OBSERVATIONAL TOPOLOGICAL FUZZY SUBGROUPS

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Abstract. In this paper relative topological fuzzy subgroups as an extension of topological fuzzy subgroups are considered. In fact topological fuzzy subgroups from the viewpoint of an observer are considered. Three methods for constructing relative topological fuzzy subgroups are presented.

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1. Introduction

One of the mathematical models for an observer of a set, has been constructed via fuzzy theory [5] in 2002. In fact if X is a nonempty set, then a one dimensional observer of X is a fuzzy subset of X. This notion has been extended for multi-dimensional observers in [2]. In this paper we restrict ourself to one dimensional observers. Let $\mu : X \to [0,1]$ be an observer of X. A relative topology for X or a μ topology [3] is a collection of members of $[0,1]^X$, which are subsets of μ with the following properties:

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i) μ, $\chi_\phi \in \tau_\mu$, where χ is the characteristic function;
ii) If λ_1, $\lambda_2 \in \tau_\mu$, then $\lambda_1 \cap \lambda_2 \in \tau_\mu$;
iii) If $\{\lambda_i : i \in \Gamma\} \subset \tau_\mu$, then $\bigcup_{i\in\Gamma} \lambda_i \in \tau_\mu$.
τ_μ is a relative mathematical modeling of the notion of topology from the view point of an observer μ.
Let (X, τ_μ) and (Y, τ_γ) be two relative topological spaces. Then a mapping $f : X \to Y$ is called a (μ, γ)-continuous mapping [3] if $f^{-1}(\eta) \cap \mu \in \tau_\mu$ for all $\eta \in \tau_\gamma$, where $f^{-1}(\eta)(x) = \eta(f(x))$.
Remark 1.1. Let X be a set and let $f : X \to X$ be a map. Moreover let τ_μ be a relative topology for X. Then $\tau_{\mu of} = \{\eta of \ : \ \eta \in \tau_\mu\}$ is a relative topology for X and f is $(\mu, \mu of)$-continuous map.
We use of the word *relative* to denote an observational approach. For example a relative topological space means an observational topological space. Now we are going to present an observational model for topological fuzzy subgroups. The notion of fuzzy subgroup [1] at first has been considered by Rosenfeld [4] in 1971. In this paper we are going to consider this notion from an observer viewpoint via topological means. In section three we present three methods to construct new relative topological fuzzy subgroups.

2. Relative topological fuzzy subgroups

In this section we assume that $(G, .)$ is a group, $\mu : G \to [0, 1]$ is an observer of G such that $\mu(e) \geq \mu(g)$ for all $g \in G$ (e is the identity of G). Moreover we assume that τ_μ is a relative topology. $(G, ., \tau_\mu)$ is called a relative topological group if it satisfies the following two conditions:
(i) The product $. : G \times G \to G$ is a $(\mu \times \mu, \mu)$-continuous mapping, where $\tau_{\mu\times\mu} = \{\delta \times \eta \ : \ \delta, \eta \in \tau_\mu \textit{ and } (\delta \times \eta)(g, h) = \delta(g)\}$.
(ii) The inverse map $g \mapsto g^{-1}$ is a (μ, μ)-continuous map.
Moreover we assume that T_1 and T_2 are two continuous $*$-norms with $T_1 \leq T_2$. In fact a continuous mapping $T : [0, 1]^2 \to [0, 1]$ which is also an $*$-norm is called a continuous $*$-norm. We recall that T is an $*$-norm [1] if it satisfies the following properties:
(i) $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in [0, 1]$;
(ii) $T(x, y) = T(y, x)$ for all $x, y \in [0, 1]$;

(iii) If $x, y, z \in [0, 1]$ and $y \leq z$, then $T(x, y) \leq T(x, z)$ and $T(y, x) \leq T(z, x)$;
(iv) $T(x, 1) = x$ for all $x \in [0, 1]$;
(v) If $A \subseteq [0, 1]$ and $B \subseteq [0, 1]$, then $T(supA, supB) = supT(A \times B)$, and $T(infA, infB) = infT(A \times B)$, where *sup* and *inf* are the abbreviations of *supremum* and *infimum* respectively.

Definition 2.1. A fuzzy subset H of μ is called a relative topological $(T_1, T_2) - \mu$-fuzzy subgroup of G if it satisfies the following conditions:
(i) $T_1((H \cap \mu)(g), (H \cap \mu)(s)) \leq (H \cap \mu)(gs) \leq T_2((H \cap \mu)(g), (H \cap \mu)(s))$, for all $g, s \in G$;
(ii) If $g \in \mu^{-1}(0, 1]$ then $T_1((H \cap \mu)(g), \mu(e)) \leq (H \cap \mu)(g^{-1}) \leq T_2((H \cap \mu)(g), \mu(e))$.

Theorem 2.1. Assume that H is a relative topological $(T_1, T_2) - \mu$-fuzzy subgroup of G, and $f : G \to G$ is a group homomorphism. If $(\mu of)^{-1}[0, 1]$ is a subgroup of G with the identity e, then Hof is a relative topological $(T_1, T_2) - \mu of$-fuzzy subgroup of G.

Proof. Remark 1.1 implies that $(G, ., \tau_{\mu of})$ is a relative topological group. For given $g, s \in G$ we have $T_1((Hof \cap \mu of)(g), (Hof \cap \mu of)(s)) = T_1((H \cap \mu)(f(g)), (H \cap \mu)(f(s))) \leq (H \cap \mu)(f(gs)) = (Hof \cap \mu of)(gs) \leq T_2((H \cap \mu)(f(g)), (H \cap \mu)(f(s))) = T_2((Hof \cap \mu of)(g), (Hof \cap \mu of)(s))$. Moreover $T_1((Hof \cap \mu of)(g), \mu of(e)) = T_1((H \cap \mu)(f(g)), \mu(f(e))) \leq (H \cap \mu)((f(g))^{-1}) = (H \cap \mu)(f(g^{-1})) = (Hof \cap \mu of)(g^{-1}) \leq T_2((H \cap \mu)(f(g)), \mu(f(e))) = T_2((Hof \cap \mu of)(g), \mu of(e))$. □

Theorem 2.2. If H is a relative topological $(T_1, T_2) - \mu$-fuzzy subgroup of G, then $T_1(0, (H \cap \mu)(e)) \leq (H \cap \mu)(g) \leq T_2(\mu(e), (H \cap \mu)(e))$, for all $g \in G$.

Proof. If $g \in G$ then

$$T_1(0, (H \cap \mu)(e)) \leq T_1((H \cap \mu)(g), (H \cap \mu)(e)) \leq$$

$$(H \cap \mu)(ge) = (H \cap \mu)(g) \leq T_2((H \cap \mu)(g), (H \cap \mu)(e)) \leq$$

$$T_2(\mu(g), (H \cap \mu)(e)) \leq T_2(\mu(e), (H \cap \mu)(e)). \ \square$$

3. THREE METHODS FOR CONSTRUCTING RELATIVE TOPOLOGICAL SUBGROUPS

Let Γ be an index set. For given $\gamma \in \Gamma$ we assume that G_γ is a group, μ_γ is an observer on G_γ, and H_γ is a relative topological $(T_1, T_2) - \mu_\gamma$-fuzzy subgroup of G_γ. We define an observer μ on the product group $\Pi_{\gamma\in\Gamma} G_\gamma$ by $\mu(f) = inf\{f(\gamma) \ : \ \gamma \in \Gamma\}$. We also define $H : \Pi_{\gamma\in\Gamma} G_\gamma \to [0, 1]$ by $H(f) = inf\{H_\gamma(f(\gamma)) \ : \ \gamma \in \Gamma\}$. With these assumptions we have the next theorem.

Theorem 3.1. H is a relative topological $(T_1, T_2) - \mu$-fuzzy subgroup of $\Pi_\gamma G_\gamma$.

Proof. Remark 1.1 implies that $(\Pi_{\gamma\in\Gamma} G_\gamma, ., \tau_\mu)$ is a relative topological group. If $f \in \Pi_{\gamma\in\Gamma} G_\gamma$, then $\mu_\gamma(f(\gamma)) \le \mu_\gamma(e(\gamma))$. So $\mu(f) \le \mu(e)$ where e is the identity of $\Pi_{\gamma\in\Gamma} G_\gamma$.

If $g, f \in \Pi_{\gamma\in\Gamma} G_\gamma$, then

$$\begin{gathered}
T_1((H \cap \mu)(g), (H \cap \mu)(f)) = \\
T_1(inf\{H_\gamma(g(\gamma)),\ \mu_\gamma(g(\gamma)) \ : \ \gamma \in \Gamma\}, inf\{H_\gamma(f(\gamma)),\ \mu_\gamma(f(\gamma)) \ : \ \gamma \in \Gamma\}) = \\
T_1(inf\{(H_\gamma \cap \mu_\gamma)(g(\gamma)) \ : \ \gamma \in \Gamma\}, inf\{(H_\gamma \cap \mu_\gamma)(f(\gamma)) \ : \ \gamma \in \Gamma\}) = \\
inf T_1(\{(H_\gamma \cap \mu_\gamma)(g(\gamma)) \ : \ \gamma \in \Gamma\} \times \{(H_\gamma \cap \mu_\gamma)(f(\gamma)) \ : \ \gamma \in \Gamma\}) \le \\
inf\{(H_\gamma \cap \mu_\gamma)(g(\gamma)f(\gamma)) \ : \ \gamma \in \Gamma\} = (H \cap \mu)(gf) \le \\
T_2(inf\{(H_\gamma \cap \mu_\gamma)(g(\gamma)) \ : \ \gamma \in \Gamma\}, inf\{(H_\gamma \cap \mu_\gamma)(f(\gamma)) \ : \ \gamma \in \Gamma\}) = \\
T_2(inf\{H_\gamma(g(\gamma)),\ \mu_\gamma(g(\gamma)) \ : \ \gamma \in \Gamma\}, inf\{H_\gamma(f(\gamma)),\ \mu_\gamma(f(\gamma)) \ : \ \gamma \in \Gamma\}) = \\
T_2((H \cap \mu)(g), (H \cap \mu)(f)).
\end{gathered}$$

Moreover

$$\begin{gathered}
T_1((H \cap \mu)(g), \mu(e)) = \\
T_1(inf\{(H_\gamma \cap \mu_\gamma)(g(\gamma)) \ : \ \gamma \in \Gamma\}, inf\{\mu_\gamma(e(\gamma)) \ : \ \gamma \in \Gamma\}) = \\
inf T_1(\{(H_\gamma \cap \mu_\gamma)(g(\gamma)) \ : \ \gamma \in \Gamma\} \times \{\mu_\gamma(e(\gamma)) \ : \ \gamma \in \Gamma\}) \le \\
inf\{T_1(H_\gamma \cap \mu_\gamma)(g(\gamma)), \mu_\gamma(e(\gamma)) \ : \ \gamma \in \Gamma\}) \le \\
inf\{(H_\gamma \cap \mu_\gamma)(g^{-1}(\gamma)) \ : \ \gamma \in \Gamma\} = \\
(H \cap \mu)(g^{-1}) \le inf\{T_2((H_\gamma \cap \mu_\gamma)(g(\gamma)), \mu_\gamma(e(\gamma))) \ : \ \gamma \in \Gamma\} \\
\le T_2((H \cap \mu)(g), \mu(e)). \ \square
\end{gathered}$$

Now we assume that G is a relative topological group with an observer μ. Moreover we assume that K is a group and $\varphi : G \to K$ is a group homomorphism. We can define two observers λ and η on K by:

$$\lambda(y) = \begin{cases} inf\{\mu(x) \ : \ x \in \varphi^{-1}(y)\} \ if \ \varphi^{-1}(y) \neq \emptyset \\ 0 \ otherwise \end{cases} ,$$

$$\eta(y) = \begin{cases} sup\{\mu(x) \ : \ x \in \varphi^{-1}(y)\} \ if \ \varphi^{-1}(y) \neq \emptyset \\ 0 \ otherwise \end{cases} .$$

Let θ be a subset of μ. Then we define

$$\varphi i(\theta) : K \to [0,1] \ and \ \varphi s(\theta) : K \to [0,1]$$

by:

$$\varphi i(\theta)(y) = \begin{cases} inf\{\theta(x) \ : \ x \in \varphi^{-1}(y)\} \ if \ \varphi^{-1}(y) \neq \emptyset \\ 0 \ otherwise \end{cases} ,$$

$$\varphi s(\theta)(y) = \begin{cases} sup\{\theta(x) \ : \ x \in \varphi^{-1}(y)\} \ if \ \varphi^{-1}(y) \neq \emptyset \\ 0 \ otherwise \end{cases} .$$

Let τ_λ be the relative topology generated by $\{\varphi i(\theta) \ : \ \theta \in \tau_\mu\}$ and let τ_η be the relative topology generated by $\{\varphi s(\theta) \ : \ \theta \in \tau_\mu\}$. Moreover let $(K, ., \tau_\lambda)$, and $(K, ., \tau_\eta)$ be relative topological groups. Then we have the next theorem:

Theorem 3.2. If H is a relative topological $(T_1, T_2) - \mu$-fuzzy subgroup of G and $\varphi : G \to K$ is a group homomorphism, then:
(a) $\varphi i(H)$ is a relative topological $(T_1, T_2) - \lambda$-fuzzy subgroup of K, and
(b) $\varphi s(H)$ is a relative topological $(T_1, T_2) - \eta$-fuzzy subgroup of K.

Proof. (a) Let $k \in K$. Then

$$(\varphi i(H) \cap \lambda)(k) = min\{\varphi i(H)(k), \lambda(k)\} =$$

$$min\{inf\{H(x) \ : \ x \in \varphi^{-1}(k)\} \ , \ inf\{\mu(x) \ : \ x \in \varphi^{-1}(k)\}\} =$$

$$inf\{min\{H(x), \mu(x)\} \ : \ x \in \varphi^{-1}(k)\} = inf\{(H \cap \mu)(x) \ : \ \varphi^{-1}(k)\} = \varphi i(H \cap \mu)(k).$$

So $\varphi i(H) \cap \lambda = \varphi i(H \cap \mu)$.

If $k, s \in K$ then

$$T_1((\varphi i(H) \cap \lambda)(k), (\varphi i(H) \cap \lambda)(s)) = T_1(\varphi i(H \cap \mu)(k), \varphi i(H \cap \mu)(s)) \leq$$

$$inf\{T_1((H \cap \mu)(x), (H \cap \mu)(y) \ : \ x \in \varphi^{-1}(k), \ y \in \varphi^{-1}(s)\} \leq$$

$$inf\{(H\cap\mu)(xy)\ :\ x\in\varphi^{-1}(k),\ y\in\varphi^{-1}(s)\}=\varphi i(H\cap\mu)(ks)=$$
$$(\varphi i(H)\cap\lambda)(ks)\leq inf\{T_2((H\cap\mu)(x),(H\cap\mu)(y)\ :\ x\in\varphi^{-1}(k),\ y\in\varphi^{-1}(s)\}\leq$$
$$T_2(\varphi i(H\cap\mu)(k),\varphi i(H\cap\mu)(s))=T_2((\varphi i(H)\cap\lambda)(k),(\varphi i(H)\cap\lambda)(s)).$$

If $k\in K$, then

$$(\varphi i(H)\cap\lambda)(k)=\varphi i(H\cap\mu)(k)=$$
$$inf\{(H\cap\mu)(x)\ :\ x\in\varphi^{-1}(k)\}=inf\{(H\cap\mu)(x^{-1})\ :\ x^{-1}\in\varphi^{-1}(k^{-1})\}=$$
$$\varphi i(H\cap\mu)(k^{-1})=(\varphi i(H)\cap\lambda)(k^{-1}).$$

Thus $\varphi i(H)$ is a $(T_1,T_2)-\lambda$-fuzzy subgroup of K.

(b) In the proof of (a) replace sup, φs and η with inf, φi, and λ respectively. □.

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