A LOCAL CONSTRUCTION OF HYPERKÄHLER METRICS ON THE FERMAT SURFACE

KWANG-SOON PARK
DEPARTMENT OF MATHEMATICAL SCIENCES
SEOUL NATIONAL UNIVERSITY, SEOUL 151-747, KOREA
PARKKSN@SNU.AC.KR

Abstract. Since S. T. Yau proves the Calabi-Yau theorem, there are only examples of Calabi-Yau metrics on the noncompact manifolds but no one on the compact manifolds. In this paper, we give some examples of Calabi-Yau metrics on a compact manifold.

AMS Classification: 53C26
Keywords: Holomorphic symplectic form, Hyperkähler metric, K3 surface.

1. Introduction

A 2-dimensional complex manifold S is said to be $K3$ if S is simply connected with its first Chern class $c_1 = 0$. A Riemannian manifold (M, g) is *hyperkähler* if there exist three complex structures I, J, and K on M such that $a)$ $I \circ J = -J \circ I = K$ and $b)$ I, J, and K are g-parallel. We say that a closed holomorphic 2-form Ω is *holomorphic symplectic* on the complex manifold (M, I) if it is everywhere nondegenerate. For any $K3$ surface S, it is well-known that $\dim H^{2,0}(S) = 1$ and the cohomology group $H^{2,0}(S)$ is generated by a holomorphic symplectic form. Let $x_i, 0 \leq i \leq 3$, be homogeneous coordinates of the complex projective space $\mathbb{P}^3$. Let X be the zero locus of a homogeneous polynomial $x_0^4 + x_1^4 + x_2^4 + x_3^4$ in $\mathbb{P}^3$. Then X

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 9, NUMBER 2 (2011) 185-200.

is called *Fermat surface*. Throughout this paper, we will denote by X the Fermat surface. It is also well-known that the Fermat surface X is $K3$ and every $K3$ surface is hyperkählerian (recall that a manifold is *hyperkählerian* if it admits a hyperkähler metric) ([2], [5], [7]).

Theorem 1.1 (Calabi-Yau Theorem). *Let M be a compact Kähler manifold, ω its Kähler form, $c_1(M)$ the real first Chern class of M. Any closed real 2-form of type $(1,1)$ belonging to $2\pi c_1(M)$ is the Ricci form of one and only one Kähler metric in the Kähler class of ω ([2], [9]).* $\square$

For the noncompact manifolds there are many examples of Calabi-Yau metrics (i.e. their holonomy groups are of type $\mathrm{SU}(n)$ [2]). But in the compact cases the author believes that there are no examples of them. So, since $\mathrm{Sp}(1) = \mathrm{SU}(2)$, this work gives us a chance to see some examples of Calabi-Yau metrics on the compact manifold.

Consider an open covering $\{\widetilde{U}_i\}_{i=0}^3$ of the Fermat surface X, where $\widetilde{U}_i := X \cap U_i$ and $U_i := \{x \in \mathbb{P}^3 \mid x = (x_0 : x_1 : x_2 : x_3), x_i \neq 0\}$ for $0 \leq i \leq 3$. Throughout this paper, we will use these notations.

Let $z_i := \dfrac{x_i}{x_0}$, $1 \leq i \leq 3$. Then $z_i, 1 \leq i \leq 3$, form natural Euclidean holomorphic coordinates on $U_0 (\cong \mathbb{C}^3)$. Consider the real coordinates $u_1, \cdots, u_6$ on U_0 such that $z_1 = u_5 + \sqrt{-1}u_6$, $z_2 = u_1 + \sqrt{-1}u_2$, and $z_3 = u_3 + \sqrt{-1}u_4$. We obtain that for each point $z = (z_1, z_2, z_3) \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$,

$$T_z X = \mathbb{R} < Z_1, Z_2, Z_3, Z_4 >$$

with

$$\begin{cases} Z_1 = a\dfrac{\partial}{\partial u_5} + b\dfrac{\partial}{\partial u_6} + \dfrac{\partial}{\partial u_1} \\ Z_2 = -b\dfrac{\partial}{\partial u_5} + a\dfrac{\partial}{\partial u_6} + \dfrac{\partial}{\partial u_2} \\ Z_3 = c\dfrac{\partial}{\partial u_5} + d\dfrac{\partial}{\partial u_6} + \dfrac{\partial}{\partial u_3} \\ Z_4 = -d\dfrac{\partial}{\partial u_5} + c\dfrac{\partial}{\partial u_6} + \dfrac{\partial}{\partial u_4}, \end{cases}$$

where $-\dfrac{z_2^3}{z_1^3} = a + \sqrt{-1}b$ and $-\dfrac{z_3^3}{z_1^3} = c + \sqrt{-1}d$ for some real values $a, b, c, d \in \mathbb{R}$ (See (3.6)). Then we have

Theorem 1.2 (Main Theorem)**.** *Let X be the Fermat surface in $\mathbb{P}^3$. Then any hyperkähler metrics g on the complex manifold X can be locally written down as follows.*

For each $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0 \subset X$, $z = (z_1, z_2, z_3)$, let $g_{ij} := g(Z_i, Z_j)$ for $1 \le i, j \le 4$. Let $S := \dfrac{|z_1|^6 + |z_2|^6 + |z_3|^6}{|z_1|^6}$ and A, B are the real and the imaginary parts of $\dfrac{z_1^3}{|z_1|^6}$, respectively. Then we have

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 & g_{13} & g_{14} \\ 0 & g_{11} & -g_{14} & g_{13} \\ g_{13} & -g_{14} & g_{33} & 0 \\ g_{14} & g_{13} & 0 & g_{33} \end{pmatrix}$$

with

$$\begin{cases} g_{11} = \lambda S(A^2 + B^2) f \\ g_{13} = -\lambda S(A f_1 + B f_2) \\ g_{14} = \lambda S(A f_2 - B f_1) \\ g_{33} = -\lambda S(A^2 + B^2) h \end{cases}$$

for some positive constant λ and some real-valued functions $f_1, f_2, f,$ and h on $\widetilde{U}_0 \cap \widetilde{U}_1$ satisfying the following conditions:

$$f_1^2 + f_2^2 + (A^2 + B^2) f h = -1, \quad f > 0, \text{ and } h < 0. \quad \Box$$

Remark 1.1. (1) Note that $\widetilde{U}_i \cap \widetilde{U}_j$ is an open dense subset of X for $0 \le i < j \le 3$.

(2) We easily get

$$\det(g_{ij}) = \frac{S^4}{|z_1|^{12}} = \frac{(|z_1|^6 + |z_2|^6 + |z_3|^6)^4}{|z_1|^{36}} \quad \text{on } \widetilde{U}_0 \cap \widetilde{U}_1.$$

(3) For any subsets $\widetilde{U}_i \cap \widetilde{U}_j$, $0 \le i \ne j \le 3$, we can also have the similar statement like the Main Theorem.

(4) At page 11, we can see the corresponding expression of the Kähler metric h, which is induced from both the metric g and the given complex structure I on X.

The paper is organized as follows. Section 2 has the explicit construction of a holomorphic symplectic form on the Fermat surface X. We need this work, because the key idea of the paper lies in the relation (3.1). In Section 3 we study the tangent spaces of X to be able to prove our Main Theorem. Finally we conclude with some remarks.

2. A HOLOMORPHIC SYMPLECTIC FORM ON X

In order to construct a holomorphic symplectic form on X, first of all, we will construct a holomorphic symplectic form on each open subsets $\widetilde{U}_i$, $0 \leq i \leq 3$. And then we will show that they are compatible on each subsets $\widetilde{U}_i \cap \widetilde{U}_j$, $0 \leq i, j \leq 3$. Denote by $Z(P)$ the zero locus of a polynomial P in $\mathbb{C}^3$.

Fact. For $\widetilde{U}_0$, we know

$$\widetilde{U}_0 = U_0 \cap X = Z(1 + z_1^4 + z_2^4 + z_3^4) \subset \mathbb{C}^3 \subset \mathbb{P}^3,$$

where $z_i := \dfrac{x_i}{x_0}$, $1 \leq i \leq 3$ and z_1, z_2, z_3 are Euclidean holomorphic coordinates of $\mathbb{C}^3$. So,

$$1 + z_1^4 + z_2^4 + z_3^4 = 0 \quad \text{on } \widetilde{U}_0. \tag{2.0.1}$$

Applying the exterior differential d to (2.0.1), we obtain

$$z_1^3 dz_1 + z_2^3 dz_2 + z_3^3 dz_3 = 0 \quad \text{on } \widetilde{U}_0. \tag{2.0.2}$$

From (2.0.2),

$$z_1^3 dz_1 \wedge dz_2 = z_3^3 dz_2 \wedge dz_3 \text{ and } z_1^3 dz_1 \wedge dz_3 = -z_2^3 dz_2 \wedge dz_3 \quad \text{on } \widetilde{U}_0. \tag{2.0.3}$$

Define a 2-form Ω_0 on $\widetilde{U}_0$ as follows:

$$\begin{aligned} \Omega_0|_{V_1} &= \frac{dz_2 \wedge dz_3}{z_1^3}, \\ \Omega_0|_{V_2} &= \frac{dz_3 \wedge dz_1}{z_2^3}, \\ \Omega_0|_{V_3} &= \frac{dz_1 \wedge dz_2}{z_3^3}, \end{aligned}$$

where $V_i := \widetilde{U}_0 \cap \widetilde{U}_i$, $1 \leq i \leq 3$. Here $\Omega_0|_{V_i}$ means the restriction of the 2-form Ω_0 to the open subset V_i, $1 \leq i \leq 3$.

By (2.0.3) and $(0,0,0) \notin \widetilde{U}_0$, Ω_0 is well-defined on $\widetilde{U}_0$. Now, we claim that Ω_0 is closed in $\widetilde{U}_0$. Since $\Omega_0|_{V_1} = (dz_2 \wedge dz_3)/z_1^3$, we have

$$\begin{aligned} d(\Omega_0|_{V_1}) &= ((-3/z_1^4)dz_1) \wedge dz_2 \wedge dz_3 \\ &= (-3/z_1^4)((z_3^3/z_1^3)dz_2 \wedge dz_3) \wedge dz_3 = 0, \quad \text{by (2.0.3)}. \end{aligned}$$

Similarly,

$$d(\Omega_0|_{V_2}) = d(\Omega_0|_{V_3}) = 0.$$

We get the result. Therefore, Ω_0 is a holomorphic symplectic form on $\widetilde{U}_0$. $\square$

In a similar way, we can also obtain a holomorphic symplectic form Ω_i on $\widetilde{U}_i$, $1 \leq i \leq 3$, as follows:

For $\widetilde{U}_1 = Z(y_0^4 + 1 + y_2^4 + y_3^4) \subset \mathbb{C}^3 \subset \mathbb{P}^3$, define a form Ω_1 on $\widetilde{U}_1$ as follows:

$$\begin{aligned} \Omega_1|_{A_0} &= \frac{dy_3 \wedge dy_2}{y_0^3}, \\ \Omega_1|_{A_2} &= \frac{dy_0 \wedge dy_3}{y_2^3}, \\ \Omega_1|_{A_3} &= \frac{dy_2 \wedge dy_0}{y_3^3}, \end{aligned}$$

where $A_i := \widetilde{U}_1 \cap \widetilde{U}_i$ and $y_i := \dfrac{x_i}{x_1}$ for $i = 0, 2, 3$.

For $\widetilde{U}_2 = Z(w_0^4 + w_1^4 + 1 + w_3^4) \subset \mathbb{C}^3 \subset \mathbb{P}^3$, define a form Ω_2 on $\widetilde{U}_2$ as follows:

$$\begin{aligned} \Omega_2|_{B_0} &= \frac{dw_1 \wedge dw_3}{w_0^3}, \\ \Omega_2|_{B_1} &= \frac{dw_3 \wedge dw_0}{w_1^3}, \\ \Omega_2|_{B_3} &= \frac{dw_0 \wedge dw_1}{w_3^3}, \end{aligned}$$

where $B_i := \widetilde{U}_2 \cap \widetilde{U}_i$ and $w_i := \dfrac{x_i}{x_2}$ for $i = 0, 1, 3$.

For $\widetilde{U}_3 = Z(v_0^4 + v_1^4 + v_2^4 + 1) \subset \mathbb{C}^3 \subset \mathbb{P}^3$, define a form Ω_3 on $\widetilde{U}_3$ as follows:

$$\Omega_3|_{C_0} = \frac{dv_2 \wedge dv_1}{v_0^3},$$
$$\Omega_3|_{C_1} = \frac{dv_0 \wedge dv_2}{v_1^3},$$
$$\Omega_3|_{C_2} = \frac{dv_1 \wedge dv_0}{v_2^3},$$

where $C_i := \widetilde{U}_3 \cap \widetilde{U}_i$ and $v_i := \dfrac{x_i}{x_3}$ for $i = 0, 1, 2$.

Now, we constructed holomorphic symplectic forms Ω_i on the open subsets $\widetilde{U}_i$, $0 \leq i \leq 3$. Define a form Ω on X such that the restriction of Ω to the open subset $\widetilde{U}_i$ is equal to the form Ω_i, i.e.,

$$\Omega|_{\widetilde{U}_i} = \Omega_i, \qquad 0 \leq i \leq 3.$$

Then we have

Theorem 2.1. *Let X be the Fermat surface in $\mathbb{P}^3$. Then Ω is a well-defined holomorphic symplectic form on X.*

Proof. By the definition of Ω, it suffices to show that Ω is well-defined on X. So we have to check 6 cases:

1. $\Omega_0 = \Omega_1$ on $\widetilde{U}_0 \cap \widetilde{U}_1$
2. $\Omega_0 = \Omega_2$ on $\widetilde{U}_0 \cap \widetilde{U}_2$
3. $\Omega_0 = \Omega_3$ on $\widetilde{U}_0 \cap \widetilde{U}_3$
4. $\Omega_1 = \Omega_2$ on $\widetilde{U}_1 \cap \widetilde{U}_2$
5. $\Omega_1 = \Omega_3$ on $\widetilde{U}_1 \cap \widetilde{U}_3$
6. $\Omega_2 = \Omega_3$ on $\widetilde{U}_2 \cap \widetilde{U}_3$.

Case 1. $\Omega_0 = \Omega_1$ on $\widetilde{U}_0 \cap \widetilde{U}_1$.

We know

$$\begin{cases} y_0 = \dfrac{x_0}{x_1} = \dfrac{1}{z_1} \\ y_2 = \dfrac{x_2}{x_1} = \dfrac{z_2}{z_1} \\ y_3 = \dfrac{x_3}{x_1} = \dfrac{z_3}{z_1} \end{cases} \quad \text{and} \quad \begin{cases} dy_2 = \dfrac{z_1 dz_2 - z_2 dz_1}{z_1^2} \\ dy_3 = \dfrac{z_1 dz_3 - z_3 dz_1}{z_1^2}. \end{cases}$$

Thus,

$$\begin{aligned}\Omega_1|_{\widetilde{U}_0\cap\widetilde{U}_1} &= \frac{z_1^4 dz_3\wedge dz_2 - z_1^3 z_3 dz_1\wedge dz_2 - z_1^3 z_2 dz_3\wedge dz_1}{z_1^3}\\ &= \frac{-z_1^4 dz_2\wedge dz_3 - z_3^4 dz_2\wedge dz_3 - z_2^3 dz_2\wedge dz_3}{z_1^3} \quad \text{(by (2.0.3))}\\ &= \Omega_0|_{\widetilde{U}_0\cap\widetilde{U}_1} \quad \text{(by (2.0.1))}.\end{aligned}$$

As the same way, we can also check the other cases. Therefore, Ω is well-defined on X. □

Remark 2.1. 1. We have constructed a holomorphic symplectic form Ω on the Fermat surface X, which is defined by

$$\begin{cases}\Omega|_{\widetilde{U}_0} = \dfrac{dz_2\wedge dz_3}{z_1^3} = \dfrac{dz_3\wedge dz_1}{z_2^3} = \dfrac{dz_1\wedge dz_2}{z_3^3} & \text{on } \widetilde{U}_1\cap\widetilde{U}_2\cap\widetilde{U}_3,\ z_i = \dfrac{x_i}{x_0}, i=1,2,3\\ \Omega|_{\widetilde{U}_1} = \dfrac{dy_3\wedge dy_2}{y_0^3} = \dfrac{dy_0\wedge dy_3}{y_2^3} = \dfrac{dy_2\wedge dy_0}{y_3^3} & \text{on } \widetilde{U}_0\cap\widetilde{U}_2\cap\widetilde{U}_3,\ y_i = \dfrac{x_i}{x_1}, i=0,2,3\\ \Omega|_{\widetilde{U}_2} = \dfrac{dw_1\wedge dw_3}{w_0^3} = \dfrac{dw_3\wedge dw_0}{w_1^3} = \dfrac{dw_0\wedge dw_1}{w_3^3} & \text{on } \widetilde{U}_0\cap\widetilde{U}_1\cap\widetilde{U}_3,\ w_i = \dfrac{x_i}{x_2}, i=0,1,3\\ \Omega|_{\widetilde{U}_3} = \dfrac{dv_2\wedge dv_1}{v_0^3} = \dfrac{dv_0\wedge dv_2}{v_1^3} = \dfrac{dv_1\wedge dv_0}{v_2^3} & \text{on } \widetilde{U}_0\cap\widetilde{U}_1\cap\widetilde{U}_2,\ v_i = \dfrac{x_i}{x_3}, i=0,1,2.\end{cases}$$

2. As we know, there are a lot of algebraic submanifolds M of the n-dimensional complex projective manifold $\mathbb{P}^n$ so that we may construct explicitly holomorphic k-forms on M for $0\le k\le n$. But the author believes that only a few of such examples are known only in some easy cases. So, the above explicit holomorphic 2-form Ω is meaningful to us.

3. Proof of the Main Theorem

Remind that we know both the complex structure I and the holomorphic symplectic form Ω on X. Given a hyperkähler metric g on the complex manifold (X, I), by the definition of g, we can choose a hyperkähler structure (g, I, J, K) on X. i.e., g is Kähler with respect to any complex structures $R\in\{I,J,K\}$ and $I\circ J = -J\circ I = K$. Consider the Kähler forms $\omega_R := g(R\ ,\ )$ for $R\in\{I,J,K\}$. Then it is easy to see that

$$\omega := \omega_J + \sqrt{-1}\omega_K$$

is a holomorphic symplectic form on (X, I). By the property $\dim H^{2,0}(X, I) = 1$, there exists a nonzero constant $c \in \mathbb{C}$ such that

$$\Omega = c \cdot \omega.$$

Denote by c_1 and c_2 the real and the imaginary parts of c, respectively. Consider another hyperkähler structure $(\widetilde{g}, I, \widetilde{J}, \widetilde{K})$ on X such that

$$\begin{aligned}
\widetilde{g} &= |c| \cdot g, \\
\widetilde{J} &= \frac{1}{|c|}(c_1 J - c_2 K), \\
\widetilde{K} &= \frac{1}{|c|}(c_2 J + c_1 K),
\end{aligned}$$

where $|c| = \sqrt{c_1^2 + c_2^2}$. Then with easy calculation, we have

$$\Omega = \widetilde{g}(\widetilde{J}\ ,\ ) + \sqrt{-1}\widetilde{g}(\widetilde{K}\ ,\ ). \tag{3.1}$$

For the convenience, we may assume $c = 1$. In order to prove the Main Theorem, we have to use the relation (3.1). It is the key point of the proof of the main theorem.

With the notations of Section 2, on the open subset $\widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$, we have

$$1 + z_1^4 + z_2^4 + z_3^4 = 0 \tag{3.2}$$

$$z_1^3 dz_1 + z_2^3 dz_2 + z_3^3 dz_3 = 0 \tag{3.3}$$

for $z = (z_1, z_2, z_3) \in \widetilde{U}_0 \cap \widetilde{U}_1$. Let the expression "$\widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$" denote that we intend to handle the subset $\widetilde{U}_0 \cap \widetilde{U}_1$ with the coordinates which come from the Euclidean space $U_0 \cong \mathbb{C}^3$.

Consider the real coordinates $u_1, \cdots, u_6$ on U_0 such that $z_1 = u_5 + \sqrt{-1}u_6$, $z_2 = u_1 + \sqrt{-1}u_2$, and $z_3 = u_3 + \sqrt{-1}u_4$. Keep in mind these notations. We will use them throughout this paper. Conveniently, denote by (b_1, b_2, b_3) the holomorphic vector $b_1\dfrac{\partial}{\partial z_1} + b_2\dfrac{\partial}{\partial z_2} + b_3\dfrac{\partial}{\partial z_3}$.

Proposition 3.1. *Let T_zX be the (real) tangent space of X at z. Then, for any $z = (z_1, z_2, z_3) \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$, we have*

$$\begin{aligned} T_zX = \mathbb{R} <& a\frac{\partial}{\partial u_5} + b\frac{\partial}{\partial u_6} + \frac{\partial}{\partial u_1},\ -b\frac{\partial}{\partial u_5} + a\frac{\partial}{\partial u_6} + \frac{\partial}{\partial u_2}, \\ & c\frac{\partial}{\partial u_5} + d\frac{\partial}{\partial u_6} + \frac{\partial}{\partial u_3},\ -d\frac{\partial}{\partial u_5} + c\frac{\partial}{\partial u_6} + \frac{\partial}{\partial u_4} >, \end{aligned}$$

where $-\frac{z_2^3}{z_1^3} = a + \sqrt{-1}b$ and $-\frac{z_3^3}{z_1^3} = c + \sqrt{-1}d$ for $a, b, c, d \in \mathbb{R}$.

Proof. Let T'_zX and $T'_z\mathbb{P}^3$ be the holomorphic tangent spaces of X at z and of $\mathbb{P}^3$ at z, respectively, for $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$. Since T'_zX is the codimension 1 subspace of $T'_z\mathbb{P}^3$, by (3.3), we have

$$T'_zX = \{a_1\frac{\partial}{\partial z_1} + a_2\frac{\partial}{\partial z_2} + a_3\frac{\partial}{\partial z_3} \in T'_z\mathbb{P}^3 \mid a_1z_1^3 + a_2z_2^3 + a_3z_3^3 = 0 \text{ and } a_1, a_2, a_3 \in \mathbb{C}\}.$$

By the elementary calculation,

$$\begin{aligned} T'_zX &= \{(a_1, a_2, a_3) \in T'_z\mathbb{P}^3 \mid a_1 = -\frac{z_2^3}{z_1^3}a_2 - \frac{z_3^3}{z_1^3}a_3 \text{ and } a_1, a_2, a_3 \in \mathbb{C}\} \\ &= \{a_2(-\frac{z_2^3}{z_1^3}, 1, 0) + a_3(-\frac{z_3^3}{z_1^3}, 0, 1) \mid a_2, a_3 \in \mathbb{C}\} \\ &= \mathbb{C} < V_1, V_2 >, \end{aligned}$$

where $V_1 := (-\frac{z_2^3}{z_1^3}, 1, 0)$ and $V_2 := (-\frac{z_3^3}{z_1^3}, 0, 1)$.

Since $\{V_1, V_2\}$ is a basis of T'_zX, we know that $\{V_1 + \bar{V}_1, \sqrt{-1}(V_1 - \bar{V}_1), V_2 + \bar{V}_2, \sqrt{-1}(V_2 - \bar{V}_2)\}$ is a basis of T_zX, where $\bar{V}_i$ is the complex conjugate of V_i for $i = 1, 2$. Therefore, the result follows. $\square$

Let $V_3 := (1, \overline{(\frac{z_2^3}{z_1^3})}, \overline{(\frac{z_3^3}{z_1^3})})$. With the above notations,

Proposition 3.2. *The set $\{V_1, V_2, V_3\}$ is a basis of the space $T'_z\mathbb{P}^3$ for $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$.*

Proof. Obviously, we know

$$T'_z\mathbb{P}^3 = \mathbb{C} < \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3} > \quad \text{for } z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0.$$

Thus it is sufficient to show that $\det A \neq 0$, where

$$A := (V_1, V_2, V_3) = \begin{pmatrix} -\frac{z_2^3}{z_1^3} & -\frac{z_3^3}{z_1^3} & 1 \\ 1 & 0 & \overline{(\frac{z_2^3}{z_1^3})} \\ 0 & 1 & \overline{(\frac{z_3^3}{z_1^3})} \end{pmatrix}.$$

But $\det A = 1 + |\frac{z_2}{z_1}|^6 + |\frac{z_3}{z_1}|^6 > 0$ for $z \in \widetilde{U}_0 \cap \widetilde{U}_1$, finished. $\square$

Now, we obtained

$$T'_z X = \mathbb{C} < V_1, V_2 > \text{ and } T'_z \mathbb{P}^3 = \mathbb{C} < V_1, V_2, V_3 > \quad \text{for } z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0.$$

Denote by $(T'_z X)^*$ and $(T'_z \mathbb{P}^3)^*$ the dual spaces of $T'_z X$ and $T'_z \mathbb{P}^3$, respectively.

Proposition 3.3. *For any $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$, we have*

$$(T'_z X)^* = \mathbb{C} < w_1, w_2 > \text{ and } (T'_z \mathbb{P}^3)^* = \mathbb{C} < w_1, w_2, w_3 >,$$

where

$$\begin{cases} w_1 = -\overline{(\frac{z_2^3}{z_1^3})}\alpha dz_1 + (1 - |\frac{z_2^3}{z_1^3}|^2 \alpha) dz_2 - \overline{(\frac{z_2^3}{z_1^3})}(\frac{z_3^3}{z_1^3})\alpha dz_3 \\ w_2 = -\overline{(\frac{z_3^3}{z_1^3})}\alpha dz_1 - (\frac{z_2^3}{z_1^3})\overline{(\frac{z_3^3}{z_1^3})}\alpha dz_2 + (1 - |\frac{z_3^3}{z_1^3}|^2 \alpha) dz_3 \\ w_3 = \alpha dz_1 + \frac{z_2^3}{z_1^3}\alpha dz_2 + \frac{z_3^3}{z_1^3}\alpha dz_3 \end{cases}$$

with $\alpha = 1/(1 + |\frac{z_2^3}{z_1^3}|^2 + |\frac{z_3^3}{z_1^3}|^2)$.

Proof. By the duality relations: $w_i(V_j) = \delta_i^j$, where δ_i^j is the Kronecker delta for $1 \leq i, j \leq 3$, the result follows. $\square$

Proposition 3.4. *For $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$, $z = (z_1, z_2, z_3)$, we have*

$$\bar{z}_1^3 \frac{\partial}{\partial z_1} + \bar{z}_2^3 \frac{\partial}{\partial z_2} + \bar{z}_3^3 \frac{\partial}{\partial z_3} = 0 \tag{3.4}$$

on X

Proof. By Proposition 3.3, for $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$, $z = (z_1, z_2, z_3)$, we have

$$\begin{aligned}(T'_zX)^* &= \{mw_1 + nw_2 \mid m, n \in \mathbb{C}\}\\ &= \{a_1dz_1 + a_2dz_2 + a_3dz_3 \mid a_1 = -m\overline{(\frac{z_2^3}{z_1^3})}\alpha - n\overline{(\frac{z_3^3}{z_1^3})}\alpha, a_2 = m + \frac{z_2^3}{z_1^3}a_1,\\ &\quad a_3 = n + \frac{z_3^3}{z_1^3}a_1, \alpha = 1/(1 + |\frac{z_2^3}{z_1^3}|^2 + |\frac{z_3^3}{z_1^3}|^2), \text{ and } m, n \in \mathbb{C}\}.\end{aligned}$$

But

$$\begin{aligned}&a_1 = -m\overline{(\frac{z_2^3}{z_1^3})}\alpha - n\overline{(\frac{z_3^3}{z_1^3})}\alpha,\ a_2 = m + \frac{z_2^3}{z_1^3}a_1, \text{ and } a_3 = n + \frac{z_3^3}{z_1^3}a_1\\ \Longleftrightarrow\ &a_1 = -(a_2 - \frac{z_2^3}{z_1^3}a_1)\overline{(\frac{z_2^3}{z_1^3})}\alpha - (a_3 - \frac{z_3^3}{z_1^3}a_1)\overline{(\frac{z_3^3}{z_1^3})}\alpha\\ \Longleftrightarrow\ &a_1 + \overline{(\frac{z_2^3}{z_1^3})}a_2 + \overline{(\frac{z_3^3}{z_1^3})}a_3 = 0 \text{ (since } \alpha = 1/(1 + |\frac{z_2^3}{z_1^3}|^2 + |\frac{z_3^3}{z_1^3}|^2)).\\ \Longleftrightarrow\ &\bar{z}_1^3a_1 + \bar{z}_2^3a_2 + \bar{z}_3^3a_3 = 0 \text{ (since } z_1 \neq 0).\end{aligned}$$

Thus we get

$$(T'_zX)^* = \{a_1dz_1 + a_2dz_2 + a_3dz_3 \mid \bar{z}_1^3a_1 + \bar{z}_2^3a_2 + \bar{z}_3^3a_3 = 0 \text{ and } a_1, a_2, a_3 \in \mathbb{C}\}$$

for $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$. Therefore, the result follows. □

Corollary 3.1. *For any* $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$, $z = (z_1, z_2, z_3)$, *we have*

$$\begin{aligned}\frac{\partial}{\partial u_5} &= a\frac{\partial}{\partial u_1} - b\frac{\partial}{\partial u_2} + c\frac{\partial}{\partial u_3} - d\frac{\partial}{\partial u_4}\\ \frac{\partial}{\partial u_6} &= b\frac{\partial}{\partial u_1} + a\frac{\partial}{\partial u_2} + d\frac{\partial}{\partial u_3} + c\frac{\partial}{\partial u_4}\end{aligned} \tag{3.5}$$

on X, *where* $-\frac{z_2^3}{z_1^3} = a + \sqrt{-1}b$ *and* $-\frac{z_3^3}{z_1^3} = c + \sqrt{-1}d$ *for* $a, b, c, d \in \mathbb{R}$.

Remark 3.1. For $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$, $z = (z_1, z_2, z_3)$, we obtained two equations on X:

$$z_1^3dz_1 + z_2^3dz_2 + z_3^3dz_3 = 0 \quad \text{and} \quad \bar{z}_1^3\frac{\partial}{\partial z_1} + \bar{z}_2^3\frac{\partial}{\partial z_2} + \bar{z}_3^3\frac{\partial}{\partial z_3} = 0.$$

The former gives the holomorphic tangent space T'_zX and the latter determines the holomorphic cotangent space $(T'_zX)^*$.

Proof of the Main Theorem We know that for any $z = (z_1, z_2, z_3) \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$,

$$T_zX = \mathbb{R} < Z_1, Z_2, Z_3, Z_4 >$$

with

$$\begin{cases} Z_1 = a\dfrac{\partial}{\partial u_5} + b\dfrac{\partial}{\partial u_6} + \dfrac{\partial}{\partial u_1} \\ Z_2 = -b\dfrac{\partial}{\partial u_5} + a\dfrac{\partial}{\partial u_6} + \dfrac{\partial}{\partial u_2} \\ Z_3 = c\dfrac{\partial}{\partial u_5} + d\dfrac{\partial}{\partial u_6} + \dfrac{\partial}{\partial u_3} \\ Z_4 = -d\dfrac{\partial}{\partial u_5} + c\dfrac{\partial}{\partial u_6} + \dfrac{\partial}{\partial u_4}, \end{cases} \tag{3.6}$$

where $-\frac{z_2^3}{z_1^3} = a + \sqrt{-1}b$ and $-\frac{z_3^3}{z_1^3} = c + \sqrt{-1}d$ for $a, b, c, d \in \mathbb{R}$ (See Proposition 3.1).

Consider the holomorphic symplectic form Ω on $\widetilde{U}_0 \cap \widetilde{U}_1$ with the relation (3.1):

$$\Omega|_{\widetilde{U}_0 \cap \widetilde{U}_1} = \frac{dz_2 \wedge dz_3}{z_1^3} = g(J\ ,\ ) + \sqrt{-1}g(K\ ,\ ).$$

Dividing the second term of the above equations as the sum of the real part and the imaginary part, we see that

$$g(J\ ,\ ) = \frac{\mathrm{Re}(z_1^3)}{|z_1|^6}(du_1 \wedge du_3 - du_2 \wedge du_4) + \frac{\mathrm{Im}(z_1^3)}{|z_1|^6}(du_1 \wedge du_4 + du_2 \wedge du_3) \tag{3.7}$$

$$g(K\ ,\ ) = \frac{\mathrm{Re}(z_1^3)}{|z_1|^6}(du_1 \wedge du_4 + du_2 \wedge du_3) - \frac{\mathrm{Im}(z_1^3)}{|z_1|^6}(du_1 \wedge du_3 - du_2 \wedge du_4), \tag{3.8}$$

where $\mathrm{Re}(z_1^3)$ and $\mathrm{Im}(z_1^3)$ are the real and the imaginary parts of z_1^3, respectively. Applying (3.5) to (3.6) yields

$$\begin{cases} Z_1 = (a^2 + b^2 + 1)\dfrac{\partial}{\partial u_1} + (ac + bd)\dfrac{\partial}{\partial u_3} + (bc - ad)\dfrac{\partial}{\partial u_4} \\ Z_2 = (a^2 + b^2 + 1)\dfrac{\partial}{\partial u_2} - (bc - ad)\dfrac{\partial}{\partial u_3} + (ac + bd)\dfrac{\partial}{\partial u_4} \\ Z_3 = (ac + bd)\dfrac{\partial}{\partial u_1} - (bc - ad)\dfrac{\partial}{\partial u_2} + (c^2 + d^2 + 1)\dfrac{\partial}{\partial u_3} \\ Z_4 = (bc - ad)\dfrac{\partial}{\partial u_1} + (ac + bd)\dfrac{\partial}{\partial u_2} + (c^2 + d^2 + 1)\dfrac{\partial}{\partial u_4}. \end{cases} \tag{3.9}$$

Let $A := \frac{\mathrm{Re}(z_1^3)}{|z_1|^6}$ and $B := \frac{\mathrm{Im}(z_1^3)}{|z_1|^6}$. Since $J \in \mathrm{End}(TX)$, $J^2 = -1$, and $I \circ J = -J \circ I$, these conditions force that there exist some real-valued functions f_1, f_2, f, h on $\widetilde{U}_0 \cap \widetilde{U}_1$, which satisfy

$$f_1^2 + f_2^2 + (A^2 + B^2)fh = -1, \quad f > 0, \quad h < 0,$$

such that

$$\begin{cases} J(Z_1) = f_1 Z_1 + f_2 Z_2 + AfZ_3 + BfZ_4 \\ J(Z_2) = f_2 Z_1 - f_1 Z_2 + BfZ_3 - AfZ_4 \\ J(Z_3) = AhZ_1 + BhZ_2 + h_3 Z_3 + h_4 Z_4 \\ J(Z_4) = BhZ_1 - AhZ_2 + h_4 Z_3 - h_3 Z_4, \end{cases} \tag{3.10}$$

where

$$\begin{cases} h_3 = \dfrac{-A^2+B^2}{A^2+B^2} f_1 - \dfrac{2AB}{A^2+B^2} f_2 \\ h_4 = \dfrac{-2AB}{A^2+B^2} f_1 + \dfrac{A^2-B^2}{A^2+B^2} f_2. \end{cases}$$

Since g is Hermitian with respect to the complex structures I, J, and K, we must have the following form:

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 & g_{13} & g_{14} \\ 0 & g_{11} & -g_{14} & g_{13} \\ g_{13} & -g_{14} & g_{33} & 0 \\ g_{14} & g_{13} & 0 & g_{33} \end{pmatrix},$$

where $g_{ij} := g(Z_i, Z_j)$ for $1 \le i, j \le 4$. Hence, using (3.7), (3.9), and (3.10), let $S := \dfrac{|z_1|^6 + |z_2|^6 + |z_3|^6}{|z_1|^6}$, by some complicated computation, we obtain that for any $z \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0 \subset X$, $z = (z_1, z_2, z_3)$,

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 & g_{13} & g_{14} \\ 0 & g_{11} & -g_{14} & g_{13} \\ g_{13} & -g_{14} & g_{33} & 0 \\ g_{14} & g_{13} & 0 & g_{33} \end{pmatrix}$$

where

$$\begin{cases} g_{11} = S(A^2+B^2)f \\ g_{13} = -S(Af_1 + Bf_2) \\ g_{14} = S(Af_2 - Bf_1) \\ g_{33} = -S(A^2+B^2)h. \end{cases}$$

This finishes the proof of the Main Theorem. □

Remark 3.2. With the above notations, we get that for any $z = (z_1, z_2, z_3) \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$,

$$T_z X = \mathbb{R} < Z_1', Z_2', Z_3', Z_4' >,$$

where

$$\begin{cases} Z'_1 = Z_1 \\ Z'_2 = I(Z_1) = Z_2 \\ Z'_3 = J(Z_1) = f_1 Z_1 + f_2 Z_2 + A f Z_3 + B f Z_4 \\ Z'_4 = K(Z_1) = -f_2 Z_1 + f_1 Z_2 - B f Z_3 + A f Z_4. \end{cases}$$

Let $g'_{ij} := g(Z'_i, Z'_j)$ for $1 \leq i, j \leq 4$. Then we easily have

$$\begin{pmatrix} g'_{11} & g'_{12} & g'_{13} & g'_{14} \\ g'_{21} & g'_{22} & g'_{23} & g'_{24} \\ g'_{31} & g'_{32} & g'_{33} & g'_{34} \\ g'_{41} & g'_{42} & g'_{43} & g'_{44} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{11} & 0 \\ 0 & 0 & 0 & g_{11} \end{pmatrix}.$$

i.e., the metric g is locally conformally equivalent to the Euclidean metric with respect to the basis $\{Z'_1, Z'_2, Z'_3, Z'_4\}$.

4. Remarks

In this section we will also use the notations of section 2 and section 3. From both the proof of Proposition 3.2 and (3.6), we know

$$\begin{cases} I(Z_1) = Z_2 \\ I(Z_2) = -Z_1 \\ I(Z_3) = Z_4 \\ I(Z_4) = -Z_3 \end{cases}$$

and $T'_z X = \mathbb{C} < V_1, V_2 >$ for $z = (z_1, z_2, z_3) \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$. Let $h_{i\bar{j}} := g(V_i, \bar{V}_j)$ for $1 \leq i, j \leq 2$. Then we have the corresponding complex Hermitian metric h on (X, I) as follows:

$$\begin{cases} h_{1\bar{1}} = \dfrac{1}{2} \dfrac{\lambda S}{|z_1|^6} f \\ h_{1\bar{2}} = \dfrac{1}{2} \dfrac{\lambda S}{|z_1|^6} (-z_1^3 f_1 + \sqrt{-1} z_1^3 f_2) \\ h_{2\bar{1}} = \overline{h_{1\bar{2}}} \\ h_{2\bar{2}} = -\dfrac{1}{2} \dfrac{\lambda S}{|z_1|^6} h. \end{cases}$$

It induces

$$\det(h_{i\bar{j}})(z) = \frac{\lambda^2 (|z_1|^6 + |z_2|^6 + |z_3|^6)^2}{4|z_1|^{18}}.$$

Since g is a hyperkähler metric on (X, I), its Ricci form ρ is zero. But we know

$$\rho = \sqrt{-1} d'' d' \log(\det(h_{i\bar{j}}))$$

and $d'' d' \log(|z_1|^{18}) = 0$ [6]. Therefore, we have the identity

$$d'' d' \log(|z_1|^6 + |z_2|^6 + |z_3|^6) = 0 \quad \text{on } \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0.$$

From Proposition 3.3, on $\widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$ we obtain the Kähler metric

$$h = \frac{1}{2} \frac{\lambda S}{|z_1|^6} (f \omega_1 \otimes \overline{\omega}_1 + (-z_1^3 f_1 + \sqrt{-1} z_1^3 f_2) \omega_1 \otimes \overline{\omega}_2 -$$
$$(\overline{z}_1^3 f_1 + \sqrt{-1} \overline{z}_1^3 f_2) \omega_2 \otimes \overline{\omega}_1 - h \omega_2 \otimes \overline{\omega}_2)$$

and its Kähler form

$$\begin{aligned} \omega_I = g(I\ ,\ ) = & \frac{\sqrt{-1}}{2} \frac{\lambda S}{|z_1|^6} (f \omega_1 \wedge \overline{\omega}_1 + (-z_1^3 f_1 + \sqrt{-1} z_1^3 f_2) \omega_1 \wedge \overline{\omega}_2 \\ & - (\overline{z}_1^3 f_1 + \sqrt{-1} \overline{z}_1^3 f_2) \omega_2 \wedge \overline{\omega}_1 - h \omega_2 \wedge \overline{\omega}_2), \end{aligned}$$

where

$$\begin{cases} w_1 = - \overline{(\frac{z_2^3}{z_1^3})} \alpha dz_1 + (1 - |\frac{z_2^3}{z_1^3}|^2 \alpha) dz_2 - \overline{(\frac{z_2^3}{z_1^3})} (\frac{z_3^3}{z_1^3}) \alpha dz_3 \\ w_2 = - \overline{(\frac{z_3^3}{z_1^3})} \alpha dz_1 - (\frac{z_2^3}{z_1^3}) \overline{(\frac{z_3^3}{z_1^3})} \alpha dz_2 + (1 - |\frac{z_3^3}{z_1^3}|^2 \alpha) dz_3 \\ w_3 = \alpha dz_1 + \frac{z_2^3}{z_1^3} \alpha dz_2 + \frac{z_3^3}{z_1^3} \alpha dz_3 \end{cases}$$

with $\alpha = 1/(1 + |\frac{z_2^3}{z_1^3}|^2 + |\frac{z_3^3}{z_1^3}|^2)$.

Hence, we get the volume form

$$\frac{\omega_I^2}{2!} = \frac{1}{2} \frac{\lambda^2 S^2}{|z_1|^6} \omega_1 \wedge \omega_2 \wedge \overline{\omega}_1 \wedge \overline{\omega}_2.$$

Furthermore, we can obtain the corresponding Hermitian connection D as follows.

Let $Ds = s\omega$, in the matrix notation, where $s = (V_1, V_2)$ and ω is the connection form of D. Then we know [6]

$$\omega^t = d' h \cdot h^{-1}.$$

That is, we have $\omega_j^i = \sum h^{i\bar{k}} d' h_{j\bar{k}}$, where $(h^{i\bar{j}})$ is the inverse transpose matrix of $(h_{i\bar{j}})$. Since

$$h = \frac{\lambda S}{2|z_1|^6} \begin{pmatrix} f & -z_1^3 f_1 + \sqrt{-1} z_1^3 f_2 \\ -\overline{z}_1^3 f_1 - \sqrt{-1} \overline{z}_1^3 f_2 & -h \end{pmatrix}$$

with $S = \dfrac{|z_1|^6 + |z_2|^6 + |z_3|^6}{|z_1|^6}$, we get

$$\omega(Q) = \begin{pmatrix} \omega_{1\bar{1}} & \omega_{1\bar{2}} \\ \omega_{2\bar{1}} & \omega_{2\bar{2}} \end{pmatrix}(Q)$$

at a point $Q := (\frac{1}{\sqrt{2}}(1+\sqrt{-1}), 0, 0) \in \widetilde{U}_0 \cap \widetilde{U}_1 \subset \widetilde{U}_0$, where

$$\begin{cases} \omega_{1\bar{1}} = -hd'f + (f_1 + \sqrt{-1}f_2)(-d'f_1 + \sqrt{-1}d'f_2) + 3\overline{z}_1 fhdz_1 \\ \omega_{1\bar{2}} = z_1^3(f_1 - \sqrt{-1}f_2)d'f + z_1^3 f(-d'f_1 + \sqrt{-1}d'f_2) + 3f(f_1 - \sqrt{-1}f_2)dz_1 \\ \omega_{2\bar{1}} = \overline{z}_1^3 h(d'f_1 + \sqrt{-1}d'f_2) - \overline{z}_1^3(f_1 + \sqrt{-1}f_2)d'h \\ \omega_{2\bar{2}} = -(f_1 - \sqrt{-1}f_2)(d'f_1 + \sqrt{-1}d'f_2) - fd'h - 3\overline{z}_1 dz_1. \end{cases}$$

References

[1] A. Beauville, Variétés Kähleriennes dont la lère classe de Chern est nulle, J. Diff. Geom., Vol 18, 755-782 (1983).

[2] A. Besse, Einstein Manifolds, Springer-Verlag, New York, 1987.

[3] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley-Interscience, New York, 1978.

[4] N.J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, Hyperkähler metrics and Supersymmetry, Commun. Math. Phys., Vol 108, 535-589 (1987).

[5] D. Huybrechts, Compact hyperkähler manifolds: basic results, Invent. Math., Vol 135, 63-113 (1999).

[6] S. Kobayashi, Differential geometry of complex vector bundles, Princeton University Press, Princeton, 1987.

[7] Y.T. Siu, Every K3 surface is kähler, Invent. math., Vol 73, 139-150 (1983).

[8] M. Verbitsky, Hyperholomorphic sheaves and new examples of hyperkähler manifolds, preprint, 1997.

[9] S.T. Yau, On the Ricci curvature of a compact kähler manifold and the complex Monge-Ampère equation I, Com. Pure. and Appl. Math., Vol 31, 339-411 (1978).