 - p_{22})w_2 \Rightarrow w_2 = \frac{p_{12}}{p_{21}}w_1$$

So we get the same result

$$w_2 = \frac{p_{12}}{p_{21}}w_1,$$

then by substituting w_2 in $w_1 + w_2 = 1$ we get

$$w_1 = \frac{p_{21}}{p_{21} + p_{12}} \text{ and } w_2 = \frac{p_{12}}{p_{21} + p_{12}}.$$

Now,

$$\begin{aligned} \frac{\partial w_1}{\partial p_{21}} &= \frac{p_{12}}{(p_{21} + p_{12})^2} > 0, \quad \frac{\partial w_1}{\partial p_{12}} = \frac{-p_{21}}{(p_{21} + p_{12})^2} < 0, \\ \frac{\partial w_2}{\partial p_{21}} &= \frac{-p_{12}}{(p_{21} + p_{12})^2} < 0 \text{ and } \frac{\partial w_2}{\partial p_{12}} = \frac{p_{21}}{(p_{21} + p_{12})^2} > 0. \end{aligned}$$

If

$$\bar{\bar{p}}_{21}[\alpha] = [p_{211}(\alpha), p_{212}(\alpha)]$$

and

$$\bar{\bar{p}}_{12}[\alpha] = [p_{121}(\alpha), p_{122}(\alpha)],$$

then by restricted fuzzy matrix multiplication $\bar{P}^n \rightarrow \bar{\Pi}$ where each row in $\bar{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2)$ and $\bar{\pi}_1[\alpha] = [\pi_{11}(\alpha), \pi_{12}(\alpha)]$ and $\bar{\pi}_2[\alpha] = [\pi_{21}(\alpha), \pi_{22}(\alpha)]$ with

$$\pi_{11}(\alpha) = \min\left\{\frac{p_{21}}{p_{21} + p_{12}} : (p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]\right\} = \frac{p_{211}(\alpha)}{p_{211}(\alpha) + p_{122}(\alpha)},$$

$$\pi_{12}(\alpha) = \max\left\{\frac{p_{21}}{p_{21} + p_{12}} : (p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]\right\} = \frac{p_{212}(\alpha)}{p_{212}(\alpha) + p_{121}(\alpha)},$$

$$\pi_{21}(\alpha) = \min\left\{\frac{p_{12}}{p_{21} + p_{12}} : (p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]\right\} = \frac{p_{121}(\alpha)}{p_{121}(\alpha) + p_{212}(\alpha)},$$

and

$$\pi_{22}(\alpha) = \max\left\{\frac{p_{12}}{p_{21} + p_{12}} : (p_{11}, p_{12}, p_{21}, p_{22}) \in Dom[\alpha]\right\} = \frac{p_{122}(\alpha)}{p_{122}(\alpha) + p_{211}(\alpha)}.$$

Therefore,

$$(1) \quad \bar{\pi}_1[\alpha] = \left[\frac{p_{211}(\alpha)}{p_{211}(\alpha) + p_{122}(\alpha)}, \frac{p_{212}(\alpha)}{p_{212}(\alpha) + p_{121}(\alpha)}\right],$$

and

$$(2) \quad \bar{\pi}_2[\alpha] = \left[\frac{p_{121}(\alpha)}{p_{121}(\alpha) + p_{212}(\alpha)}, \frac{p_{122}(\alpha)}{p_{122}(\alpha) + p_{211}(\alpha)}\right]$$

for all $0 \leq \alpha \leq 1$.

5. TRAPEZOIDAL-PARABOLIC (SHAPED) FUZZY NUMBERS

Consider a fuzzy number that is determined by five real number a_1, a_2, a_3, a_4 and c such that $a_1 < a_2 < a_3 < a_4$ and $0 < c < 1$, denoted by $\bar{N}_c = (a_1/a_2/a_3/a_4)$ or $(a_1, a_2, a_3, a_4)_c$ whose membership function is given by

$$\bar{N}_c(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{c}{a_2 - a_1}(x - a_1), & a_1 \leq x \leq a_2 \\ 1 - \frac{1-c}{(a_2 - a_3)^2}(2x - a_2 - a_3)^2, & a_2 \leq x \leq a_3 \\ \frac{-c}{a_4 - a_3}(x - a_4), & a_3 \leq x \leq a_4 \\ 0, & x \geq a_4 \end{cases}$$

(1) $\bar{N}_c(x)$ is a line from $(a_1, 0)$ to (a_2, c) .

(2) $\bar{N}_c(x)$ is a parabola from (a_2, c) to (a_3, c) whose vertex is $\left(\frac{a_2 + a_3}{2}, 1\right)$ and

$$\text{focus is } \left(\frac{a_2 + a_3}{2}, 1 - \frac{(a_3 - a_2)^2}{16(1 - c)}\right).$$

(3) $\bar{N}_c(x)$ is a line from (a_3, c) to $(a_4, 0)$.

Call such a fuzzy number a Trapezoidal-Parabolic Fuzzy Number. If it is determined by five numbers and it is not of this form call it a Trapezoidal-Parabolic Shaped Fuzzy Number.

Consider the cases: $\alpha = \frac{c}{a_2 - a_1}(x - a_1)$, $\alpha = 1 - \frac{1 - c}{(a_2 - a_3)^2}(2x - a_2 - a_3)^2$ and $\alpha = \frac{-c}{a_4 - a_3}(x - a_4)$ and then evaluate for x in each case we get the α -cuts of $\bar{\bar{N}}_c$. Therefore,

$$\bar{\bar{N}}_c[\alpha] = \left[\frac{(a_2 - a_1)}{c}\alpha + a_1, \frac{-(a_4 - a_3)}{c}\alpha + a_4 \right]$$

for $0 \leq \alpha \leq c$ and

$$\bar{\bar{N}}_c[\alpha] = \left[\frac{-(a_3 - a_2)}{2}\sqrt{\frac{1 - \alpha}{1 - c}} + \frac{a_2 + a_3}{2}, \frac{(a_3 - a_2)}{2}\sqrt{\frac{1 - \alpha}{1 - c}} + \frac{a_2 + a_3}{2} \right]$$

for $c \leq \alpha \leq 1$.

Remarks on the arithmetic of trapezoidal-parabolic fuzzy numbers:

- (1) The sum and the difference of two trapezoidal-parabolic fuzzy numbers having the same c is also a trapezoidal-parabolic fuzzy number with the same c .
- (2) The product and the quotient of two trapezoidal-parabolic fuzzy numbers having the same c need not be a trapezoidal-parabolic fuzzy number, here is an example:

Example 4: Let $\bar{\bar{A}}_{0.2} = (1/2/3/4)_{0.2}$. Find $\bar{\bar{B}} = \bar{\bar{A}}_{0.2} \cdot \bar{\bar{A}}_{0.2}$ and $\bar{\bar{C}} = \bar{\bar{A}}_{0.2} / \bar{\bar{A}}_{0.2}$. Now $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$ and $c = 0.2$. Therefore,

$$\bar{\bar{A}}_{0.2}[\alpha] = [5\alpha + 1, -5\alpha + 4] \text{ for } 0 \leq \alpha \leq 0.2$$

and

$$\bar{\bar{A}}_{0.2}[\alpha] = \left[\frac{-1}{2}\sqrt{1.25(1 - \alpha)} + \frac{5}{2}, \frac{1}{2}\sqrt{1.25(1 - \alpha)} + \frac{5}{2} \right]$$

for $0.2 \leq \alpha \leq 1$.

$$\bar{\bar{B}}[\alpha] = [(5\alpha + 1)^2, (-5\alpha + 4)^2] \text{ for } 0 \leq \alpha \leq 0.2$$

and

$$\bar{\bar{B}}[\alpha] = \left[\left(\frac{-1}{2} \sqrt{1.25(1-\alpha)} + \frac{5}{2} \right)^2, \left(\frac{1}{2} \sqrt{1.25(1-\alpha)} + \frac{5}{2} \right)^2 \right]$$

for $0.2 \leq \alpha \leq 1$.

Hence, $\bar{\bar{B}}[0] = [1, 16]$, $\bar{\bar{B}}[0.2] = [4, 9]$ and $\bar{\bar{B}}[1] = \frac{25}{4}$, which is a

Trapezoidal- Parabolic Shaped Fuzzy number, denote it by

$$\bar{\bar{B}} \approx (1/4/9/16)_{0.2}.$$

$$\bar{\bar{C}}[\alpha] = \left[\frac{5\alpha + 1}{-5\alpha + 4}, \frac{-5\alpha + 4}{5\alpha + 1} \right] \text{ for } 0 \leq \alpha \leq 0.2$$

and

$$\bar{\bar{C}}[\alpha] = \left[\frac{-\frac{1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}}{\frac{1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}}, \frac{\frac{1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}}{-\frac{1}{2}\sqrt{1.25(1-\alpha)} + \frac{5}{2}} \right]$$

for $0.2 \leq \alpha \leq 1$.

Hence,

$$\bar{\bar{C}}[0] = \left[\frac{1}{4}, 4 \right], \bar{\bar{C}}[0.2] = \left[\frac{2}{3}, \frac{3}{2} \right] \text{ and } \bar{\bar{C}}[1] = 1,$$

again, we get a trapezoidal-parabolic shaped fuzzy number,

$$\bar{\bar{C}} \approx \left(\frac{1}{4}/\frac{2}{3}/\frac{3}{2}/4 \right)_{0.2}.$$

- (3) If $\bar{\bar{N}}_{c_1}$ and $\bar{\bar{N}}_{c_2}$ are two trapezoidal-parabolic fuzzy numbers with $c_1 \neq c_2$ then the addition, subtraction, multiplication, and quotient of $\bar{\bar{N}}_{c_1}$ and $\bar{\bar{N}}_{c_2}$ are not defined.

6. PARABOLIC (SHAPED) FUZZY NUMBERS

If we put $c = 0$ in the membership function of the trapezoidal-parabolic fuzzy number, in section 5, then we get another fuzzy number, call it parabolic fuzzy number, and is defined by two real numbers a_1 and a_2 with $a_1 < a_2$, denoted by $\bar{\bar{N}} = (a_1/a_2)$ or $\bar{\bar{N}} = (a_1, a_2)$. The membership function $\bar{\bar{N}}(x)$ is given by

$$\bar{\bar{N}}(x) = \begin{cases} 0, & x \leq a_1 \\ 1 - \frac{1}{(a_1 - a_2)^2} (2x - a_1 - a_2)^2, & a_1 \leq x \leq a_2 \\ 0, & x \geq a_2 \end{cases}$$

- (1) $\bar{\bar{N}}(x)$ is parabolic in $[a_1, a_2]$ whose vertex is $(\frac{a_1 + a_2}{2}, 1)$ and focus $(\frac{a_1 + a_2}{2}, 1 - \frac{(a_1 - a_2)^2}{16})$.
- (2) If $\bar{\bar{N}}(x)$ is not parabolic in $[a_1, a_2]$ then it is called a parabolic shaped fuzzy number and is denoted by $\bar{\bar{N}} \approx (a_1/a_2)$ or $\bar{\bar{N}} \approx (a_1, a_2)$.
- (3) $\bar{\bar{N}}[\alpha] = [-\frac{(a_2 - a_1)}{2}\sqrt{1 - \alpha} + \frac{a_2 + a_1}{2}, \frac{(a_2 - a_1)}{2}\sqrt{1 - \alpha} + \frac{a_2 + a_1}{2}]$ for $0 \leq \alpha \leq 1$.
- (4) $\bar{\bar{N}} \approx (a_1/\frac{a_1 + a_2}{2}/a_2)$ is a triangular shaped fuzzy number.

Remarks on the arithmetic of parabolic fuzzy numbers:

- (1) The sum and the difference of two parabolic fuzzy numbers is also a parabolic fuzzy number.
- (2) The product and the quotient of two parabolic fuzzy numbers need not be a parabolic fuzzy number.
- (3) Let $\bar{\bar{A}} = (-a/a)$ and $\bar{\bar{B}} = (-b/b)$ where $a, b > 0$, be two parabolic fuzzy numbers then

$$\begin{aligned}\bar{\bar{A}}[\alpha] &= [-a\sqrt{1 - \alpha}, a\sqrt{1 - \alpha}] \text{ and} \\ \bar{\bar{B}}[\alpha] &= [-b\sqrt{1 - \alpha}, b\sqrt{1 - \alpha}] \text{ for } 0 \leq \alpha \leq 1.\end{aligned}$$

Let $\bar{\bar{C}} = \bar{\bar{A}} \cdot \bar{\bar{B}}$ then $\bar{\bar{C}}[\alpha] = [-ab(1 - \alpha), ab(1 - \alpha)]$ for $0 \leq \alpha \leq 1$.

Now $\bar{\bar{C}}[0] = [-ab, ab]$ and $\bar{\bar{C}}[1] = 0$, so $\bar{\bar{C}} = (-ab/0/ab)$ is a triangular fuzzy number. This is interesting since we find two triangular shaped fuzzy numbers whose product is a triangular fuzzy number.

7. APPLYING TRAPEZOIDAL-PARABOLIC AND PARABOLIC FUZZY NUMBERS TO REGULAR 2x2 FUZZY MARKOV CHAINS

Proposition 5: Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

be a regular crisp transition matrix with $0 < p_{ij} < 1$ for $i = 1, 2, j = 1, 2$. Let $0 < \delta < \min\{p_{ij} : i = 1, 2, j = 1, 2\}$. Let

$$\bar{\bar{P}} = \begin{bmatrix} \bar{\bar{p}}_{11} & \bar{\bar{p}}_{12} \\ \bar{\bar{p}}_{21} & \bar{\bar{p}}_{22} \end{bmatrix}$$

where $\bar{\bar{p}}_{ij} = (p_{ij} - \delta, p_{ij} - \frac{\delta}{m}, p_{ij} + \frac{\delta}{m}, p_{ij} + \delta)_c$ and $m > 1, 0 < c < 1$ (trapezoidal-parabolic fuzzy numbers) then by restricted matrix multiplication $\bar{\bar{P}}^n \rightarrow \bar{\bar{\Pi}}$ where each row in $\bar{\bar{\Pi}}$ is $\bar{\bar{\pi}} = (\bar{\bar{\pi}}_1, \bar{\bar{\pi}}_2)$ and

$$\begin{aligned} \bar{\bar{\pi}}_1 &= \left(\frac{p_{21} - \delta}{p_{21} + p_{12}}, \frac{p_{21} - \delta + \frac{1}{c}(\delta - \frac{\delta}{m})}{p_{21} + p_{12}}, \frac{p_{21} + \delta - \frac{1}{c}(\delta - \frac{\delta}{m})}{p_{21} + p_{12}}, \frac{p_{21} + \delta}{p_{21} + p_{12}} \right)_c \\ \bar{\bar{\pi}}_2 &= \left(\frac{p_{12} - \delta}{p_{21} + p_{12}}, \frac{p_{12} - \delta + \frac{1}{c}(\delta - \frac{\delta}{m})}{p_{21} + p_{12}}, \frac{p_{12} + \delta - \frac{1}{c}(\delta - \frac{\delta}{m})}{p_{21} + p_{12}}, \frac{p_{12} + \delta}{p_{21} + p_{12}} \right)_c \end{aligned}$$

Proof.

$$p_{ij}[\alpha] = \left[\frac{1}{c}(\delta - \frac{\delta}{m})\alpha + p_{ij} - \delta, -\frac{1}{c}(\delta - \frac{\delta}{m})\alpha + p_{ij} + \delta \right]$$

for all $0 \leq \alpha \leq c$, and

$$p_{ij}[\alpha] = \left[-\frac{\delta}{m}\sqrt{\frac{1-\alpha}{1-c}} + p_{ij}, \frac{\delta}{m}\sqrt{\frac{1-\alpha}{1-c}} + p_{ij} \right]$$

for all $c \leq \alpha \leq 1$.

Now

$$\bar{\bar{\pi}}_1[\alpha] = \left[\frac{\frac{1}{c}(\delta - \frac{\delta}{m})\alpha + p_{21} - \delta}{p_{21} + p_{12}}, \frac{-\frac{1}{c}(\delta - \frac{\delta}{m})\alpha + p_{21} + \delta}{p_{21} + p_{12}} \right]$$

for $0 \leq \alpha \leq c$ so

$$\bar{\bar{\pi}}_1[0] = \left[\frac{p_{21} - \delta}{p_{21} + p_{12}}, \frac{p_{21} + \delta}{p_{21} + p_{12}} \right]$$

and

$$\bar{\bar{\pi}}_1[c] = \left[\frac{p_{21} - \frac{\delta}{m}}{p_{21} + p_{12}}, \frac{p_{21} + \frac{\delta}{m}}{p_{21} + p_{12}} \right].$$

For $c \leq \alpha \leq 1$

$$\bar{\bar{\pi}}_1[\alpha] = \left[\frac{-\frac{\delta}{m}\sqrt{\frac{1-\alpha}{1-c}} + p_{21}}{p_{21} + p_{12}}, \frac{\frac{\delta}{m}\sqrt{\frac{1-\alpha}{1-c}} + p_{21}}{p_{21} + p_{12}} \right].$$

So $\bar{\pi}_1[1] = \frac{p_{21}}{p_{21} + p_{12}}$. And

$$\bar{\pi}_2[\alpha] = \left[\frac{\frac{1}{c}(\delta - \frac{\delta}{m})\alpha + p_{12} - \delta}{p_{21} + p_{12}}, \frac{-\frac{1}{c}(\delta - \frac{\delta}{m})\alpha + p_{12} + \delta}{p_{21} + p_{12}} \right]$$

for $0 \leq \alpha \leq c$ so

$$\bar{\pi}_2[0] = \left[\frac{p_{12} - \delta}{p_{21} + p_{12}}, \frac{p_{12} + \delta}{p_{21} + p_{12}} \right]$$

and

$$\bar{\pi}_2[c] = \left[\frac{p_{12} - \frac{\delta}{m}}{p_{21} + p_{12}}, \frac{p_{12} + \frac{\delta}{m}}{p_{21} + p_{12}} \right].$$

For $c \leq \alpha \leq 1$,

$$\bar{\pi}_2[\alpha] = \left[\frac{-\frac{\delta}{m}\sqrt{\frac{1-\alpha}{1-c}} + p_{12}}{p_{21} + p_{12}}, \frac{\frac{\delta}{m}\sqrt{\frac{1-\alpha}{1-c}} + p_{12}}{p_{21} + p_{12}} \right]$$

so $\bar{\pi}_2[1] = \frac{p_{12}}{p_{21} + p_{12}}$.

Therefore, both $\bar{\pi}_1$ and $\bar{\pi}_2$ are trapezoidal-parabolic fuzzy numbers with the form given in the proposition, and this completes the proof.

Proposition 6: Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

be a regular crisp transition matrix with $0 < p_{ij} < 1$ for $i = 1, 2, j = 1, 2$. Let $0 < \delta < \min\{p_{ij} : i = 1, 2, j = 1, 2\}$. Let

$$\bar{P} = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$$

where $\bar{p}_{ij} = (p_{ij} - \delta/p_{ij} + \delta)$ (parabolic fuzzy numbers) then by restricted matrix multiplication $\bar{P} \rightarrow \bar{\Pi}$ where each row in $\bar{\Pi}$ is $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2)$ and $\bar{\pi}_1 = \left(\frac{p_{21} - \delta}{p_{21} + p_{12}} / \frac{p_{21} + \delta}{p_{21} + p_{12}} \right)$, $\bar{\pi}_2 = \left(\frac{p_{12} - \delta}{p_{21} + p_{12}} / \frac{p_{12} + \delta}{p_{21} + p_{12}} \right)$.

Proof:

$$\bar{p}_{ij}[\alpha] = [-\delta\sqrt{1-\alpha} + p_{ij}, \delta\sqrt{1-\alpha} + p_{ij}]$$

for $0 \leq \alpha \leq 1$.

$$\bar{\pi}_1[\alpha] = \left[\frac{-\delta\sqrt{1-\alpha} + p_{21}}{p_{21} + p_{12}}, \frac{\delta\sqrt{1-\alpha} + p_{21}}{p_{21} + p_{12}} \right]$$

for $0 \leq \alpha \leq 1$. Hence, $\bar{\pi}_1[0] = \left[\frac{p_{21} - \delta}{p_{21} + p_{12}}, \frac{p_{21} + \delta}{p_{21} + p_{12}} \right]$ and $\bar{\pi}_1[1] = \frac{p_{21}}{p_{21} + p_{12}}$

Also

$$\bar{\pi}_2[\alpha] = \left[\frac{-\delta\sqrt{1-\alpha} + p_{12}}{p_{21} + p_{12}}, \frac{\delta\sqrt{1-\alpha} + p_{12}}{p_{21} + p_{12}} \right]$$

for $0 \leq \alpha \leq 1$. Hence $\bar{\pi}_2[0] = \left[\frac{p_{12} - \delta}{p_{21} + p_{12}}, \frac{p_{12} + \delta}{p_{21} + p_{12}} \right]$ and $\bar{\pi}_2[1] = \frac{p_{12}}{p_{21} + p_{12}}$.

Therefore, both $\bar{\pi}_1$ and $\bar{\pi}_2$ are parabolic fuzzy numbers with the form given in the proposition and this completes the proof.

8. COCLUSIONS

Given any set of real numbers $\{a_1, a_2, \dots, a_n\}$ such that $a_1 < a_2 < \dots < a_n$ and a constant c , $0 \leq c \leq 1$, we can construct particular fuzzy numbers determined by these real numbers and use them in many applications. In this paper, we used the introduced fuzzy numbers in regular 2x2 fuzzy Markov chains. These fuzzy numbers can be used in the future in regular fuzzy Markov chains of different sizes and other types of fuzzy Markov chains. Besides these fuzzy numbers can be used in fuzzy analysis and fuzzy optimization.

REFERENCES

- [1] K.E. Avrachenkov and E. Sanchez, Fuzzy Markov Chains: Specificities and Properties, Proc. IPMU, July 3-7 CityplaceMadrid, country-regionSpain, 2000, pp. 1851-1856.
- [2] K.E. Avrachenkov and E. Sanchez, Fuzzy Markov Chains and Decision –making, Fuzzy Optimization and Decision Making 1, 143-159 (2002).
- [3] R.E. Bellman and L. A. Zadeh, Decision-Making in a Fuzzy Environment, Management Science 17 (1970), pp. 141-164.
- [4] P. Billingsley, Probability and Measure, 1986, Wiley, StateplaceNew York.
- [5] J. Buckley, placeE. Eslami, Fuzzy Markov Chains: Uncertain Probabilities, Mathware and Soft Computing, 9, (2002), 33-41.
- [6] J. Buckley, Fuzzy Probabilities; New Approach and Applications, 2005, Springer-Verlag, CityplaceHeidelberg.
- [7] W. Feller, An Introduction to Probability Theory and its Applications, vol. 1, 1968, Wiley, StateplaceNew York.
- [8] J.C.F. Garcia et al., A Simulation Study on Fuzzy Markov Chains, D.-S. Huang et al. (Eds.): ICIC 2008, CCIS 15, pp. 109-117, 2008, CityplaceSpringer-Verlag, StateBerlin Heidelberg 2008.
- [9] Y. Give'on, Lattice matrices, Information and Control, 7 (1964), 477-484.
- [10] R.M. Kleye and A. DeKorvin, Constructing One-Step and Limiting Fuzzy Transition Probabilities for Finite Markov Chains, J. Intelligent and Fuzzy Systems, 6 (1998), pp. 223-235.

- [11] R. Kruse, R. Buck-Emden and R. Cordes, Processor Power Considerations- An Application of Fuzzy Markov Chains, *Fuzzy Sets and Systems*, 21 (1987), pp. 289-299.
- [12] M. Kurano, M. Yasuda, J. Nakagami and Y. Yoshida, Markov-type Fuzzy Decision Processes with a Discounted Reward on a Closed Interval, *European J. Oper. Res.*, 92 (1996), pp. 649-662.
- [13] S.F. Mallak, Mohammad Mara"Beh and Abdelhalim Zaiqan, A Particular Class of Ergodic Finite Fuzzy Markov Chains, *Advances in Fuzzy Mathematics*, Volume 6, Number 2 (2011), pp. 253-268.
- [14] S.F. Mallak, Mohammad Mara"Beh and Abdelhalim Zaiqan, Further Particular Classes of Ergodic Finite Fuzzy Markov Chains, *Advances in Fuzzy Mathematics*, Volume 6, Number 2 (2011), pp. 269-281.
- [15] S.F. Mallak, Mohammad Mara"Beh and Abdelhalim Zaiqan, Finite Fuzzy Markov Chains: Uncertain Probabilities, A Deeper Look on the Limit of the 2x2 Regular Case (submitted)
- [16] I. C. Resnick , *Adventures in Stochastic Processes*, 1992, Birkhauser, CityplaceBoston.
- [17] E. Sanchez, Resolution of Eigen Fuzzy Sets Equations, *Fuzzy Sets and Systems*, 1, 69-74 (1978).
- [18] E. Sanchez, Eigen Fuzzy Sets and Fuzzy Relations, *J. Math. Anal. Appl.* 81, 399-442 (1981).
- [19] R. Sujatha et al., Long Term Behavior of Cyclic Non-Homogeneous Fuzzy Markov Chain, *Int. J. Contemp. Math. Sciences*, Vol. 5, 2010, no. 22, 1077-1089.
- [20] M.G. Thomason, Convergence of Powers of a Fuzzy Matrix, *J. Math. Anal. Appl.* 57, 476-480 (1977).
- [21] Y. Yoshida, Markov Chains with a Transition Possibility Measure and Fuzzy Dynamic Programming, *Fuzzy Sets and Systems*, 66 (1994), pp. 39-57.
- [22] L.A. Zadeh, Fuzzy Sets, *Information and Control*, 1965, 8 :338-353.
- [23] L.A. Zadeh, Probability Measures of Fuzzy Events, *Journal of Mathematical Analysis and Applications*, 1968, 23 :421-427.
- [24] H.J. Zimmerman, *Fuzzy Set Theory and its Applications*, Springer 4th edition, 2006.