

WAVELET ANALYSIS OF SELF SIMILAR FUNCTIONS

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ABSTRACT. The self-similarity property of some kind of fractals is studied by using Harmonic Wavelets. The scale invariance of fractals is compared with the scale dependence of wavelets. Harmonic wavelets are complex values wavelets, with sharp compact support in frequency domain, which show some kind of self-similarity with respect to scale changes. Due to their self similarity property and scale dependence, harmonic wavelets can offer an expedient tool for a good approximation and analysis of a suitable class of deterministic (finite energy) fractals. It is shown that (localized, deterministic) fractals are characterized by their wavelet coefficients. In particular, the Riemann-Weirstrass and Riemann-Cell  rier function can be easily represented in terms of harmonic wavelets, in the sense that their wavelet coefficients can be easily (and analytically) computed. Some equations for generating self-similar functions are eventually given as well.

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1. INTRODUCTION

One of the main properties of fractals is self-similarity or scale invariance, in the sense that some patches replicates themselves at each scale. Analogously, wavelets is a family of functions depending on a scale parameter, in a such way that the transition from one scale to another is achieved by scaled instances of the basic function.

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In both cases, these functions are characterized by the scale of observation. The concept of scale is a characteristic feature of physics, and usually mathematical models which try to describe complex phenomena must be adapted to different scales and specifically defined, for each scale, in a different way. Therefore any physical theory is characterized by a scale and by a model, which holds true only for a given scale. On the other hand, the concept of scale has a direct meaning in the interpretation of complexity. Many complex phenomena, like biological signals, geological deformations, natural phenomena as clouds, tree branches evolution, leaves formation, stocks and financial data, show the scale invariance [1, 2, 11, 12, 13, 21] and/or dependence thereon. From mathematical point of view, self-similarity, or scale invariance, can be detected in some kind of singular (no-where differentiable) functions and mostly in fractals. These functions are mathematically defined through tabular and recursive-analytical formulas both deterministic and stochastic thus being represented by a very large family of functions and they behave as if the function is identical to any of its rescaled instances up to a suitable renormalization of the amplitude.

Nevertheless there is another class of functions, namely wavelets, which became very popular in last year (see e.g. [10, 16, 21]), and are based on the concept of scale and scaling. Wavelets are some special functions [8, 10, 16, 21] which depend on a scale parameter, thus being a suitable candidate for the analysis of scale depending problems. Moreover they depend on a second localization parameter which enable them to be localized (in space or frequency). They form the fundamental tool for building up the so-called multi-resolution (or multi-scale) analysis of energy bounded functions. Any finite energy function can be represented by an infinite series of wavelets whose coefficients describe the projection into a given scale by “local” oscillations.

In the following these two concepts of scale and oscillation, arising in self-similar functions, will be compared with analogous features which characterize wavelets. The aim is to better understand fractals and multifractals by using wavelets sharply localized in frequency. Although there are many publications on fractals only a few attempts were done on fractals and wavelets (see e.g. [5, 7, 11, 13, 14, 15, 21]) so that a satisfactory definition of fractal wavelets has not yet been achieved. In

[14, 15] the authors give a definition of fractal wavelets by using some fractal interpolation functions, thus being dependent on a discretization of the space variable. In the following, instead an analytical definition of fractal wavelet will be proposed which is based on complex value wavelets. In fact, this makes a better analogy with the Weierstrass-Riemann which is based on complex values and harmonic functions (trigonometric). It has been recently observed that also some kind of highly oscillating functions, random-like or fractal-like functions can be easily reconstructed and studied by using wavelets [1, 2, 3, 5, 7, 11, 13, 21]. In particular, the analysis of the self-similarity will be proposed, in the following, by using harmonic complex valued functions, complex Littlewood Paley bases [4, 8, 5, 6, 7, 18, 19]. They not only offer a good tool for studying localized deterministic fractals (as Weierstrass-Riemann, Weierstrass-Cell  rier functions), but can also be used to define a new classes of fractals.

2. SELF-SIMILAR FUNCTIONS

Let $f(x), x \in A \subset \mathbb{R}$ be a scale invariant function with a scaling exponent $H, (0 < H < 1)$ (Hausdorff dimension) and $\mu > 0$ such that it is

$$(1) \quad f(\mu x) = \mu^H f(x) .$$

By assuming

$$(2) \quad \mu = 2^C \Rightarrow C = \log_2 \mu ,$$

we get

$$(3) \quad f(2^C x) = 2^{CH} f(x) .$$

Indeed, for self-similar functions the invariance is satisfied by only few values of μ and the corresponding weakened version of scale invariance is called discrete scale invariance [1, 3]. An example of scale-invariant function is the generalized Riemann-Weierstrass function (Fig. 1)

$$(4) \quad f(x) \stackrel{\text{def}}{=} \frac{1}{\pi} \sum_{k=1}^{\infty} k^{-a} e^{i\pi k^a x} \quad (a > 1) ,$$

which is, in trigonometric form,

$$f(x) \stackrel{\text{def}}{=} \frac{1}{\pi} \sum_{k=1}^{\infty} [k^{-a} (\cos \pi k^a x + i \sin \pi k^a x)] , \quad (a > 1) .$$

In practical computation the upper limit of the series runs up to a finite value $n_{\max} < \infty$. It should be also noticed that the fractal behaviour of this real function with complex value can be much better appreciated in the complex plane $\mathbb{C}^1$ (Fig. 1).

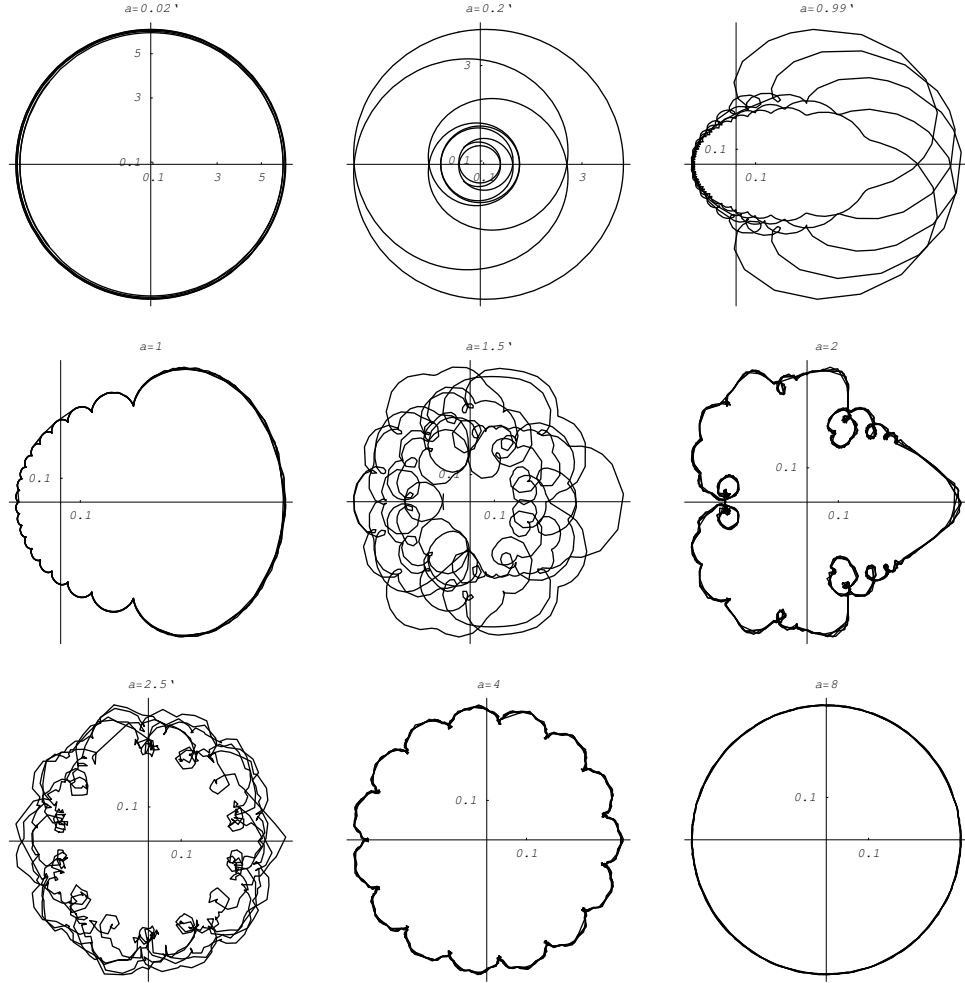


FIGURE 1. The generalized Riemann-Weierstrass function (4) in $\mathbb{C}^1$ for different values of the parameter a , with $n_{\max} = 20$, $x \in [-5, 5]$.

From the real part of Eq. (4) it can be derived the Weierstrass function [20]

$$(5) \quad f(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} a^k \cos 2\pi b^k x, \quad \left(0 < a < 1, b = 2h + 1 > 1, ab > 1 + \frac{3}{2}\pi, h \in \mathbb{N} \right),$$

and the Mandelbrot-Weierstrass function [3]

$$f(x) = \sum_{k=-\infty}^{\infty} a^{-kH} (1 - \cos a^k x), \quad (a > 1, 0 < H < 1).$$

The imaginary part of (4) is the so-called Cell  rier (or Weierstrass-Cell  rier) function [9] (see Fig. 2)

$$(6) \quad f(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} a^{-k} \sin 2\pi a^k x, \quad a > 1.$$

The Mellin function (also known as chirp), is an example of self-similar function which is not a fractal

$$f(x) = x^{H+mi} \quad (|m| \geq 10, H \leq 1).$$

A common feature of the above functions is that they show some high frequency oscillations, thus suggesting us that somehow it should be possible to connect the analysis of fractals with high oscillating functions. Moreover, we have that

- Concerning with the complex Riemann-Weierstrass function, its fractal behavior can be easily and better represented in $\mathbb{C}^1$. Furthermore, by increasing the parameter a which is somehow linked with a scale factor we can pass from a circle (top left of Fig. 1) to fractals (self-similar pictures) which correspond to $a = 2$ and $a = 2.5$, and again to another circle $a = 8$. So that by manipulating the scaling factor we can focus on the fractal nature.
- With respect to the Cell  rier function, its fractal behavior appears when $a > 1$ (see Fig. 2). In particular, by adding more instances of the series, i.e. by increasing $n_{\max}$ one can see (bottom row of Fig. 2) that there is a kind of approximation of the basic instance $\sin 2\pi x$ corresponding to $n_{\max} = 0$ in the series (2).

Together with the scale dependence, and the concept of approximation, in the self-similar functions a fundamental role is played also by the basis of harmonic functions (sinus and cosinus). There are many analogies with wavelet theory where

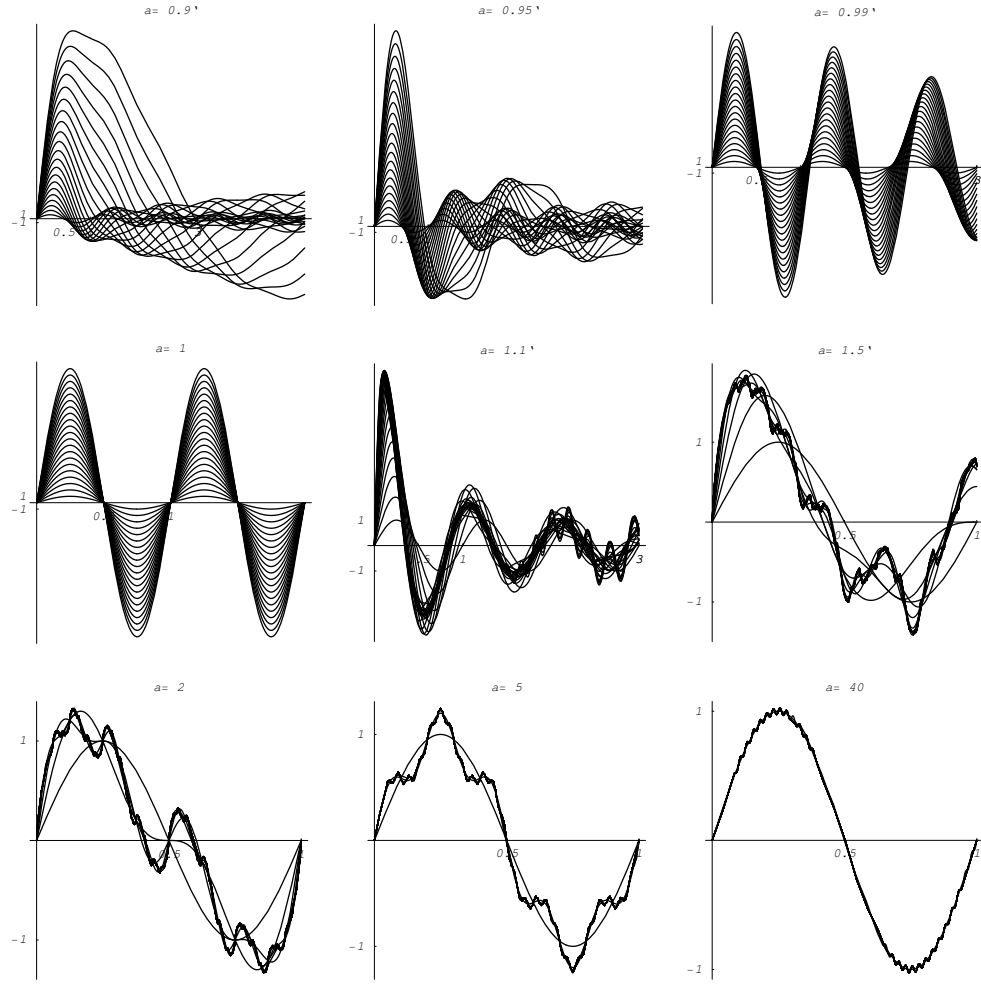


FIGURE 2. The Cellier function (6) in $\mathbb{R}^2$ for different values of the parameter a , with $0 \leq n_{\max} \leq 20$, $x \in [0, 1]$.

a difference is represented by the basis (which are wavelets). However, as we will see, with harmonic wavelets there is a better similarity between fractals and wavelets.

In the following we will restrict our investigation to a family of deterministically self-similar functions $f(x)$, such that $f(x) \in L_2(\mathbb{R}, \mathbb{C})$ with

$$f(x) \in L_2(\mathbb{R}, \mathbb{C}) \text{ if } f(x) : \mathbb{R} \rightarrow \mathbb{C} \text{ and } \left[\int_{-\infty}^{\infty} |f(x)|^2 dx \right]^{1/2} < \infty .$$

In fact, in general, a self-similar function $f(x) : \mathbb{R} \rightarrow \mathbb{C}$ does not belong to $L_2(\mathbb{R}, \mathbb{C})$ [5, 7], in the sense that there not exists a self-similar function which is defined over the entire $\mathbb{R}$ and belongs to $L_2(\mathbb{R}, \mathbb{C})$. To have a bounded energy function we should limit ourselves to a bounded interval where the self-similar function is $f(x) : [a, b] \subset \mathbb{R} \rightarrow \mathbb{C}$ (see also [11]).

For example, starting from a slightly modified version of the Weierstrass function (5)

$$f(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \lambda^{nH} \cos \lambda^n x \quad , (\lambda > 1, 0 \leq H \leq 1) ,$$

as fractal function, it could be considered the following modification (Fig. 3)

$$(7) \quad f(x) \stackrel{\text{def}}{=} \begin{cases} \sum_{n=-\infty}^{\infty} \lambda^{nH} \cos \lambda^n x , & x \in [a, b] \\ 0 , & x \notin [a, b] \end{cases} , \quad (\lambda > 1, 0 \leq H \leq 1) .$$

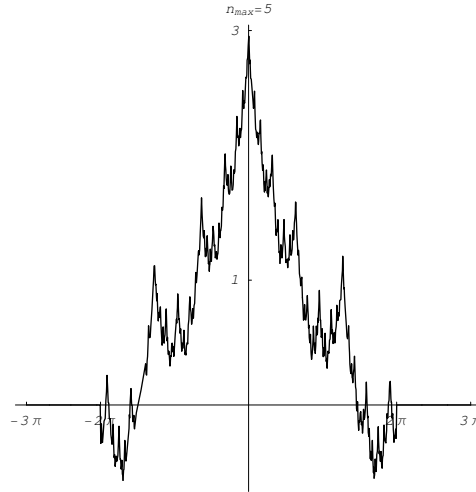


FIGURE 3. The bounded self-similar function (7), with $\lambda = 2$, $H = 1/2$, $|n_{\max}| = 5$.

Indeed, this function replicates itself (Fig. 3) with many peaks (oscillations) and it vanishes outside a bounded interval.

3. HARMONIC WAVELET BASIS

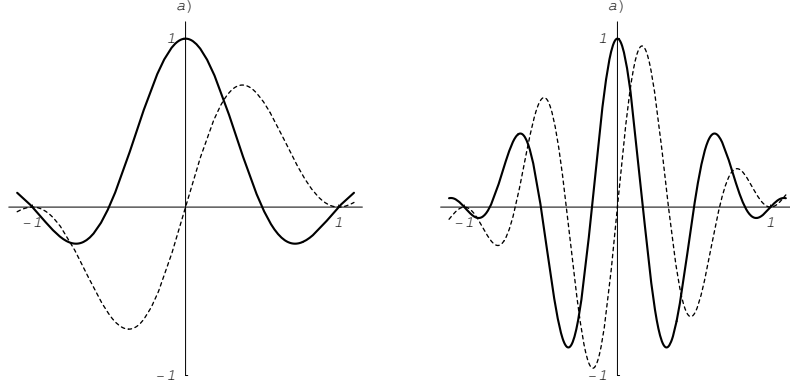


FIGURE 4. Fundamental instances of the scaling and harmonic wavelets: a) the real part $\Re[\varphi_0^0(x)]$ (plain) and dashed the imaginary part $\Im[\varphi_0^0(x)]$; b) $\Re[\psi_0^0(x)]$ (plain) and dashed $\Im[\psi_0^0(x)]$.

The dilated and translated instances of the scaling $\varphi(x) = \varphi_0^0(x)$ and harmonic wavelet $\psi(x) = \psi_0^0(x)$ functions are [4, 5, 6, 7, 8, 18, 19]

$$(8) \quad \begin{cases} \varphi_k^n(x) \stackrel{\text{def}}{=} 2^{n/2} \frac{e^{2\pi i(2^n x - k)} - 1}{2\pi i(2^n x - k)}, \\ \psi_k^n(x) \stackrel{\text{def}}{=} 2^{n/2} \frac{e^{4\pi i(2^n x - k)} - e^{2\pi i(2^n x - k)}}{2\pi i(2^n x - k)}, \end{cases}$$

with (see Figs. 4,5)

$$\varphi_k^n(x) : \mathbb{R} \rightarrow \mathbb{C} \quad , \quad \psi_k^n(x) : \mathbb{R} \rightarrow \mathbb{C} \quad (n \in \mathbb{N}, k \in \mathbb{Z}).$$

It should be noticed that, in the complex plane $\mathbb{C}^1$ these functions show some oscillations (Fig. 5) with basins of attraction.

3.1. Properties of the Harmonic wavelets. Functions (8) are not scale invariant, however, comparing with the definition (1), it is

$$(9) \quad \varphi_k^n(\mu x) = \mu^{-\mu} \varphi_k^{n-1}(x) \quad , \quad \psi_k^n(\mu x) = \mu^{-\mu} \psi_k^{n-1}(x)$$

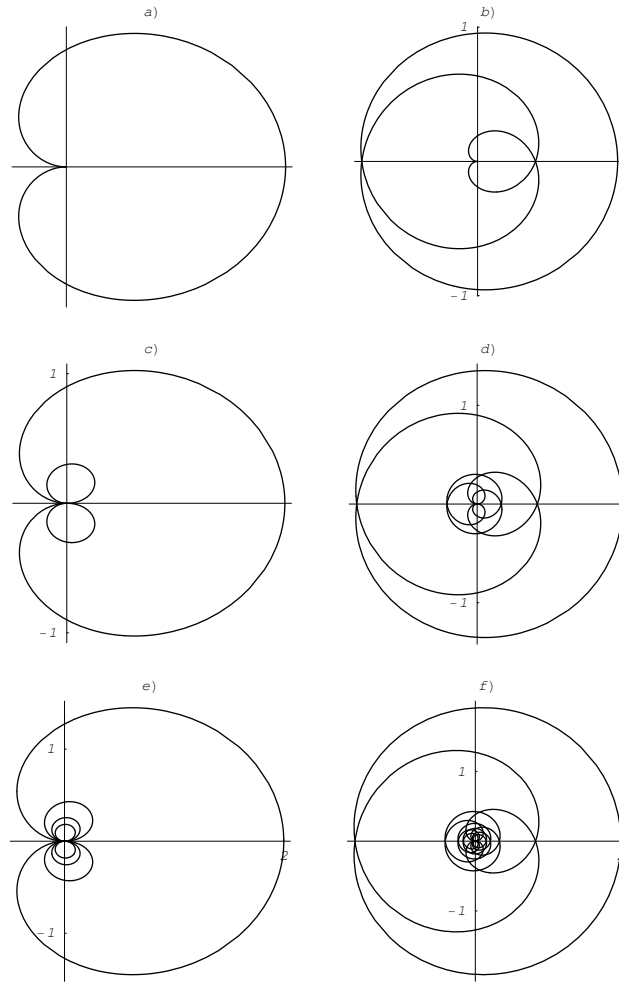


FIGURE 5. Orbits in $\mathbb{C}^1$ of the scaling (left column) and harmonic wavelets: a) $\varphi_0^0(x)$; b) $\psi_0^0(x)$; c) $\varphi_0^1(x)$; d) $\psi_0^1(x)$; e) $\varphi_0^2(x)$; f) $\psi_0^2(x)$.

with $\mu = 2^{-1}$ and, when $n = 0, k = 0$ it is assumed $\mu = 1$. By iterating,

$$(10) \quad \varphi_k^n(\mu^n x) = \mu^{-n\mu} \varphi_k^0(x) \quad , \quad \psi_k^n(\mu^n x) = \mu^{-n\mu} \psi_k^0(x) \quad .$$

Analogously, by taking into account the definition (3) it is also

$$(11) \quad \varphi_k^n(2^C x) = 2^{-C/2} \varphi_k^{n+C}(x) \quad , \quad \psi_k^n(2^C x) = 2^{-C/2} \psi_k^{n+C}(x) \quad .$$

By using the Fourier transform:

$$\widehat{\varphi}(\omega) = \mathcal{F}[\varphi(x)] \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x) e^{-i\omega x} dx$$

we get the wavelet basis (8) in the Fourier domain:

$$(12) \quad \begin{cases} \widehat{\varphi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi(2\pi + \omega/2^n) \\ \widehat{\psi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi(\omega/2^n) \end{cases}$$

where $\chi(\omega)$ is the box (characteristic) function

$$(13) \quad \chi(\omega) \stackrel{\text{def}}{=} \begin{cases} 1, & 2\pi \leq \omega \leq 4\pi, \\ 0, & \text{elsewhere} . \end{cases}$$

3.2. Inner product. The space of the $L_2(\mathbb{R}, \mathbb{C})$ functions is an Hilbert space. We can define the inner product, which fulfills the Parseval equality, as follows (the bar stands for complex conjugation):

$$(14) \quad \langle f, g \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = 2\pi \int_{-\infty}^{\infty} \widehat{f}(\omega) \overline{\widehat{g}(\omega)} d\omega = 2\pi \langle \widehat{f}, \widehat{g} \rangle ,$$

for any $f(x) \in L_2(\mathbb{R}, \mathbb{C})$ and $g(x) \in L_2(\mathbb{R}, \mathbb{C})$.

It can be shown [4, 5, 6, 7, 19] that harmonic wavelets are orthonormal functions, in the sense that

$$(15) \quad \langle \psi_k^n(x), \psi_h^n(x) \rangle = \delta^{nm} \delta_{hk} ,$$

where δ^{nm} (δ_{hk}) is the Kronecker symbol.

Analogously it is

$$(16) \quad \left\{ \begin{array}{l} \langle \varphi_k^n(x), \varphi_h^m(x) \rangle = \delta^{nm} \delta_{kh}, \quad \langle \overline{\varphi}_k^n(x), \overline{\varphi}_h^m(x) \rangle = \delta^{nm} \delta_{kh}, \\ \langle \varphi_k^n(x), \overline{\varphi}_h^m(x) \rangle = 0, \\ \langle \overline{\psi}_k^n(x), \overline{\psi}_h^m(x) \rangle = \delta^{nm} \delta_{kh}, \quad \langle \psi_k^n(x), \overline{\psi}_h^m(x) \rangle = 0, \\ \langle \varphi_k^n(x), \overline{\psi}_h^m(x) \rangle = 0, \quad \langle \overline{\varphi}_k^n(x), \psi_h^m(x) \rangle = 0. \end{array} \right.$$

Some simple technical computations also show that the harmonic scaling function and the harmonic wavelets fulfill the multiresolution conditions

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1, \quad \int_{-\infty}^{\infty} \psi_k^n(x) dx = 0.$$

4. HARMONIC WAVELET RECONSTRUCTION

Let W_M^N be the subspace of $L_2(\mathbb{R}, \mathbb{C})$ defined as

$$W_M^N = \text{Span} [\varphi_k^n(x), \overline{\varphi}_k^n(x), \psi_k^n(x), \overline{\psi}_k^n(x)]_{n \in \mathbb{N}, k \in \mathbb{Z}}, \quad (N \leq \infty, M \leq \infty)$$

being

$$\text{Span} [\varphi_k^n(x), \overline{\varphi}_k^n(x), \psi_k^n(x), \overline{\psi}_k^n(x)] \stackrel{\text{def}}{=} \{ \alpha_k^n \varphi_k^n(x) + \alpha_k^{*n} \overline{\varphi}_k^n(x) + \beta_k^n \psi_k^n(x) + \beta_k^{*n} \overline{\psi}_k^n(x) \}$$

with

$$|\alpha_k^n| < \infty, \quad |\alpha_k^{*n}| < \infty, \quad |\beta_k^n| < \infty, \quad |\beta_k^{*n}| < \infty, \quad n \in \mathbb{N}, k \in \mathbb{Z}.$$

For each $f(x) \in L_2(\mathbb{R}, \mathbb{C})$ the wavelet projection operator is defined as

$$\Pi_M^N : L_2(\mathbb{R}, \mathbb{C}) \rightarrow W_M^N \subseteq L_2(\mathbb{R}, \mathbb{C})$$

and maps a function $f(x) \in L_2(\mathbb{R}, \mathbb{C})$ into its wavelet approximation (or reconstruction).

$$f(x) \mapsto \Pi_M^N f(x).$$

Explicitly, the projection is represented by (see e.g. [4, 5, 6, 7, 19])

$$(17) \quad \begin{aligned} \Pi_M^N f(x) &\stackrel{\text{def}}{=} \left[\sum_{k=-M}^M \alpha_k \varphi_k^0(x) + \sum_{n=0}^N \sum_{k=-M}^M \beta_k^n \psi_k^n(x) \right] \\ &+ \left[\sum_{k=-M}^M \alpha_k^* \overline{\varphi}_k^0(x) + \sum_{n=0}^N \sum_{k=-M}^M \beta_k^{*n} \overline{\psi}_k^n(x) \right], \quad (M \leq \infty, N \leq \infty) \end{aligned}$$

where, thanks to the orthonormality of the basis (15),(16), the wavelet coefficients can be computed by

$$(18) \quad \left\{ \begin{array}{l} \alpha_k = \langle f(x), \varphi_k^0(x) \rangle = \int_{-\infty}^{\infty} f(x) \overline{\varphi_k^0(x)} dx \\ \alpha_k^* = \langle f(x), \overline{\varphi_k^0(x)} \rangle = \int_{-\infty}^{\infty} f(x) \varphi_k^0(x) dx \\ \beta_k^n = \langle f(x), \psi_k^n(x) \rangle = \int_{-\infty}^{\infty} f(x) \overline{\psi_k^n(x)} dx \\ \beta_k^{*n} = \langle f(x), \overline{\psi_k^n(x)} \rangle = \int_{-\infty}^{\infty} f(x) \psi_k^n(x) dx . \end{array} \right.$$

For $f(x) \in L_2(\mathbb{R}, \mathbb{C})$, it is [19]

$$\lim_{N \rightarrow \infty, M \rightarrow \infty} |f(x) - \Pi_M^N f(x)| = 0 ,$$

whereas for finite values of N, M we obtain only a wavelet approximation of $f(x)$.

It should be noticed that the above integrals (18) are finitely defined for a wide class of functions which belong for example to $L_2(\mathbb{R}, \mathbb{C})$ but also $L_1(\mathbb{R}, \mathbb{C})$, and the space of periodic functions with period $b - a$ that is $L_1([a, b], \mathbb{C})$. In general, the existence of the Fourier transform is a sufficient condition for the existence of the wavelet coefficients, and this is true e.g. for the Schwartz space functions, i.e. the space of smooth functions on compact support.

The conjugate of the reconstruction (17) it is

$$\begin{aligned} \overline{f}(x) &= \left[\sum_{k=-M}^M \overline{\alpha_k} \overline{\varphi_k^0(x)} + \sum_{n=0}^N \sum_{k=-M}^M \overline{\beta_k^n} \overline{\psi_k^n(x)} \right] \\ &+ \left[\sum_{k=-M}^M \overline{\alpha_k^*} \overline{\varphi_k^0(x)} + \sum_{n=0}^N \sum_{k=-M}^M \overline{\beta_k^{*n}} \overline{\psi_k^n(x)} \right] \\ &= \left[\sum_{k=-M}^M \overline{\alpha_k^*} \varphi_k^0(x) + \sum_{n=0}^N \sum_{k=-M}^M \overline{\beta_k^{*n}} \psi_k^n(x) \right] \\ &+ \left[\sum_{k=-M}^M \overline{\alpha_k} \overline{\varphi_k^0(x)} + \sum_{n=0}^N \sum_{k=-M}^M \overline{\beta_k^n} \overline{\psi_k^n(x)} \right] \end{aligned}$$

so that for a real function $f(x) = \overline{f}(x)$ we have

$$\alpha_k^*(x) = \alpha_k(x) \quad , \quad \beta_k^{*n}(x) = \beta_k^n(x) \quad .$$

4.1. Reconstruction in the Fourier domain. The wavelet coefficients, according to (14), can be easily computed also in the Fourier domain [4, 5, 6, 7],

$$(19) \quad \left\{ \begin{array}{lcl} \alpha_k & = & 2\pi \langle \widehat{f(x)}, \widehat{\varphi_k^0(x)} \rangle = \int_{-\infty}^{\infty} \widehat{f}(\omega) \overline{\widehat{\varphi_k^0}(\omega)} d\omega = \int_0^{2\pi} \widehat{f}(\omega) e^{i\omega k} d\omega \\ \alpha_k^* & = & 2\pi \langle \widehat{f(x)}, \widehat{\overline{\varphi_k^0}(x)} \rangle = \dots = \int_0^{2\pi} \widehat{f}(\omega) e^{-i\omega k} d\omega \\ \beta_k^n & = & 2\pi \langle \widehat{f(x)}, \widehat{\psi_k^n(x)} \rangle = \dots = 2^{-n/2} \int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega) e^{i\omega k/2^n} d\omega \\ \beta_k^{*n} & = & \langle \widehat{f(x)}, \widehat{\overline{\psi_k^n}(x)} \rangle = \dots = 2^{-n/2} \int_{2^{n+1}\pi}^{2^{n+2}\pi} \widehat{f}(\omega) e^{-i\omega k/2^n} d\omega \quad , \end{array} \right.$$

being

$$\widehat{\overline{f(x)}} = \overline{\widehat{f(-\omega)}} \quad .$$

If we apply the Fourier transform to (17), we get

$$\begin{aligned} \widehat{f_M^N}(\omega) \stackrel{\text{def}}{=} [\Pi_M^N \widehat{f(x)}] &= \left[\sum_{k=0}^M \alpha_k \widehat{\varphi_k^0}(\omega) + \sum_{n=-N}^N \sum_{k=-M}^M \beta_k^n \widehat{\psi_k^n}(\omega) \right] \\ &+ \left[\sum_{k=0}^M \alpha_k^* \widehat{\overline{\varphi_k^0}}(\omega) + \sum_{n=-N}^N \sum_{k=-M}^M \beta_k^{*n} \widehat{\overline{\psi_k^n}}(\omega) \right] \end{aligned}$$

and, according to (12),

(20)

$$\begin{aligned} \widehat{f_M^N}(\omega) &= \left[\frac{1}{2\pi} \chi(2\pi + \omega) \sum_{k=0}^M \alpha_k e^{-i\omega k} + \sum_{n=0}^N \frac{2^{-n/2}}{2\pi} \chi(2\pi + \omega/2^n) \sum_{k=-M}^M \beta_k^n e^{-i\omega k/2^n} \right] \\ &+ \left[\frac{1}{2\pi} \chi(2\pi + \omega) \sum_{k=0}^M \alpha_k^* e^{i\omega k} + \sum_{n=0}^N \frac{2^{-n/2}}{2\pi} \chi(2\pi + \omega/2^n) \sum_{k=-M}^M \beta_k^{*n} e^{i\omega k/2^n} \right] \end{aligned}$$

so that for a real function, it is

$$\begin{aligned}
\widehat{f}_M^N(\omega) &= \frac{1}{2\pi} \chi(2\pi + \omega) \sum_{k=0}^M \alpha_k \cos(\omega k) \\
(21) \quad &+ \sum_{n=0}^N \frac{2^{-n/2}}{\pi} \chi(2\pi + \omega/2^n) \sum_{k=-M}^M \beta_k^n \cos(\omega k/2^n)
\end{aligned}$$

This equation is interesting and should be commented in comparison with deterministic fractals. In fact, in a finite interval, let say $0 \leq \omega \leq 2\pi$, it becomes

$$\widehat{f}_M^N(\omega) = \frac{1}{2\pi} \sum_{k=0}^M \alpha_k \cos(\omega k)$$

from where, by the substitution

$$\alpha_k \rightarrow \lambda^{kH}, \quad k \rightarrow 2^k, \quad M \rightarrow \infty$$

we re-obtain the Weierstrass function (5). Therefore the Weierstrass function is a special case of the harmonic wavelet reconstruction in the Fourier domain. Thus we can say that:

- (1) Harmonic wavelets (8) have some kind of fractal behaviour.
- (2) Some fractal-like functions, as the Weierstrass function can be reconstructed by harmonic wavelets.
- (3) Weierstrass function is a special case of the harmonic wavelet series. Thus suggesting us that formula (21) (for real functions) and (20) (for complex functions) can be used to define other kind of fractals. It can be easily seen that (20) it includes also the Cell  rier function (6).

5. SCALE INVARIANCE AND WAVELET COEFFICIENTS

Let us compute the wavelet coefficients (18) for a self-similar function (3). It can be shown that (see also [5, 7, 21])

Theorem 1. *The scaling coefficients of a self-similar function (3) are not independent on the lower scale ($n = 0$) scaling coefficients since it is*

$$(22) \quad \alpha_k^C = 2^{-C(H+1/2)} \alpha_k, \quad \forall k \in \mathbb{Z}$$

with C defined by (2).

Proof: From (18)₁, by the substitution $x = 2^C X$ it is

$$\alpha_k = 2^C \int_{-\infty}^{\infty} f(2^C X) \overline{\varphi}_k^0(2^C X) dX$$

and, by taking into account the definition (11),

$$\overline{\varphi}_k^0(2^C X) = 2^{-C/2} \overline{\varphi}_k^C(X) .$$

Thus, we have

$$\begin{aligned} \alpha_k &= 2^C \int_{-\infty}^{\infty} f(2^C X) 2^{-C/2} \overline{\varphi}_k^C(X) dX \stackrel{(3)}{=} 2^{C/2} \int_{-\infty}^{\infty} 2^{CH} f(X) \overline{\varphi}_k^C(X) dX \\ &= 2^{C(H+1/2)} \int_{-\infty}^{\infty} f(X) \overline{\varphi}_k^C(X) dX \end{aligned}$$

that is

$$(23) \quad \alpha_k = 2^{C(H+1/2)} \alpha_k^C$$

so that (22) follows. ■

Analogously, for the detail coefficients, it can be shown by a similar proof, from (18)₃ that

$$(24) \quad \beta_k^{n+C} = 2^{-C(H+1/2)} \beta_k^n , \quad \forall n \in \mathbb{N}, k \in \mathbb{Z}$$

with C defined by (2), and

$$\beta_k^{*n} = 2^{C(H+1/2)} \beta_k^{*n+C} .$$

In particular, Eq. (24) holds [7] in the form

$$(25) \quad \beta_k^n = F(2^n, k)$$

with $F(2^n, k)$ self-similar function of the scale

$$(26) \quad F(\mu 2^n, k) = \mu^S F(2^n, k) ,$$

being

$$\mu = 2^C , \quad S = - \left(H + \frac{1}{2} \right) ,$$

so that

$$(27) \quad F(2^{n+C}, k) = 2^{-C(H+1/2)} F(2^n, k) .$$

Analogously we have $\beta_k^{*n} = \overline{F}(2^n, k)$.

For example, according to (27), we can take

$$F(2^n, k) = 2^{-n(H+1/2)} g(k)$$

so that

$$(28) \quad \beta_k^n = F\left(2^{-n(H+1/2)} g(k)\right),$$

and, in particular,

$$(29) \quad \beta_k^n = 2^{-n(H+1/2)} g(k).$$

As a special case it can be assumed

$$(30) \quad F(2^n, k) = 2^{-n(H+1/2)} k^q, q \in \mathbb{R}$$

which fulfills (27) and the corresponding wavelet coefficients are

$$(31) \quad \beta_k^n = 2^{-n(H+1/2)} k^q, q \in \mathbb{R},$$

that is

$$\log_2 \beta_k^n = -n(H + 1/2) + \log_2 k^q, q \in \mathbb{R}.$$

So that in the log-plane of coordinates $(n, \log_2 \beta_k^n)$ we have a straight line with the slope given by $-(H + 1/2)$. In this way it is possible to compute the Hurst coefficient by the slope of the wavelet coefficients towards scale n .

6. FRACTAL BASED ON HARMONIC WAVELETS

In this section some examples of fractals generated by harmonic wavelets are explicitly given.

6.1. Wavelet representation of the Weierstrass function. We consider the Weierstrass-like function (7) on a bounded interval and up to a finite fixed value $|S| = |k_{\max}| < \infty$.

The Fourier transform of (7) it is

$$\widehat{f}(\omega) = \mathcal{F}[f(x)] = \sum_{k=-S}^S 2^{-kH} \mathcal{F}[\cos 2^k x] = \sum_{k=-S}^S 2^{-kH} \frac{1}{2} [\delta(\omega - 2^k) + \delta(\omega + 2^k)],$$

so that according to (12),(13),(19), we get

$$\alpha_k = \int_0^{2\pi} \widehat{f}(\omega) e^{-i\omega k} d\omega = \dots = \sum_{h=-S}^S 2^{-hH} \cos(k2^h) \stackrel{(7)}{=} f(k) ,$$

and, analogously,

$$(32) \quad \alpha_k^* = -f(k) , \quad \beta_k^n = f\left(\frac{k}{2^n}\right) , \quad \beta_k^{*n} = -f\left(\frac{k}{2^n}\right) .$$

So that equation (28) is fulfilled with

$$F(x) = \sum_{h=-S}^S 2^{-hH} \cos(2^h x) , \quad H = 1/2 , \quad g(k) = k .$$

The reconstruction formula (17) for the Weierstrass function (7) gives

(33)

$$f(x) = \sum_{k=-M}^M f(k) [\varphi_k^0(x) + \overline{\varphi}_k^0(x)] + \sum_{n=0}^N \sum_{k=-M}^M f\left(\frac{k}{2^n}\right) [\psi_k^n(x) + \overline{\psi}_k^n(x)] .$$

6.2. Wavelet representation of the Cell  rier function. We consider the Cell  rier function (6) on a bounded interval and up to a finite fixed value $|S| = |k_{\max}| < \infty$.

The Fourier transform of (6) it is

$$\widehat{f}(\omega) = \mathcal{F}[f(x)] = \sum_{k=-S}^S 2^{-kH} \mathcal{F}[\sin 2^k x] = \sum_{k=-S}^S 2^{-kH} \frac{i}{2} [\delta(\omega - 2^k) - \delta(\omega + 2^k)] ,$$

so that according to (12),(13),(19), we get

$$\alpha_k = \int_0^{2\pi} \widehat{f}(\omega) e^{-i\omega k} d\omega = \dots = \sum_{h=-S}^S 2^{-hH} \sin(k2^h) \stackrel{(7)}{=} f(k) ,$$

and, analogously,

$$(34) \quad \alpha_k^* = -f(k) , \quad \beta_k^n = f\left(\frac{k}{2^n}\right) , \quad \beta_k^{*n} = -f\left(\frac{k}{2^n}\right) .$$

So that equation (28) is fulfilled with

$$F(x) = \sum_{h=-S}^S 2^{-hH} \sin(2^h x) , \quad H = 1/2 , \quad g(k) = k ,$$

and the reconstruction formula (17) for the Cell  rier function (6) gives

$$f(x) = \sum_{k=-M}^M f(k) [\varphi_k^0(x) + \overline{\varphi}_k^0(x)] + \sum_{n=0}^N \sum_{k=-M}^M f\left(\frac{k}{2^n}\right) [\psi_k^n(x) + \overline{\psi}_k^n(x)] .$$

6.3. Fractals generation by wavelet expansion. According to Eqs. (17),(25), (and the example (33)) it is possible to define new classes of fractals as follows. Given a self similar function of the scale $F(n, k)$ we can take as wavelet coefficients

$$\alpha_k \stackrel{\text{def}}{=} F(0, k) , \quad \alpha^* \stackrel{\text{def}}{=} \bar{F}(0, k) , \quad \beta_k^n \stackrel{\text{def}}{=} F(n, k) , \quad \beta_k^{*n} \stackrel{\text{def}}{=} \bar{F}(n, k)$$

so that the fractals can be constructed according to

$$(35) \quad \begin{aligned} f(x) &= \sum_{k=-M}^M F(0, k) \varphi_k(x) + \sum_{n=0}^N \sum_{k=-M}^M F(n, k) \psi_k^n(x) \\ &+ \sum_{k=-M}^M \bar{F}(0, k) \bar{\varphi}_k(x) + \sum_{n=0}^N \sum_{k=-M}^M \bar{F}(n, k) \bar{\psi}_k^n(x) \end{aligned}$$

with $N \leq \infty$, $|M| \leq \infty$. In particular, according to Eqs. (29),(31) we can have

$$(36) \quad \begin{aligned} f(x) &= \sum_{k=-M}^M \bar{g}(k) \varphi_k(x) + \sum_{n=0}^N \sum_{k=-M}^M 2^{-n(H+1/2)} g(k) \psi_k^n(x) \\ &+ \sum_{k=-M}^M \bar{g}(k) \bar{\varphi}_k(x) + \sum_{n=0}^N \sum_{k=-M}^M 2^{-n(H+1/2)} \bar{g}(k) \bar{\psi}_k^n(x) \\ f(x) &= \sum_{k=-M}^M k^q \varphi_k(x) + \sum_{n=0}^N \sum_{k=-M}^M 2^{-n(H+1/2)} k^q \psi_k^n(x) \\ &+ \sum_{k=-M}^M k^q \bar{\varphi}_k(x) + \sum_{n=0}^N \sum_{k=-M}^M 2^{-n(H+1/2)} k^q \bar{\psi}_k^n(x) . \end{aligned}$$

6.4. Fractals Wavelets. Due to their self-similarity property, from Eq. (8) by taking into account (10) it is possible to define some wavelet based self similar functions defined as follows,

$$\Phi_A^N(x) \stackrel{\text{def}}{=} \sum_{n=0}^N \varphi_A^n(x)$$

and

$$\Psi_A^N(x) \stackrel{\text{def}}{=} \sum_{n=0}^N \psi_A^n(x) .$$

The self-similarity can be easily shown from (9):

$$\Phi_A^N(\mu x) = \sum_{n=0}^N \varphi_A^n(\mu x) \stackrel{(9)}{=} \sum_{n=0}^N \mu^{-\mu} \varphi_A^{n-1}(x) = \mu^{-\mu} \Phi_A^{N-1}(x) ,$$

as well as for $\Psi_A^N(x)$.

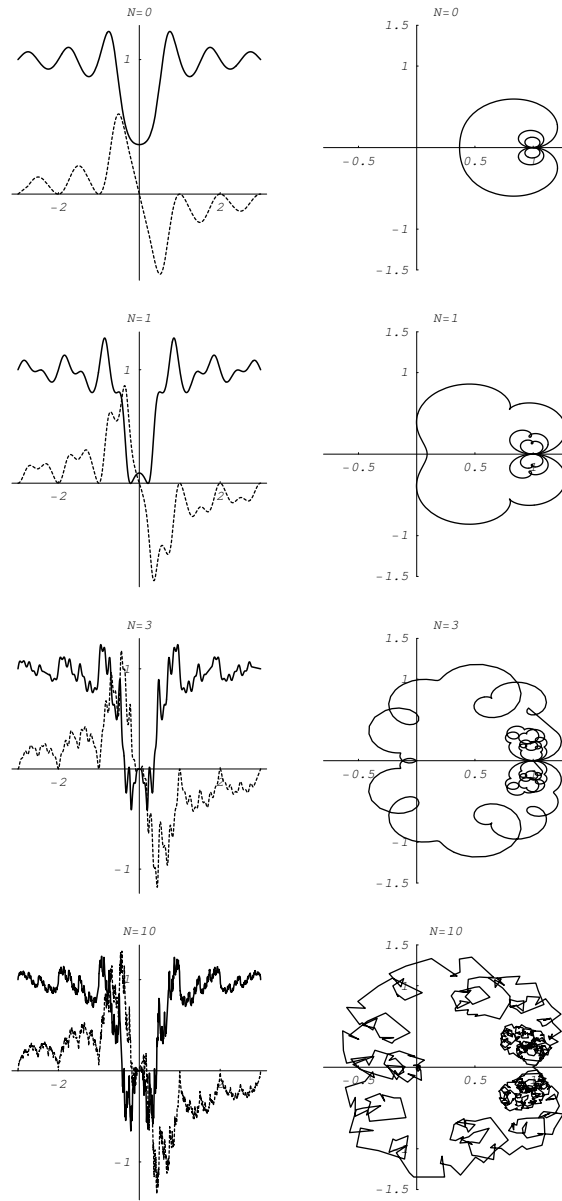


FIGURE 6. Plots (left column) in $\mathbb{R}^2$ and orbits (right column) in $\mathbb{C}^1$ of the scaling function $\Phi_0^N(x)$ of Eq. (37) in the interval $x \in [-1, 1]$.

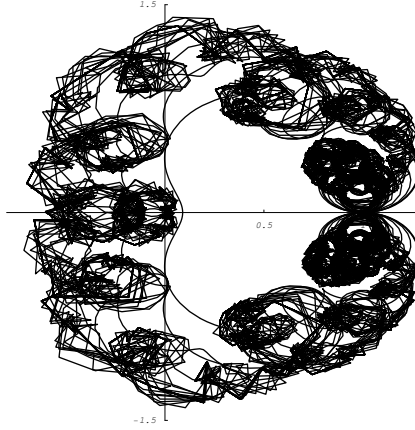


FIGURE 7. Orbits in $\mathbb{C}^1$ of the self similar scaling function $\Phi_0^N(x)$ of Eq. (37) in the interval $x \in [-1, 1]$, $0 \leq N \leq 30$.

However, some other fractal-like functions with a weak self-similarity can be defined on harmonic wavelets (Figs. 6,7,8,9):

$$(37) \quad \Phi_A^N(x) \stackrel{\text{def}}{=} \exp \left[- \sum_{n=0}^N \varphi_A^n(x) \right]$$

and

$$(38) \quad \Psi_A^N(x) \stackrel{\text{def}}{=} \exp \left[- \sum_{n=0}^N \psi_A^n(x) \right] .$$

We can observe the fractal behavior both in the variable and in the complex plane, both for the scaling (Fig. 6,7) and harmonic wavelet (Fig. 8,9) functions (in the interval $x \in [-1, 1]$).

It should be noticed, however that from their definition (37),(38), the fractal shape in the complex plane depends also on the range of the parameter x .

CONCLUSION

In this paper the wavelet coefficients for self-similar localized functions have been explicitly computed and the scale invariance of some fractals has been characterized by some relations on wavelet coefficients. Harmonic wavelets have been used to generate some scale self-similar wavelets and a formula based on wavelet expansion for the generation of self similar functions has been also given, thus opening new perspectives in the analysis of fractals.

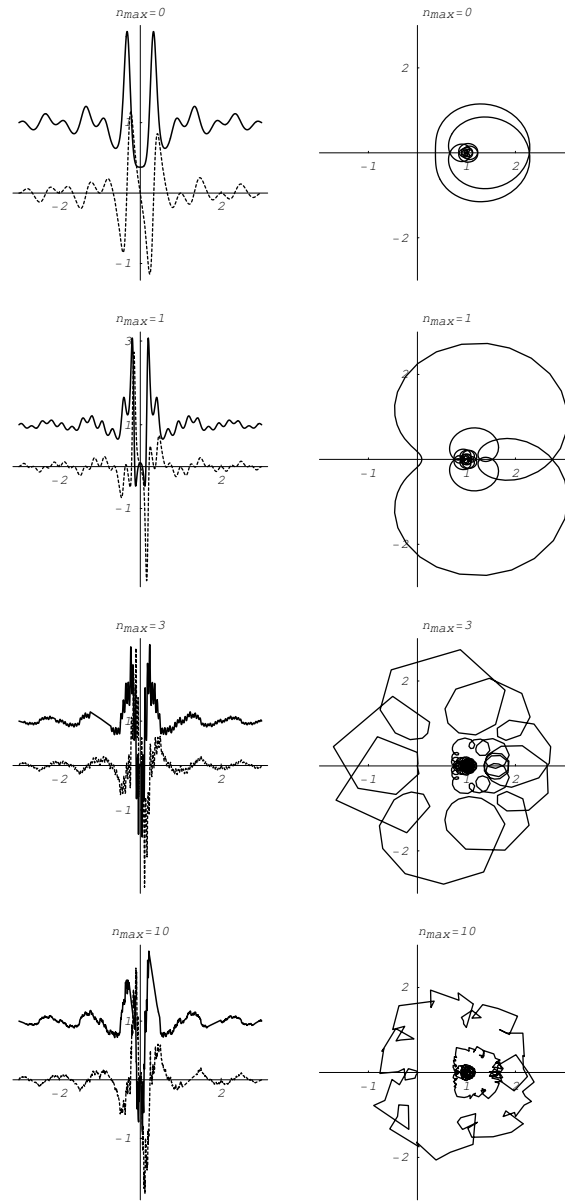


FIGURE 8. Plots (left column) in $\mathbb{R}^2$ and orbits (right column) in $\mathbb{C}^1$ of the self similar wavelet function $\Psi_0^N(x)$ of Eq. (38) in the interval $x \in [-1, 1]$.

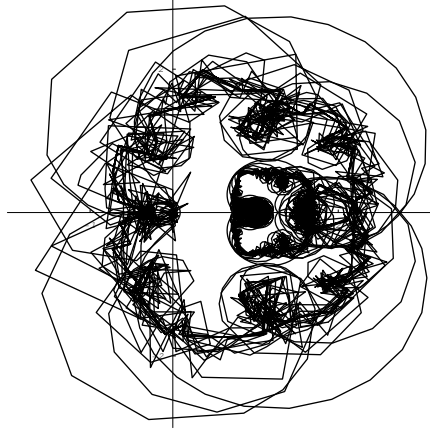


FIGURE 9. Orbits in $\mathbb{C}^1$ of the self similar wavelet function $\Psi_0^N(x)$ of Eq. (38) in the interval $x \in [-1, 1]$, $0 \leq N \leq 30$.

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