# ON LARGE DEVIATIONS FOR SOME DYNAMICAL SYSTEMS AND FOR GIBBS STATES AT ZERO TEMPERATURE

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Abstract. In this article we analyze two issues related with large deviations in dynamical systems:

1.We show that the level-2 large deviation principle established by Comman and Rivera-Letelier[1], is satisfied by maps with a specification property and, with some additional condition, also by those with the *almost property product,* which is weaker than specification. The earlier mentioned authors proved that their principle, which is a generalization of previous results by Kifer, is verified by a class of hyperbolic rational maps.

2. In a previous article[5] we have considered a family of Gibbs states $\{\mu_q\}_{q\geq 1}$, which had an accumulation point $\mu_\infty$ ( zero temperature limit since the interpretation of $q$ as the inverse of the temperature). We proved that $\mu_\infty$ is a maximizing measure for more general systems than symbolics. In a recent article by Lopes and Mengue[4] were considered similar families of states and proved, for symbolic dynamics, that an accumulation point of the family was maximizing. Also they established a large deviation principle. In this note we show how to use the results of our previous work to describe a large deviation process in a more general context than[4].

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## 1. Introduction

A large deviation process is described in the following way: let $\mathcal{X}$ be a Hausdorff topological vector space and let $(\nu_n)$ be a sequence of measures in $\mathcal{X}$ , a *level-2 large deviation principle* for the sequence $(\nu_n)$ is satisfied when there exists a lower semi-continuous map $I : \mathcal{X} \to [0, +\infty]$ such that

$\lim_{n\to\infty} \sup \dfrac{1}{n} \log \nu_n(F) \leq -\inf \{I(x) : x \in F\}$ if $F$ is a closed subset of $\mathcal{X}$

$\lim_{n\to\infty} \inf \dfrac{1}{n} \log \nu_n(U) \geq -\inf \{I(x) : x \in U\}$ if $U$ is an open subset of.$\mathcal{X}$.

The map $I$ is called the *rate function.*

Let $(\nu_n)$ be supported on a compact subset of $\mathcal{X}$ and satisfying a large deviation principle in $\mathcal{X}$ with rate function $I$, then given any Hausdorff topological space $\mathcal{Y}$ and any map $\phi : \mathcal{X} \to \mathcal{Y}$, a large deviation process in $\mathcal{Y}$ with rate function defined by the functional $y \mapsto \inf \{I(x) : \phi(x) = y\}$ is satisfied for the sequence $(\phi(\nu_n))$. This result is known as the *contraction principle.*

Let us consider the special cases $\mathcal{X} = \widetilde{\mathcal{M}}(X)$, the set of signed measures on a compact set $X$, $K = \mathcal{M}(X)$ a compact subset, $\mathcal{Y} = \mathbf{R}$ and $\phi(\mu) = \phi_\psi(\mu) = \int \psi d\mu$, for some map $\psi \in C(X, \mathbf{R})$. In this situations the functional is defined as $\alpha \mapsto \inf \left\{I(\mu) : \int \psi d\mu = \alpha\right\}$. Deviation principles obtained from the contraction principle are called *level-1 large deviation principle.*

In [1] a level-2 large deviation principle was formulated as follows: let $f : X \to X$ be a continuous map, such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, let us fix a map $\varphi \in C(X, \mathbf{R})$ and let $W$ be a dense subset of $C(X, \mathbf{R})$, with the property that for any $\psi \in W$ the map $\varphi + \psi$ has an unique equilibrium state. Let $(\nu_n)$ be a sequence of Borel measures in $\mathcal{M}(X)$ such that

$\dfrac{1}{n} \log \int_{\mathcal{M}(X)} \exp(n \int \psi d\mu) d\nu_n$ converges, as $n \to \infty$, to $P(\varphi + \psi) - P(\varphi)$, then a level-2 large deviation process is satisfied with rate function

$I(\mu) = \begin{cases} P(\varphi) - \int \varphi d\mu - h_\mu(f) & \text{if } \mu \in \mathcal{M}(X,f) \\ \infty & \text{if } \mu \in \mathcal{M}(X) - \mathcal{M}(X,f) \end{cases}$, where $P(\varphi)$ is the topological pressure of $\varphi$ and $\mathcal{M}(X,f)$ denotes the set of $f-$invariant measures on $X$.

In [1] is established that sequences of measures supported on periodic points or preimages of points of non-uniformly hyperbolic rational maps satisfy this principle. The conditions imposed on the potentials $\varphi$ are that to be Hölder continuous.

The aim of the first part of this article is to study level-2 deviation principles for dynamical systems with the properties of expansiveness and specification (not necessarily with approximation by periodic orbits) and for a special class of potentials. Also will be shown how can be obtained results with a weaker condition than specification. In [6] we proved that a level-1 large deviation process was satisfied for these systems, there was considered $I(\mu) = h_\mu(f) - h_{top}(f)$.

In the second part is consider another kind of problem in large deviations which is posed as: Let $f : X \to X$ be a homeomorphism with $(X,d)$ a compact metric space, if $d_n(x,y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i = 0,1,...,n-1\right\}$ then let us denote by $B_{n,\varepsilon}(x)$ the ball of centre $x$ and radius $\varepsilon$ in the metric $d_n$. Now the problem is for a family of measures$\{\mu_q\}_{q\geq 1}$ to find the rate function $I$ to describe the process

$$\lim_{q\to\infty} \frac{1}{q} \log \mu_q\left(B_{n,\varepsilon}(x)\right), \text{ for any } n, \varepsilon \tag{1}$$

In [5] we have considered a family $\{\mu_q\}_{q\geq 1}$, where each $\mu_q$ is the Gibbs state associated to the potential $\varphi_q := q\varphi - P(q\varphi)$ where $P$ is the topological pressure and with the potential $\varphi$ belonging to a special class. The sequence $\{\mu_q\}$ has, with the weak topology of the space of measures, an accumulation point $\mu_\infty$. This measure can be called the *Gibbs ground state or Gibbs state at zero temperature* (due the interpretation of $q$ as the inverse of the temperature). In our previous mentioned article was proved that

- $\mu_\infty$ is maximizing for the potential $\psi := \varphi_1 = \varphi - P(\varphi)$, recall that a measure $\mu$ is maximizing for a potential $\psi$ if

$\int \psi d\mu \geq \int \psi d\nu$ for any measure $\nu$. Besides $\int \varphi d\mu_\infty = \lim_{q\to\infty} \int \varphi d\mu_q$.

- $h_{\mu_\infty}(f) = \lim_{q\to\infty} h_{\mu_q}(f)$.

- $h_{\mu_\infty}(f) = \sup h_\nu(f)$, where the supreme is taken over all the $\varphi_{q-}$maximizing measures.

These results were obtained in a more general context than symbolics. In [2] results of this nature were proved in a symbolic context but for subshifts with infinite alphabet. So in [5] was generalized the compact case.

In [4] is considered a sequence of probability measures $\{\mu_q\}_{q\geq 1}$ associated to Hölder continuous potentials defined on symbolic spaces and with the probabilities supported on periodic points of the shift. It was proved in that work that the sequence has an accumulation point which is maximizing and is described the process

$\lim_{q\to\infty} \frac{1}{q} \log \mu_q (C_n)$, where $C_n$ is a cylinder of length $n$. The description is done by a function defined firstly on periodic points and the extended to the whole space of sequences.

In this work we give a description of a large deviation process in a more general situation than symbolics, using our previous results of [5].

## 2. Previous definitions

We recall some definitions and facts in order to set here in which conditions we shall work, always $X$ is a compact metric space:

- A homeomorphism $f : X \to X$ is *expansive* if there is a constant $\delta > 0$, such that $d(f^n(x), f^n(y)) < \delta$, for any integer $n$ implies $x = y$.

- *A* homeomorphism $f : X \to X$ has the *specification property*, if for any $\varepsilon > 0$ there is an integer $M(\varepsilon)$ such that for any $k \geq 1$, any points $x_1, ..., x_k$ and any integers $a_1 \leq b_1 \leq ... \leq a_k \leq b_k$ with $b_{i-1} - a_i > M(\varepsilon)$, there exists a point $x_0 \in X$ such that

$d\left(f^{a_i+j}(x_0), f^j(x_i)\right) < \varepsilon$, for $0 \leq j \leq b_i - a_i$, $i = 1, 2.,,, k$.

When the point can be selected in a set $B$ we will say that the system has the specification property with respect to $B$.

For a potential $\varphi$ we put

$$S_n(\varphi)(x) = \sum_{i=0}^{n-1} \varphi\left(f^i(x)\right) \tag{2}$$

which is called the *statistical sum.*

**Definition:** A potential $\varphi$ belongs to the class $\nu_f(X)$ if there are constants $\varepsilon, K > 0$ such that

$$d_n(x,y) < \varepsilon \Longrightarrow |S_n(\varphi)(x) - S_n(\varphi)(y)| < K.$$

This class of bounded distortion maps includes Hölder continuous maps on hyperbolic sets of Riemannian manifolds.

The *topological pressure* associated to a potential $\varphi : X \to \mathbf{R}$, and a map $f : X \to X$ is defined as

(3)
$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \left[ \sup \left\{ \sum_{x\in E} \exp(S_n(\varphi)(x)) : E \text{ is } (g,n,\varepsilon) - \text{separated in } X \right\} \right].$$

An *equilibrium state* for a potential $\varphi : X \to \mathbf{R}$ is a measure $\mu = \mu_\varphi$ such that

$$P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi.$$

Let us consider the functional defined as

$$\tau_\varphi(\psi) = P(\varphi + \psi) - P(\psi),$$

whose Legendre transform $\widehat{\tau_\varphi}$ is given by [1]:

$$\widehat{\tau_\varphi}(\mu) = \begin{cases} P(\varphi) - \int \varphi d\mu - h_\mu(f) & \text{if } \mu \in \mathcal{M}(X,f) \\ \infty \quad \text{if } \mu \in \widetilde{\mathcal{M}}(X) - \mathcal{M}(X,f) \end{cases},$$ i.e. the restriction of $\widehat{\tau_\varphi}$ to $\mathcal{M}(X,f)$ agrees with the rate function for the large deviation process. So that the equilibrium states for $\varphi$ are the zeros of $\widehat{\tau_\varphi}$. Also a direct calculus leads that $\mu$ is a equilibrium state for $\varphi + \psi$ if and only if $\tau_\varphi(\psi) + \widehat{\tau_\varphi}(\mu) = \int \psi d\mu$ [1].

A *tangent functional* at $\varphi$ is a measure $\mu$ for which $\tau_\varphi(\psi) \geq \int \psi d\mu$. If the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous then the set of tangent functionals at $\varphi$ agrees with the set equilibrium states for $\varphi$[10]. For potentials in the class $\nu_f(X) \subset C(X)$, with $f$ an expansive homeomorphism with specification, there is an unique tangent functional[3] and so an unique equilibrium state. Let us recall the classic fact from Functional Analysis that a convex functional on a separable Banach space has an unique tangent functional on a dense set of points, so the uniqueness of equilibrium states can be checked by the existence of dense class in $C(X)$.

**Definition:** A map $f : X \to X$ has the *almost product property* if there is a function $g : \mathbf{N} \to \mathbf{N}$ for which $\frac{g(n)}{n} \to 0$, as $n \to \infty$, and a map $m : \mathbf{R}^+ \to \mathbf{N}$ such that for any $k \geq 1$, any points $x_1, ..., x_k$, any numbers $\varepsilon_1 > 0, ..., \varepsilon_k > 0$ and any integers $n_1 \geq m(\varepsilon_1), ..., n_k \geq m(\varepsilon_k)$ there is a point $x_0 \in X$ such that

$card\left\{j = 0, 1, ..., n_i - 1 : d\left(f^{n_1+\cdots+n_{i-1}+j}(x_0), f^j(x_i)\right) > \varepsilon_i\right\} \leq g(n_i)$ for any $i = 1, 2, .., k$. If the point $x_0$ can be selected in a set $B$, we shall say that the map has the almost product property with respect to $B$.

The specification condition implies the almost product property, in particular the $\beta-$shift has specification property for a set of values of $\beta > 1$, and almost product property for any $\beta > 1$[7].

**Definition:** A set $E \subset X$ is $(n, \varepsilon)-$*separated* if for any $x, y \in E$, $x \neq y$ holds $d\left(f^j(x), f^j(y)\right) > \varepsilon$, for $j = 0, 1, ..., n-1$. If $g : \mathbf{N} \to \mathbf{N}$ then $E \subset X$ is $(g, n, \varepsilon)-$*separated* if for any points $x, y \in E$, $x \neq y$ holds

$card\left\{j = 0, 1, ..., n-1 : d\left(f^j(x), f^j(y)\right) > \varepsilon\right\} > g(n).$

The topological pressure can be computed using $(g, n, \varepsilon)-$separated sets, with a function $g$ like of the definition of almost product product property[9], i.e. for a potential $\varphi : X \to \mathbf{R}$

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log\left[\sup\left\{\sum_{x\in E} \exp\left(S_n(\varphi)(x)\right) : E \text{ is } (g, n, \varepsilon) - \text{separated in } X\right\}\right].$$

## 3. The large deviation process

We shall consider sequence of measures in order to be proved that they verify the large deviation principle. Recall that we are working under the conditions of expansiveness and specifications for the dynamics $f : X \to X$ and potentials $\varphi : X \to \mathbf{R}$ in the bounded distortion class $\nu_f(X)$. If $B_n$ is a sequence of finite sets then let

$$\nu_{\varphi,n} = \frac{1}{\widetilde{Z}(f, \varphi, n)} \sum_{x\in B_n} \exp\left(S_n(\varphi)(x)\right) \delta_{W_n(x)} \tag{4}$$

where $\widetilde{Z}(f,\varphi,n) = \sum\limits_{x\in B_n} \exp\left(S_n(\varphi)(x)\right)$ (the "partition function") and $\delta_{W_n(x)}$ is the point mass measure at the empirical measure:

$$W_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$$

To establish that the sequence $(\nu_{\varphi,n})$, supported on sets $B_n$, satisfies the large deviation principle it should be proved that

$$P(\varphi) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{x\in B_n}\exp\left(S_n(\varphi)(x)\right)\right), \tag{5}$$

in fact

$\dfrac{1}{n}\log\int_{\mathcal{M}(X)}\exp(n\int\psi d\mu)d\nu_{n,\varphi} = \dfrac{\sum\limits_{x\in B_n}\exp\left(S_n(\varphi+\psi)(x)\right)}{\sum\limits_{y\in B_n}\exp\left(S_n(\varphi)(y)\right)}$, now once checked that $\lim\limits_{n\to\infty}\dfrac{1}{n}\log\left(\sum\limits_{x\in B_n}\exp\left(S_n(\varphi)(x)\right)\right)$ tends to $P(\varphi)$, would be proved that $\lim\limits_{n\to\infty}\dfrac{1}{n}\log\int_{\mathcal{M}(X)}\exp(n\int\psi d\mu)d\nu_{n,\varphi} = P(\varphi+\psi)-P(\varphi)$ as is required in the criteria of [1].

**Theorem 1:** Let $f: X\to X$ be an expansive homeomorphism satisfying specification with respect to $(n,\Delta)-$ separated sets $B_n$, for sufficiently small $\Delta$, and let $\varphi$ be a potential in the class $\nu_f(X)$, then sequences of measures (4) satisfy a level-2 large deviation process.

An example of sets satisfying the hypothesis are $B_n = P_n(f)$ the $n-$periodic points of $f$, i.e. $P_n(f) = \{x : f^n(x) = x\}$. The sets $P_n(f)$ are $(n,\delta)-$separated in $X$, for any $n$ and with $\delta$ the constant of expansiveness[10].

*Proof:* It must to be proved that

1) $P(\varphi) = \lim\limits_{n\to\infty}\dfrac{1}{n}\log\left(\sum\limits_{x\in B_n}\exp\left(S_n(\varphi)(x)\right)\right)$.

2) There is dense subset of $C(X)$ whose members has an unique equilibrium state.

Proof of 1) Since $B_n$ are separated sets they are finite and holds $P(\varphi) \geq \lim\limits_{n\to\infty}\dfrac{1}{n}\log\left(\sum\limits_{x\in B_n}\exp\left(S_n(\varphi)(x)\right)\right)$. To show the opposite inequality let

$\varepsilon > 0$ as in the definition of the class $\nu_f(X)$, and let $M(\varepsilon)$ be the number of the specification property, with respect to $B_n$. Let us consider for $n \geq M(\varepsilon)$ a $(n - M(\varepsilon), 3\varepsilon)$ $-$ separated set $E$, thus because $B_n$ is separated for each $x \in E$ there is a point $z = z(x) \in B_n$ such that $d_{n-M(\varepsilon)}(x, z) < \varepsilon$. Because $E$ is separated the assignation $x \mapsto z(x)$ is injective.

Therefore since the potential $\varphi$ belongs to the class $\nu_f(X)$ there is a constant $K > 0$

$$\left|S_{n-M(\varepsilon)}(\varphi)(x) - S_{n-M(\varepsilon)}(\varphi)(z)\right| \leq K, \tag{6}$$

thus

$$\left|S_n(\varphi)(z) - S_{M(\varepsilon)}(\varphi)(z) - S_{n-M(\varepsilon)}(\varphi)(x)\right| \leq K, \tag{7}$$

and

$$S_n(\varphi)(z) \geq S_{n-M(\varepsilon)}(\varphi)(x) + M(\varepsilon)\|\varphi\|_0 - K. \tag{8}$$

Therefore

$$\sum_{x \in B_n} \exp(S_n(\varphi)(z(x))) \geq C \times \sum_{x \in E} \exp(S_n(\varphi)(z)) \exp(M(\varepsilon)\|\varphi\|_0 - K) \tag{9}$$

with $C = \exp(M(\varepsilon)\|\varphi\|_0 - K)$. Thus

$$P(\varphi) \leq \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{x \in B_n} \exp(S_n(\varphi)(x))\right).$$

Proof of 2) Let $\varepsilon > 0$ as in the definition of the class $M(\varepsilon)$ be the specification number and let $t \geq n + 2M(\varepsilon)$ and $m = t - m - 2M(\varepsilon)$. We shall firstly prove that for any $y \in X$

$$A_\varepsilon(\exp(S_n(\varphi)(y)) - nP(\varphi)) \leq \nu_t(B_{n,\varepsilon}(y)) \leq B_\varepsilon(\exp(S_n(\varphi)(y)) - nP(\varphi)),$$

where

$B_{n,\varepsilon}(y) = \left\{w : d_n(w,z) := \max\left\{d\left(f^i(w), f^i(y)\right) : i = 0,1,...,n-1\right\} < \varepsilon\right\}$ and $\nu_t$ defined as in eq (4). Measures satisfying this property are known as the $t-$ *ensembles* and the accumulation points of these sequences are the *Gibbs states* for the potential $\varphi$.

Let $E_m$ be a $(m, 3\varepsilon)-$separated set with maximal cardinality, thus for any $x \in E_m$ there is a point $z = z(x) \in B_t \cap B_{n,\varepsilon}(y)$ with $d_m\left(f^{n+M(\varepsilon)}(z(x)), x\right) < \varepsilon$, with the assignation $z(x)$ to any $x$ injective, this is due, like in 1), to that $E_m$ is separated. Now is used again the bounded distortion property of the potential, and similarly to 1):

$$|S_t(\varphi)(z(x)) - S_n(\varphi)(y) - S_m(\varphi)(x)| \leq 2(M(\varepsilon)\|\varphi\|_0 + K). \tag{10}$$

Thus

$$\nu_t(B_{n,\varepsilon}(y)) = \frac{1}{\widetilde{Z}(f,\varphi,t)} \sum_{z\in B_t\cap B_{n,\varepsilon}(y)} \exp(S_n(\varphi)(z)) \geq$$
$$\frac{\exp(S_n(\varphi)(z))\exp(-2(M(\varepsilon)\|\varphi\|_0 + K))}{\widetilde{Z}(f,\varphi,t)} \times \sum_{x\in E_m} \exp(S_m(\varphi)(x)).$$

If $\varepsilon < \delta/3$ ($\delta$ the constant of expansiveness) then there are constants $m_\varepsilon, M_\varepsilon$ such that[3].

$$\frac{1}{M_\varepsilon}\exp(nP(\varphi)) \leq N_{n,\varepsilon}(\varphi, f) := \sup\left\{\sum_{x\in E}\exp(S_n(\varphi)(x)) : E \text{ is } (n,\varepsilon) - \text{separated}\right\}$$
$$\leq \frac{1}{m_\varepsilon}\exp(nP(\varphi)),$$

therefore,

$\nu_t(B_{n,\varepsilon}(y)) \geq \dfrac{\exp(-2(M(\varepsilon)\|\varphi\|_0 + K))}{\widetilde{Z}(f,\varphi,t)} \times N_{m,3\varepsilon}(\varphi, f) \times \exp(S_n(\varphi)(y))$. By 1) there is a constant $C$ such that, for sufficiently large $t$ holds

$$\nu_t(B_{n,\varepsilon}(y)) \geq C\exp(-rP(\varphi))\exp(-2(M(\varepsilon)\|\varphi\|_0 + K))\times$$
$$\times M_{3\varepsilon}^{-1}\exp(mP(\varphi))\exp(S_n(\varphi)(y)) \geq A_\varepsilon \exp(S_n(\varphi)(y))\exp(-nP(\varphi)).$$

For the other inequality proceeds in a similar way:

$\nu_t(B_{n,\varepsilon}(y)) \leq \frac{1}{\widetilde{Z}(f,\varphi,t)} \sum_{x \in B_t \cap B_{n,\varepsilon}(y)} \exp(S_n(\varphi)(x)) \leq$
$\frac{1}{\widetilde{Z}(f,\varphi,t)} \sum_{x \in B_t \cap B_{n,\varepsilon}(y)} \exp(S_n(\varphi)(y) - S_{t-n}(\varphi)(f^n(x)) + K)$,
now there is a constant $C$ such that for sufficiently large $t$
$\nu_t(B_{n,\varepsilon}(y)) \leq C\exp(K)\exp(-rP(\varphi)) \times N_{t-n,\delta}(\varphi, f)$, with $\delta$ the constant of expansiveness. Finally we get

$$\nu_t(B_{n,\varepsilon}(y)) \leq C\exp(K)m_\delta^{-1}\exp(S_n(\varphi)(y))\exp(-nP(\varphi)).$$

By the Bowen theorem the measures (4) which are $t-$ensembles for maps $f : X \to X$ with expansiveness and specifications and potentials $\varphi \in \nu_f(X)$ have an unique accumulation point which is an equilibrium state for $\varphi$. The density is consequence of the fact commented in the earlier section of that a functional on a Banach space has an unique tangent functional on a dense set of points. ■

*Remark:* The condition of that the $B_n$ be separated may be relaxed imposing the condition

$$\sum_{x \in B_n} \exp(S_n(\varphi)(x)) \leq \sum_{x \in E} \exp(S_n(\varphi)(x))$$

for some $(n, \varepsilon)$ $-$separated $E$. This requirement is fulfilled for example for $B_n = \{f^{-n}(x_0)\}$ with $x_0$ regular point of a hyperbolic rational map[1].

We prove next the result of the theorem 1, with the almost product property $(APP)$ instead of specification, but assuming the existence of the set with potentials with an unique equilibrium state. Recall that $APP$ is weaker than specification.

**Proposition 1:**Let $f : X \to X$ be an expansive homeomorphism satisfying $APP$ with respect to $(n, \delta)$ $-$ separated sets $B_n$, with $\delta$ the constant of expansiveness, and let us assume that there is a dense subset $W \subset C(X)$ whose members has an unique equilibrium state. Then sequences of measures (4) satisfy a level-2 large deviation process.

*Proof:* Let $\varphi \in W$, the inequality $P(\varphi) \geq \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{x \in B_n} \exp(S_n(\varphi)(x))\right)$ holds, like in theorem 1 since $B_n$ are separated. Let $g : \mathbf{N} \to \mathbf{N}$ like in the definition of $g-$separated sets and let $B_{n,\varepsilon}(g,x)$ be the set

$$\left\{ \begin{array}{c} y : \exists\, \Omega \subset \{0,1,,...,n-1\}\,,\ card\,(\Omega - \{0,1,,...,n-1\}) \leq g(n) \text{ with} \\ \max\left\{d\left(f^{i}\left(x\right), f^{i}\left(y\right)\right) : i \in \Omega\right\} < \varepsilon \end{array} \right\}. \tag{11}$$

Let $E$ be a $(2g,n,\delta)-$separated set, thus, by *APP,* for any $x \in E$ there exists an element $z = z(x) \in B_n$, injectively assigned, such that

$$card\left\{j = 0,1,...,n-1 : d\left(f^{j}\left(x\right), f^{j}\left(z\right)\right) > \delta/2\right\} \leq g(n).$$

We have the following result[7]: for any $K > 0$, there exists a $N = N(K)$ such that for any $n \geq N$ and $y \in B_{n,\delta}\left(g,x\right)$ holds $\left|S_n\left(\varphi\right)\left(x\right) - S_n\left(\varphi\right)\left(y\right)\right| < Kn$.Thus

$\left|S_n\left(\varphi\right)\left(x\right) - S_n\left(\varphi\right)\left(z\right)\right| < Kn$ and so

$\dfrac{1}{n}\log(\sum\limits_{x\in B_n}\exp\left(S_n\left(\varphi\right)\left(z\right)\right)) \geq \dfrac{1}{n}\log(\sum\limits_{x\in E}\exp\left(S_n\left(\varphi\right)\left(z(x)\right)\right)\exp(-K))$ and therefore

$$P(\varphi) \leq \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{x\in B_n}\exp\left(S_n\left(\varphi\right)\left(x\right)\right)\right)$$

## 4. Large deviation for Gibbs states at zero temperature

In this part we consider the family of measures $\{\mu_q\}$ with $\mu_q$ the Gibbs state associated to the potential, $q\varphi$ with $\varphi \in \nu_f\left(X\right)$. In [5] we have considered measures associated to the potentials $q\varphi - P\left(q\varphi\right)$, recall that the Gibbs states for a potential $\varphi$ and for $\varphi$ adding a constant are the same. The reason of considering potentials $q\varphi - P\left(q\varphi\right)$, instead of directly $q\varphi$, is that the first family fitted better for the multifractal analysis, object of interest there.

The rate function $I$ to be used for the description of the large deviation process is taken from [4]: for $x \in P_n\left(f\right)$,

$$I(x) = I_{\varphi}(x) := n\left[\int\varphi d\mu_{\infty} - \frac{S_n\left(\varphi\right)\left(x\right)}{n}\right],$$

in that article was set $n\left[\sup_\mu\left\{\int\varphi d\mu\right\}-\dfrac{S_n(\varphi)(x)}{n}\right]$, here we put directly $\int\varphi d\mu_\infty$, because we already know that $\mu_\infty$ is maximizing. This map can be extended to the whole space $X$ by[4]

$\widetilde{I}(x):=\lim_{\epsilon\to 0}\inf\{y:d(x,y)<\varepsilon\}.$

Now we formulate the result of this section:

**Theorem 2:** Let $f:X\to X$ be an expansive homeomorphism satisfying specification, and let $\varphi$ be a potential in the class $\nu_f(X)$, then for any periodic point $y$ is valid. Let $\{\mu_q\}$ be the family of Gibbs states as before, i.e. verifying

$A_\varepsilon(q)\left(\exp\left(S_n(\varphi)(x)\right)-nP(\varphi)\right)\le\mu_q\left(B_{n,\varepsilon}(x)\right)$

$\le B_\varepsilon(q)\left(\exp\left(S_n(\varphi)(x)\right)-nP(\varphi)\right)$, for some constants $A_\varepsilon(q),B_\varepsilon(q)$. Then for any periodic point $y$ is valid

$M-\inf\limits_{x\in B_{n,\varepsilon}(y)\cap Per}I(x)\le\lim_{q\to\infty}\dfrac{1}{q}\log\mu_q\left(B_{n,\varepsilon}(y)\right)\le N-\inf\limits_{x\in B_{n,\varepsilon}(y)\cap Per}I(x),$

where $Per=\cup_nP_n(f)$ and

$M=\lim_{n\to\infty}\dfrac{1}{n}\log A_\varepsilon(q),\;\; N=\lim_{n\to\infty}\dfrac{1}{n}\log B_\varepsilon(q)$, provided these limits exist.

*Proof:* Let $y\in P_n(f)$,

$\mu_q\left(B_{n,\varepsilon}(y)\right)\ge A_\varepsilon(q)\exp\left(S_n(q\varphi)(y)\right)\exp(-nP(q\varphi))=$

$A_\varepsilon(q)\exp\left[-n(h_{\mu_q}(f)+q\int\varphi d\mu_q)\right]\exp\left[q\left(n\int\varphi d\mu_\infty-I(y)\right)\right]=$

$A_\varepsilon(q)\exp\left[-nh_{\mu_q}(f)+qn\left(\int\varphi d\mu_\infty-\int\varphi d\mu_q\right)-qI(y)\right]$, then

$\lim_{q\to\infty}\dfrac{1}{q}\log\mu_q\left(B_{n,\varepsilon}(y)\right)\ge M+\lim_{q\to\infty}n\left(\int\varphi d\mu_\infty-\int\varphi d\mu_q\right)-I(y)=M-I(y)$, since $y$ is any generic periodic point, we have

$\lim_{q\to\infty}\dfrac{1}{q}\log\mu_q\left(B_{n,\varepsilon}(y)\right)\ge-\inf\limits_{x\in B_{n,\varepsilon}(y)\cap Per}I(x)+M$

Let $\delta>0$, since we shall consider $q\to\infty$, we can choose a number $K$ such that $q>2K$, for big $q$. Let $s\in\mathbf{R}$ with

$$h_{\mu_{q\delta}}(f)<h_{\mu_\infty}(f)+Ks.$$

Let $q_0$ such that for $q\ge q_0$

$$h_{\mu_{q_0\delta}}(f)+Ks<h_{\mu_\infty}(f)+2Ks<qs+h_{\mu_\infty}(f).$$

So that, for $q \geq q_0$

$$\mu_q\left(B_{n,\varepsilon}(y)\right) \leq B_\varepsilon(q)\exp(-nP(q\varphi))\exp\left(S_n(q\varphi)(y)\right) =$$

$$B_\varepsilon(q)\exp\left[-n\left(h_{\mu_q}(f) + q\int\varphi d\mu_q\right)\right]\exp\left(S_n(q\varphi)(y)\right) \leq$$

$$B_\varepsilon(q)\exp\left[-nq\left(\int\varphi d\mu_q - \frac{S_n(\varphi)(y))}{n}\right)\right]\exp\left[-nh_{\mu_\infty}(f)\right] \leq$$

$$B_\varepsilon(q)\exp\left[-q(1-\delta)I(y) - qn\left(\int\varphi d\mu_q - \frac{S_n(\varphi)(y))}{n}\right)\delta\right] \times$$
$$\times\exp\left[-nh_{\mu_{q_0\delta}}(f) - nKs\right] \leq$$

$$B_\varepsilon(q)\exp\left[-q(1-\delta)I(y)\right]\exp\left[-q_0\delta\left(\int\varphi d\mu_q - \frac{S_n(\varphi)(y))}{n}\right)\right] \times$$
$$\times\exp\left[-nh_{\mu_{q_0\delta}}(f) - nKs\right] \leq$$

$$B_\varepsilon(q)\exp\left[-q(1-\delta)I(y)\right]\exp\left[q_0\delta\frac{S_n(\varphi)(y))}{n}\right]\exp(-nP(q_0\varphi))\exp(-nKs).$$

Therefore

$\lim_{q\to\infty}\frac{1}{q}\log\mu_q\left(B_{n,\varepsilon}(y)\right) \leq N - (1-\delta)I(y)$, so the results follows since $\delta$ can be arbitrarily chosen. ∎

The theorem can be stated is the more general way:
$M - \inf\limits_{x\in B_{n,\varepsilon}(y)} I(x) \leq \lim_{q\to\infty}\frac{1}{q}\log\mu_q\left(B_{n,\varepsilon}(y)\right) \leq N - \inf\limits_{x\in B_{n,\varepsilon}(y)} I(x)$, this is possible from the property, established in [4], of the rate function $I$ :

$$\inf_{x\in B_{n,\varepsilon}(y)}\widetilde{I}(x) = \inf_{x\in B_{n,\varepsilon}(y)\cap Per}\widetilde{I}(x).$$

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