SHADOWING AND CHAOS

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ABSTRACT. In this short paper we show that if homeomorphism f on a compact metric space X has the shadowing property and only one chain component, then f is Σ-chaotic. As a corollary we show that generically f is Σ-chaotic.

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1. Introduction

Let (X, d) be a compact metric space and let $f : X \to X$ be a homeomorphism of X onto itself. A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called an orbit of f if for each $n \in \mathbb{Z}$ we have $x_{n+1} = f(x_n)$ and we call it a δ-pseudo-orbit of f if for each $n \in \mathbb{Z}$,

$$d(f(x_n), x_{n+1}) \leq \delta.$$

The homeomorphism f is said to have the shadowing property if for each $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit $\{x_n\}_{n\in\mathbb{Z}}$ is ϵ shadowed by the orbit $\{f^n(y) : n \in \mathbb{Z}\}$, for some y in X, i.e for all $n \in \mathbb{Z}$ we have

$$d(f^n(y), x_n) < \epsilon.$$

We denote the set $\{x \in X : x \in \omega_f(x) \cap \alpha_f(x)\}$ by $C(f)$, where $\omega_f(x)$ and $\alpha_f(x)$ are the positive and negative limit sets of x for f, respectively. Let $x, y \in X$ be

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given. We write $x \xrightarrow{f} y$ if and only if for each $\delta > 0$ there is a δ-pseudo-orbit $\{x_i\}_{i=0}^{l}$ of some lenth $l+1$ of f, such that $x = x_0, ..., x_n = y$. We write $x \overset{f}{\sim} y$ if and only if $x \xrightarrow{f} y$ and $y \xrightarrow{f} x$, and let

$$R(f) = \{x \in X : x \overset{f}{\sim} x\}.$$

A homeomorphism f is called to be P-chaotic if it has shadowing property and the closure of the set of all periodic points for f is equal to X and is said is chaotic in the sense of Devaney if

(1) f is transitive,

(2) the set of all periodic points of f is dense in X, and

(3) f has sensitive dependence on initial condition; there exists $\delta > 0$ such that for each $x \in X$ and each neighborhood U of x, there exist $y \in U$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

In [2], it has been shown that f is chaotic in the sense of Devaney if and only if f satisfies the above conditions (1) and (2).

In [1], Arai and Chinen proved that every P-chaotic map from a continuum X to itself is chaotic in the sense of Devaney.

Let $\{0,1\}^{\mathbb{Z}}$ be the space of all two sided sequences $x = \{x_i : i \in \mathbb{Z}\}$ with elements $x_i \in \{0,1\}$, endows with a metric D defined by

$$D(x,y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}},$$

where $x, y \in \{0,1\}^{\mathbb{Z}}$. We also write this space as Σ to shorten the notation. Define a shift map $\sigma : \Sigma \longrightarrow \Sigma$ by

$$\sigma(x_i) = x_{i+1} \ , \ (i \in \mathbb{Z}),$$

The homeomorphism f is said to be semiconjugated to (Σ, σ) if there exit a closed, f-invariant set $\Lambda \subset X$ and a continuous surjection $p : \Lambda \longrightarrow \Sigma$ such that $p \circ f = \sigma \circ p$.

The homeomorphism f is called chaotic if some positive iteration of f is semiconjugated to (Σ, σ) and we call such f with Σ-chaotic.

In [6], it has been shown that if a homeomorphism f has the shadowing property then it is Σ-chaotic if and only if there is a chain recurrent point which is unstable in $R(f)$.

A point $x \in A \subset X$ is called a stable point in A if for each $\epsilon > 0$ there is $\delta > 0$ such that for every $y \in A$ if $d(x, y) < \delta$ then $d(f^n(x), f^n(y)) < \epsilon$, $\forall\ n \in \mathbb{N}$.

A homeomorphism f is called stable if any point $x \in X$ is a stable point in X.

A homeomorphism f is called positively recurrent if $x \in \omega_f(x)$ for each x in X.

A property P of elements of a topological space X is said to be generic if the set of all $x \in X$ satisfying P is residual, i.e, it includes a countable intersection of open and dense subsets of X.

2. Results

X is called relatively compact if the clousure of every bounded subset of X is compact.We state the following lemma and one theorem from [3].

Lemma[3] let f be a homeomorphism of a relatively compact space X. If f has the shadowing property, then for every $\epsilon > 0$ there is $\delta > 0$ such that every periodic δ-pseudo-orbit of f can be ϵ-shadowed by some point in $C(f)$.

Theorem[3] Let f be a homeomorphism of a compact space X which has the shadowing property and $f|_{\Omega(f)}$,where $\Omega(f)$ is the nonwandering set of f, be topologically transitive, then $X = \Omega(f)$.

Corollary. *Let f be a homeomorphism of a compact metric space X which has the shadowing property. Then $\overline{C(f)} = R(f)$*

Proof. Clearly $\overline{C(f)} \subset R(f)$. For every $\epsilon > 0$ let $\delta > 0$ be as in the previous lemma. Then every $x \in R(f)$ can be a member of a periodic δ-pseudo-orbit of f and it can be ϵ-shadowed by a point in $C(f)$. Thus $x \in \overline{C(f)}$ since ϵ is arbitrary. □

Theorem 1. *Let f be a homeomorphism of a compact space X which has the shadowing property and only chain component, then $X = R(f)$.*

Proof. By the previous theorem, it is sufficient to show that $f|_{\Omega(f)}$ is topologically transitive. Let U and V be two open neighborhoods in $\Omega(f)$ and $x \in U, y \in V$ and $\epsilon > 0$ be such that $N_\epsilon(x) \subset U$ and $N_\epsilon(y) \subset V$. Let $\delta > 0$ be in previous lemma, since x and y are in $\Omega(f)$ and hence are in $R(f)$ and f has only one chain component so there are two δ-chains from x to y and from y to x. By composition this two δ-chains, we obtain a δ-chain from x to x. By previous lemma there is $z \in C(f)$ such that ϵ-shadowed this δ-chain. Hence there are $m, n \in \mathbb{Z}, n > m$ such that $f^m(z) \in N_\epsilon(x)$ and $f^n(z) \in N_\epsilon(y)$. So $f^{n-m}(U) \cap V \neq \phi$ and hence $f|_{\Omega(f)}$ is topologically transitive. $\square$

Theorem 2. *Let f be a stable homeomorphism of a compact metric space X which has the shadowing property and only chain component, then f is positively recurrent.*

Proof. By corollary and theorem 1, $X = \overline{C(f)}$. Let $x \in X$ and $\epsilon > 0$ be arbitrary. That is enough to show that there is a positive integer n such that $d(f^n(x), x) < \epsilon$. Let $0 < \delta < \frac{\epsilon}{3}$ be in the definition of stable point x for $\frac{\epsilon}{3}$. There are a $y \in C(f)$ and a positive integer n such that $d(x, y) < \delta$ and $d(f^n(y), y) < \frac{\epsilon}{3}$. So

$$d(f^n(x), x) < d(f^n(x), f^n(y)) + d(f^n(y), y) + d(x, y) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

$\square$

Theorem A[4] Let f be chain transitive and has the shadowing property. If f is a stable homeomorphism then f is weak strictly persistence.

Theorem B[4] Let f be a weak strictly persistence homeomorphism of a compact connected metric space X which is not minimal. If X is not one point, and if f is positively recurrent, then f does not have the shadowing property.

Theorem 3. *Let f be a homeomorphism of a compact metric space X which has the shadowing property and only chain component, then f is Σ-chaotic.*

Proof. Let f is not Σ-chaotic. By [6] and theorem 1, f is a stable homeomorphism and f is chain transitive so by theorem A in [4] f is weak strictly persistence. Since f has shadowing property then f is not minimal. In other hand by theorem 2, f is positively recurrent. Then by theorem B in [4] f does not have the shadowing property that is a contradiction. So f is Σ-chaotic. $\square$

Corollary. *Generically the homeomorphism f is Σ-chaotic.*

Proof. Generically f has shadowing property (see [7]) and $X = R(f)$. By a similar argument in theorem 3, f is Σ-chaotic. $\square$

Remark: If f has the periodic shadowing property then every $x \in R(f)$ is in the closure of periodic points of f. Since generically f has the periodic shadowing property (see [5]) and $X = R(f)$, so generically f is P-chaotic.

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