

ON CLOSED RULED SURFACES CONCERNED WITH DUAL FRENET AND BISHOP FRAMES

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ABSTRACT. In this paper; the pitch, the angle of pitch and the dual angle of pitch of ruled surface in $\mathbb{R}^3$ which is corresponded to a closed curve on dual unit sphere are investigated. This closed curve is formed by the vectors of Frenet and Bishop frame of a closed curve on dual unit sphere. Also the relation between dual angle of pitch and pitch, angle of pitch is obtained.

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1. INTRODUCTION

The Study Mapping corresponds the dual points of a dual unit sphere, in $\mathbb{D}$ -Module to the oriented lines in $\mathbb{R}^3$. This mapping is one to one.

A differentiable curve on the dual unit sphere, depending on a real parameter t , represents a differentiable family of straight lines in $\mathbb{R}^3$ which is called ruled surface [4]. This correspondence allows us to study the properties of a ruled surface on the geometry of dual spherical curves on a dual unit sphere.

In this paper we dwell on a curve with Frenet and Bishop frame on dual unit sphere. Each vector of Frenet and Bishop frame of this curve generates another curve on dual unit sphere by moving the vectors to the centre of the sphere. Also these representation curves correspond ruled surfaces in $\mathbb{R}^3$. We detect the pitch, the angle of pitch and the dual angle of pitch of these ruled surfaces under cover of [3, 4].

2. PRELIMINARIES

Dual numbers were defined by W.K. Clifford. The set $\mathbb{D}$ is called the dual number system and the elements of this set are in type of $A = a + \varepsilon a^*$. Here a and a^* are real numbers and $\varepsilon^2 = 0$ which is called a dual unit. The elements of the set $\mathbb{D}$ are called dual numbers. The set

$$\mathbb{D} = \{A = a + \varepsilon a^* \mid a, a^* \in \mathbb{R}\}$$

forms a commutative ring with the following operations

$$i) \quad (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)$$

$$ii) \quad (a + \varepsilon a^*). (b + \varepsilon b^*) = a.b + \varepsilon(ab^* + ba^*).$$

The division of two dual numbers $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$ provided $b \neq 0$ can be defined as

$$\frac{A}{B} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{a^*b - ab^*}{b^2}.$$

The set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \left\{ \begin{array}{l} \vec{A} \mid \vec{A} = (a_1 + \varepsilon a_1^*, a_2 + \varepsilon a_2^*, a_3 + \varepsilon a_3^*) \\ \quad = (a_1 + a_2 + a_3) + \varepsilon(a_1^* + a_2^* + a_3^*) \\ \quad = a + \varepsilon a^*, \quad a \in IR^3, a^* \in IR^3 \end{array} \right\}$$

is a module on the ring $\mathbb{D}$ which is called $\mathbb{D}$ -Module and the elements are dual vectors consisting of two real vectors. The norm $\|\vec{A}\|$ of $\vec{A}$ is defined by

$$\|\vec{A}\| = \sqrt{\langle \vec{A}, \vec{A} \rangle} = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}$$

where $a \neq 0$.

Let

$$\begin{array}{ccc} \hat{\alpha} : I & \longrightarrow & \mathbb{D}^3 \\ s & \longrightarrow & \vec{\hat{\alpha}}(s) = \vec{\alpha}(s) + \varepsilon \vec{\alpha}^*(s) \end{array}$$

be a curve with arc-length parameter s . Then

$$\frac{d\vec{\hat{\alpha}}}{ds} = \frac{d\vec{\alpha}}{ds} \cdot \frac{ds}{ds} = \vec{\hat{T}}$$

is called the unit tangent vector of $\hat{\alpha}(s)$. The norm of the vector $\frac{d\vec{\hat{T}}}{ds}$ which is given by

$$\frac{d\vec{\hat{T}}}{ds} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{ds} = \frac{d^2\vec{\hat{\alpha}}}{ds^2} = \hat{\kappa} \vec{\hat{N}}$$

is called curvature function of $\vec{\hat{\alpha}}(s)$. Here $\hat{\kappa} : I \longrightarrow \mathbb{D}$ is never pure-dual. Then the unit principal normal vector of $\vec{\hat{\alpha}}(s)$ is defined as

$$\vec{\hat{N}} = \frac{1}{\hat{\kappa}} \cdot \frac{d\vec{\hat{T}}}{ds}$$

The vector $\vec{B} = \vec{T} \times \vec{N}$ is called the binormal of $\vec{\alpha}(s)$. Also we call the vectors $\vec{T}, \vec{N}, \vec{B}$ Frenet trihedron of $\vec{\alpha}(s)$ at the point $\hat{\alpha}(s)$. The derivatives of dual Frenet vectors $\vec{T}, \vec{N}, \vec{B}$ can be written in matrix form as

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ -\hat{\kappa} & 0 & \hat{\tau} \\ 0 & -\hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

which are called Frenet formulas [7]. The function $\hat{\tau} : I \longrightarrow \mathbb{D}$ such that $\frac{d\vec{B}}{ds} = -\hat{\tau}\vec{N}$ is called the torsion of $\vec{\alpha}(s)$.

If a moving point moves on a dual space curve $\hat{\alpha}$, then the moving dual Frenet frame $\{\vec{T}, \vec{N}, \vec{B}\}$ moves along an axis consisting an instantaneous translation and an instantaneous rotation around the curve. The instantaneous rotation is expressed by an angular velocity vector which is called dual Darboux vector is given by

$$\vec{W} = \hat{\tau}\vec{T} + \hat{\kappa}\vec{B}$$

see [1]. The dual Darboux vector satisfies the following equations:

$$\begin{aligned} \vec{T}' &= \vec{W} \times \vec{T} \\ \vec{N}' &= \vec{W} \times \vec{N} \\ \vec{B}' &= \vec{W} \times \vec{B}. \end{aligned}$$

Now let us give the instructions of dual Bishop vectors. The differentiation of unit tangent vector $\vec{T}$ is given by

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{ds} = \frac{d^2\vec{\alpha}}{ds^2} = \hat{k}_1\vec{N}_1 + \hat{k}_2\vec{N}_2$$

where $\hat{k}_1, \hat{k}_2 : I \longrightarrow \mathbb{D}$; $\hat{k}_1 = k_1 + \varepsilon k_1^*$, $\hat{k}_2 = k_2 + \varepsilon k_2^*$ are natural curvatures and never pure-dual. The vectors $\vec{T}, \vec{N}_1, \vec{N}_2$ are called dual Bishop trihedron of $\hat{\alpha}(s)$. Also the dual Bishop formulas hold as it is expressed in [6]:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}_1' \\ \vec{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & \hat{k}_1 & \hat{k}_2 \\ -\hat{k}_1 & 0 & 0 \\ -\hat{k}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N}_1 \\ \vec{N}_2 \end{bmatrix}.$$

The relation between $\widehat{\kappa}, \widehat{\tau}$ and $\widehat{k}_1, \widehat{k}_2$ is given as;

$$\begin{aligned}\widehat{\kappa}(t) &= \kappa + \varepsilon\kappa^* = \sqrt{k_1^2 + k_2^2 + 2\varepsilon(k_1k_1^* + k_2k_2^*)}, \\ \widehat{\theta}(t) &= \theta + \varepsilon\theta^* = \arctan\left(\frac{\widehat{k}_2}{\widehat{k}_1}\right) = \arctan\left(\frac{k_2}{k_1} + \varepsilon\frac{k_1k_2^* - k_1^*k_2}{k_1^2}\right), \\ \widehat{\tau}(t) &= \frac{d\widehat{\theta}(t)}{dt}.\end{aligned}$$

The moving dual Bishop frame $\left\{\vec{T}, \vec{N}_1, \vec{N}_2\right\}$ forms an instantaneous rotation with dual Bishop Darboux vector which is given by

$$\vec{\omega} = -\widehat{k}_2\vec{N}_1 + \widehat{k}_1\vec{N}_2$$

(see[6]).

3. THE PITCH, THE ANGLE OF PITCH AND THE DUAL ANGLE OF PITCH OF CLOSED RULED SURFACES

A differentiable closed curve on dual unit sphere represents a closed ruled surfaces in $\mathbb{R}^3$. So in this paper all mentioned curves are closed. The vectors of Frenet and Bishop frame of the dual curve $\widehat{\alpha}$ form representation curves by moving them to the centre of dual unit sphere. We deal with closed ruled surfaces which are formed by these representation curves on dual unit sphere.

A straight line l moving along a curve α in $\mathbb{R}^3$ generate a ruled surface which can be expressed as

$$\varphi(t, v) = \alpha(t) + v\vec{X}(t)$$

where α is the base curve and $\vec{X}$ is the director vector of l . The lines l are called the rulings of the ruled surface [10].

Definition 1. For a closed ruled surface $\varphi(t, v) = \alpha(t) + v\vec{X}(t)$, the pitch of this closed ruled surfaces is the distance between first and last position of direction which is perpendicular to rulings after one period and it is defined by

$$\ell = \int dv = - \int \langle d\vec{\alpha}, \vec{X} \rangle$$

[5].

Definition 2. For a closed ruled surface $\varphi(t, v) = \alpha(t) + v\vec{X}(t)$, the angle between first and last position of direction which is perpendicular to rulings after one period is called the angle of pitch and it is defined as

$$\lambda_X = \int d\varphi$$

[5].

Theorem 1. *The dual angle of a closed ruled surface which is constructed by the dual unit vector $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ is given by*

$$\Lambda_X = -\langle \vec{D}, \vec{X} \rangle$$

where $\vec{D} = \vec{d} + \varepsilon \vec{d}^*$ is the dual Steiner vector of dual spherical motion [3].

Theorem 2. *The dual angle of pitch of a closed ruled surface can be expressed by*

$$\Lambda_X = \lambda_X - \varepsilon \ell_X$$

ℓ_X and λ_X are respectively the pitch and the angle of pitch of the closed ruled surface [3].

Remark 1. *This theorem denotes that a closed ruled surface which has two integral invariants in $\mathbb{R}^3$, can be characterized with a unique dual integral invariant on dual unit sphere.*

Definition 3. *Let $\vec{\Psi} = \vec{\psi} + \varepsilon \vec{\psi}^*$ be the instantaneous dual Pfaffian vector of the dual spherical motion. The vector*

$$\vec{D} = \vec{d} + \varepsilon \vec{d}^* = \int \vec{\Psi}$$

is called the dual Steiner vector of this dual spherical motion.[4].

The instantaneous dual Pfaffian vector has the same role with the dual Darboux vector. $\vec{\psi}$ and $\vec{\psi}^*$ are respectively the instantaneous rotational velocity vector and the instantaneous translational velocity vector of the dual spherical motion.

First of all let's find the dual Steiner vector relevant to dual Frenet frame.

$$\begin{aligned} \vec{D} &= \vec{d} + \varepsilon \vec{d}^* = \int \vec{W} \\ &= \int (\tau \vec{T} + \kappa \vec{B} + \varepsilon (\tau \vec{T}^* + \tau^* \vec{T} + \kappa \vec{B}^* + \kappa^* \vec{B})) \\ &= \vec{T} \int \tau + \vec{B} \int \kappa + \varepsilon (\vec{T}^* \int \tau + \vec{T} \int \tau^* + \vec{B}^* \int \kappa + \vec{B} \int \kappa^*). \end{aligned}$$

Also the dual Steiner vector relevant to dual Bishop frame is given by

$$\begin{aligned} \vec{\tilde{D}} &= \vec{\tilde{d}} + \varepsilon \vec{\tilde{d}}^* = \int \vec{\tilde{W}} \\ &= \int (-k_2 N_1 + k_1 N_2 + \varepsilon (k_1^* N_2 + k_1 N_2^* - k_2^* N_1 - k_2 N_1^*)) \\ &= -N_1 \int k_2 + N_2 \int k_1 + \varepsilon (N_2 \int k_1^* + N_2^* \int k_1 - N_1 \int k_2^* - N_1^* \int k_2). \end{aligned}$$

Theorem 3. *Let $\hat{\alpha} : I \longrightarrow \mathbb{D}^3$ be a closed curve on dual unit sphere and $\left\{ \vec{\tilde{T}}, \vec{\tilde{N}}, \vec{\tilde{B}} \right\}$ be the dual Frenet frame of $\hat{\alpha}$. The dual angles of pitch of closed ruled surfaces*

which are formed by representation curves of dual unit tangent vector, dual unit principal vector and dual unit binormal vector are respectively

$$\begin{aligned}\Lambda_{\hat{T}} &= - \int \tau - \varepsilon \int \tau^* \\ \Lambda_{\hat{N}} &= 0 \\ \Lambda_{\hat{B}} &= - \int \kappa - \varepsilon \int \kappa^*\end{aligned}$$

where $\hat{\tau} = \tau + \varepsilon\tau^*$ is the torsion and $\hat{\kappa} = \kappa + \varepsilon\kappa^*$ is the curvature of $\hat{\alpha}$ according to dual Frenet frame of $\hat{\alpha}$.

Proof. The dual angle of pitch of closed ruled surface formed by representation curve of dual unit tangent vector can be calculated as below, by using the Theorem 1 and the properties of dual unit vectors of Frenet frame.

$$\begin{aligned}\Lambda_{\hat{T}} &= - \left\langle \vec{D}, \vec{\hat{T}} \right\rangle \\ &= - \left\langle T \int \tau + B \int \kappa + \varepsilon(T^* \int \tau + T \int \tau^* + B^* \int \kappa + B \oint \kappa^*, T + \varepsilon T^* \right\rangle \\ &= - (\underbrace{\langle T, T \rangle}_1 \oint \tau + \underbrace{\langle B, T \rangle}_0 \oint \kappa + \varepsilon \underbrace{\langle T^*, T \rangle}_0 \oint \tau + \varepsilon \underbrace{\langle T, T \rangle}_1 \oint \tau^* + \\ &\quad \varepsilon \underbrace{\langle B^*, T \rangle}_0 \oint \kappa + \varepsilon \underbrace{\langle B, T \rangle}_0 \oint \kappa^* + \varepsilon \underbrace{\langle T, T^* \rangle}_0 \oint \tau + \varepsilon \underbrace{\langle B, T^* \rangle}_{-\langle T, B^* \rangle} \oint \kappa) \\ &= - (\oint \tau + \varepsilon \oint \tau^* + \varepsilon \langle B^*, T \rangle \oint \kappa - \varepsilon \langle T, B^* \rangle \oint \kappa) \\ &= - \oint \tau - \varepsilon \oint \tau^*\end{aligned}$$

By similar calculations $\Lambda_{\hat{N}} = 0$ and $\Lambda_{\hat{B}} = - \int \kappa - \varepsilon \int \kappa^*$ can easily be found. $\square$

Corollary 1. The angle of pitch and the pitch of these closed ruled surfaces are respectively $\lambda_{\hat{T}} = - \int \tau$, $\lambda_{\hat{N}} = 0$, $\lambda_{\hat{B}} = - \int \kappa$ and $\ell_{\hat{T}} = \int \tau^*$, $\ell_{\hat{N}} = 0$, $\ell_{\hat{B}} = \int \kappa^*$.

Proof. It is a natural result of Theorem 2. $\square$

Theorem 4. Let $\hat{\alpha} : I \longrightarrow \mathbb{D}^3$ be a closed curve on dual unit sphere and $\left\{ \vec{\hat{T}}, \vec{\hat{N}}_1, \vec{\hat{N}}_2 \right\}$ be the dual Bishop frame of $\hat{\alpha}$. The dual angles of pitch of closed ruled surfaces which are formed by representation curves of dual Bishop vectors are respectively

$$\begin{aligned}\Lambda_{\hat{T}}^* &= 0 \\ \Lambda_{\hat{N}_1} &= \int k_2 + \varepsilon \int k_2^* \\ \Lambda_{\hat{N}_2} &= - \int k_1 - \varepsilon \int k_1^*\end{aligned}$$

where $\widehat{k}_1 = k_1 + \varepsilon k_1^*$ and $\widehat{k}_2 = k_2 + \varepsilon k_2^*$ are the natural curvatures of $\widehat{\alpha}$ according to dual Bishop frame of $\widehat{\alpha}$.

Proof. The dual angle of pitch of closed ruled surface formed by representation curve of dual unit tangent vector of dual Bishop frame can be calculated as below, by using the Theorem 1 and the properties of dual unit vectors of Bishop frame.

$$\begin{aligned}
 \Lambda_T^* &= -\left\langle \overrightarrow{D}, \overrightarrow{T} \right\rangle \\
 &= -\left\langle -N_1 \int k_2 + N_2 \int k_1 + \varepsilon(N_2 \int k_1^* + N_2^* \int k_1 - N_1 \int k_2^* - \right. \\
 &\quad \left. N_1^* \int k_2), T + \varepsilon T^* \right\rangle \\
 &= -\left(-\underbrace{\langle N_1, T \rangle}_0 \oint k_2 + \underbrace{\langle N_2, T \rangle}_0 \oint k_1 + \varepsilon \underbrace{\langle N_2, T \rangle}_0 \oint k_1^* + \varepsilon \underbrace{\langle N_2^*, T \rangle}_{-\langle T^*, N_2 \rangle} \oint k_1 - \right. \\
 &\quad \left. \varepsilon \underbrace{\langle N_1, T \rangle}_0 \oint k_2^* - \varepsilon \underbrace{\langle N_1^*, T \rangle}_{-\langle T^*, N_1 \rangle} \oint k_2 - \varepsilon \langle N_1, T^* \rangle \oint k_2 + \varepsilon \langle N_2, T^* \rangle \oint k_1 \right) \\
 &= -(-\varepsilon \langle T^*, N_2 \rangle \oint k_1 + \varepsilon \langle N_2, T^* \rangle \oint k_1 + \varepsilon \langle T^*, N_1 \rangle \oint k_2 - \varepsilon \langle N_1, T^* \rangle \oint k_2) \\
 &= 0
 \end{aligned}$$

By similar calculations $\Lambda_{\widehat{N}_1} = \int k_2 + \varepsilon \int k_2^*$ and $\Lambda_{\widehat{N}_2} = -\int k_1 - \varepsilon \int k_1^*$ can easily be found. $\square$

Corollary 2. *The angle of pitch and the pitch of these closed ruled surfaces are respectively $\lambda_T^* = 0$, $\lambda_{\widehat{N}_1} = \int k_2$, $\lambda_{\widehat{N}_2} = -\int k_1$ and $\ell_T^* = 0$, $\ell_{\widehat{N}_1} = -\int k_2^*$, $\ell_{\widehat{N}_2} = \int k_1^*$.*

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