

GEOMETRIC METHOD FOR FREE OSCILLATOR UNDER TWO PARAMETRIC PERTURBATION

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ABSTRACT. The free oscillation is the harmonic at which any body tends to vibrate most freely . By geometric method of the theory of bifurcation , refined and developed of the Melnikov function; we prove the existence of the periodic orbit for the free oscillator under two parametric perturbation. In fact we use the geometric method of multi-parameter bifurcation theory in order to prove the persistence of the periodic orbit. By this method we consider the effect of forcing and nonlinear damping on the free oscillator simultaneously. Also we consider the effect of detuning on the system. At the end we apply the method on the forced Rayleigh equation.

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1. INTRODUCTION

The search for harmonic and subharmonic periodic solutions to nonlinear and linear oscillators such as Van der Pol, Duffing and free oscillator system,..., has a long and distinguished pedigree. Oscillations play a prominent role in physical, engineering and biological problems. The free oscillator is a system which has periodic orbit and there is no time-dependent force applied to the oscillator. For example:

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1). Oscillating on the spring; 2). After a large earthquake the Earth measured in minutes. These are called free oscillations because they continue without any attraction. 3). Oscillations of viscous fluids in a cylinder hypothesis theoretically by computing free and force oscillations of the atmosphere on the basis.

Since the literature with this respect is quite a lot, we only mention a few of them [1, 2, 3, 4, 5, 6, 7] for background.

In this paper we use the Melnikov function approach as refined and developed in a series of paper by Chicone [4, 5, 6, 7], to investigate the interactions forcing and nonlinear damping on the free oscillator system. The new contribution that we make to this study is the use of geometric method to understand the bifurcation of the multi-parameter families of the Melnikov function that arise.

The general framework of the technique is as follows. Let γ be a periodic orbit of period $T > 0$ for an autonomous systems of ordinary differential equations in R^n , assuming sufficiently regularity so that solutions are unique and that they are defined for all $t \in R$. Then each point of γ is a fixed point for the Poincare map ϕ_T where ϕ_t is the flow generated by the solutions of the equation. If we now perturb the system by applying a small-amplitude forcing of period T_1 to period T and perhaps at the same time changing the original equation slightly, then the map ϕ_T is replaced by a map P_ϵ which follows solutions of the perturbed system for time T_1 is therefore close to ϕ_T . Our problem is to find the fixed points (if any) of P_ϵ to γ or, to express it another way, to find which points of γ *persist* nearby as periodic points with period close to T of the perturbed system.

A standard method, which presumably goes back at least to Poincare, is to consider the perturbation as depending on a single parameter $\epsilon \in R$ and to expand the corresponding map P_ϵ in power of ϵ on a neighborhood U of γ . The *displacement map*, that is the map

$$\delta(\xi, \epsilon) = \epsilon P_\epsilon(\xi, 0) + O(\epsilon^2) = P(\xi, \epsilon) - \xi \quad (2).$$

This map vanishes at least on γ when $\epsilon = 0$, but may vanish on a larger set: indeed, in a wide class of examples in which the unperturbed system is a simple harmonic oscillator. Let $V \subset U$ denote the zero set of P_0 ; then $P_\epsilon(\xi, 0) = P^1$ is called the *bifurcation map*. Implicit function theorem provided that the zeroes of

P^1 are non-degenerate (the Jacobian matrix of P^1 is non-singular), these zeroes will persist nearby as zeroes of (2) for small $\epsilon \neq 0$, corresponding to periodic orbits of the perturbed system with period close to T .

Further question arise if:

- (a) there is more than one independent perturbation parameter, and
- (b) the map P^1 has degenerate zeroes. In fact if (a) applies then we expect there to be parameter values where (b) applies . A class of ordinary differential equation problems where this happens is studied by Hale and Tobaas [10] and a general setting for the study of such system is described in the Chicone papers [4,5,6,7].

In section 2 of this paper the general formula for the planar systems with one parameter has been given. Section 3 is devoted to the answering the questions (a) and (b). In fact in this section the multi-parameters bifurcation theory for more than one parameter has been investigated. Section 4 analyze the free oscillator system under effect of forcing and nonlinear damping simultaneously. In the last section ,we consider the forcing Rayleigh equation.

2. THE GENERAL FORMULA FOR PLANAR SYSTEMS

We consider the system

$$\dot{x} = f(x) + \epsilon g(x, t, \epsilon) \quad \epsilon \in R, \quad x \in R^2 \quad (3)$$

in a neighborhood of U of a periodic orbit γ of period $T > 0$ for (3) , where maps f, g are assumed to be smooth and g is periodic with respect to t . The period of the function g is $\tau = \frac{m}{n}T$, where m and n are prime numbers . Assuming there to be no zeros of f in U , then we have a basis $B(\xi) = \{f(\xi), f^\perp(\xi)\}$ where if $u = (u_1, u_2) \in R^2$ then $u^\perp$ denotes the vector $(-u_1, u_2)$. Let $V \subset U$ be the set of zeroes P_0 , that is the set of T - periodic orbits in U . Then we use the result in the Diliberto theorem [9], see also [4]. This theorem illustrate the important result about the *geometric* meaning of the two functions α and β ,see below, that are defined in the theorem.

Theorem 2.1 For $\xi \in V$ the map P^1 evaluated at ξ has the form

$$P^1(\xi) = (N(\xi), M(\xi))$$

with respect to the basis $B(\xi)$, where

$$\begin{aligned} N(\xi) &= \int_0^{mT} \|f\|^{-2} \left\{ f \cdot g - \frac{\alpha(t)}{\beta(t)} f^\perp \cdot g \right\} \phi_t(\xi) dt, \\ \alpha(t) &= \int_0^t \left\{ \frac{1}{\|f\|^2} (2\kappa \|f\| - \text{curl} f) \beta \right\} (s, \xi) ds, \\ M(\xi) &= \int_0^{mT} \left(\frac{1}{\beta(t)} f^\perp \cdot g \right) \|f\|^{-2} \phi_t(\xi) dt, \\ \beta(t) &= \exp \int_0^t \text{div} f(s, \xi) ds. \end{aligned}$$

Let us suppose that we have the vector field $V = (V_1, V_2)$, then we define $V^\perp = (-V_2, V_1)$ and

$$\kappa = \|V\|^{-3} (V_1 \dot{V}_2 - V_2 \dot{V}_1)$$

where the functions α and β corresponding respectively to the first order variation in time and in displacement transverse to the orbit γ of ξ for orbit of the unperturbed system (3), i.e. for $\epsilon = 0$. In case of $\alpha \neq 0$ or $\beta \neq 0$ then γ is called *normally non-degenerate*, see [7]. In this case γ is an isolated periodic orbit (limit cycle). Finding zeroes of P^1 at points of γ and checking if they are non-degenerate. Then we use the Liapunov-Schmidt reduction lemma, see [5], in order to study the zeroes of function

$$C : \gamma \longrightarrow R$$

which is called the *bifurcation function*. The case of more than one parameter has different structure to the problem.

3. MORE THAN ONE PARAMETER

Suppose that we replace the Eq. (3) by the following equation

$$\dot{x} = f(x) + \epsilon G(x, t, \epsilon) \quad (4)$$

where $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_k) \in R^k$. By setting all ϵ_i equal to zero except one of them nonzero, then we obtain the equation (3). Therefore in this case we put $P^1 = (P_1^1, P_2^1, \dots, P_k^1)$, then

$$P_i^1 : V \longrightarrow R^n$$

Hence P^1 is the map which takes the V to the space of $n \times k$ matrix, where the i^{th} column of the matrix P^1 is P_i^1 . P^1 is called the *bifurcation matrix*. Let us write $\epsilon = ps$ where $p > 0$ and s belong to unit sphere S^{k-1} in R^k . In this case we have

$$\delta(\xi, \epsilon) = \epsilon P^1(\xi) + O(\|\epsilon\|^2) = pP^1(\xi)s + O(p^2) \quad (4.1)$$

where P^1 is the derivative of $P(\epsilon, \xi)$ with respect to $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_k)$ at $\epsilon = (0, 0, \dots, 0)$. From this formula, it is clear that if s_0 does not belong to the $\ker P^1(\xi_0)$, then there are no solution to $\delta(\epsilon, \xi) = 0$ for any ξ in a neighborhood of ξ_0 and any in a cone about s_0 -axis in R^k with sufficiently small $p > 0$. Hence we are able to state the following result:

Lemma 1. A necessary condition for a solution branch to emanate from $\xi_0 \in V$ as ϵ moves from the origin in R^k in the direction of $s_0 \in S^{k-1}$ is that $s_0 \in \ker P^1(\xi_0)$. This condition is the necessary condition but it is not sufficient. However, the condition is sufficient if $s_0 \in \ker P^1(\xi_0)$ in a way that which is non-degenerate with respect to the family of $\ker P^1(\xi)$ as ξ varies near ξ_0 .

Remark. Remember the definition of displacement function $\delta(\xi, \epsilon) = P(\xi, \epsilon) - \xi$. In our problem we are working in R^2 and with two parameters. Since we want to consider the free oscillator system, then we have the non-degenerate case that is to say $\beta(t) \neq 0$ (see lemma 2). Therefore we are looking at the zeroes of the displacement function as $\epsilon \neq (0, 0)$. As we know $\delta(\xi, 0) = P(\xi, 0) - \xi = 0$. Now by writing the Taylor expansion of the Poincare map $P(\xi, \epsilon)$ with respect to $\epsilon = (\epsilon_1, \epsilon_2)$, we get

$$\delta(\xi, \epsilon) = (P_{\epsilon_1}^1(\xi), P_{\epsilon_2}^1(\xi)) \cdot (\epsilon_1, \epsilon_2) + O(\|\epsilon\|^2) \quad (5)$$

where P^1 is a 2×2 matrix. Since we want $\epsilon \neq (0, 0)$, so we look at the $\ker P^1$ which should be one-dimension. Note that equation (5) has the following meaning : $(P_{\epsilon_1}^1, P_{\epsilon_2}^1)$ at $\xi \in [a, b]$ show a curve A in R^2 , then bifurcation happen in the space ϵ when $\epsilon^\perp$ tangent to the curve A . Therefore we have bifurcation when $\epsilon^\perp$ is tangent to the curve A at $A(\xi)$.

4. FREE OSCILLATOR SYSTEM

In this section we consider the free oscillator system under forcing and nonlinear damping perturbation simultaneously. Let us consider the free oscillator system

$$\begin{aligned}\dot{x} &= -\omega_0 y + F(x, y)(r^2 - x^2 - y^2) \\ \dot{y} &= \omega_0 x + G(x, y)(r^2 - x^2 - y^2)\end{aligned}\quad (6)$$

By direct computation, we can state the following lemma

Lemma 2. Suppose that the functions $F(x, y)$ and $G(x, y)$ are C^r , $r > 1$, then

- (i)- The system (6) has the limit cycle $r^2 = x^2 + y^2$.
- (ii)- The function $\alpha(t) \equiv 0$ and $\beta(t) = \exp(-4n\pi) \neq 0$.

Now we perturb the system (6) by a small external periodic forced and nonlinear damping. Since the forced function should be periodic, then by Fourier series one can choose the forced function as $\cos \omega t$ and the nonlinear damping as $g(x, y) = a\dot{x} + b\dot{x}^2 + cx\dot{x}$. Therefore the perturbed system is

$$\begin{aligned}\dot{x} &= -\omega_0 y + x(r^2 - x^2 - y^2) + \epsilon_1 g(x, y) \\ \dot{y} &= \omega_0 x + y(r^2 - x^2 - y^2) + \epsilon_2 \cos \omega t\end{aligned}\quad (7)$$

where ϵ_1, ϵ_2 are small parameters in the space of (ϵ_1, ϵ_2) and the angular frequency ω is closed to 2π . Moreover we allow ω itself vary with ϵ_2 , so that

$$\frac{2\pi}{\omega} = 2\pi + k\epsilon_2 + O(\epsilon_2^2)$$

where we called k as the *detuning* parameter. As point out in [7] it is easy to verify the effect of detuning on the map P^1 is simply to replace the component N by $N + k$. For the system (7) we get the following *result*:

Calculating the integrals that appear in the formula for N and M (see the theorem 2.1) we have:

The limit cycle is

$$\Gamma = (r \cos(t + \theta), r \sin(t + \theta))$$

with initial value

$$(r \cos \theta, r \sin \theta)$$

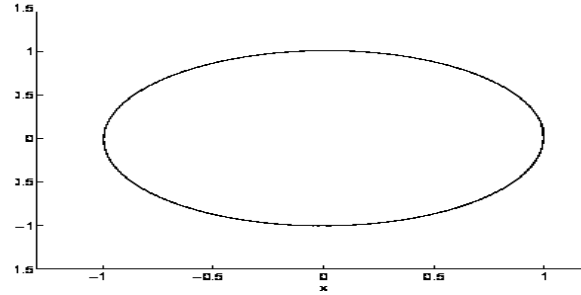


FIGURE 1. Periodic orbit for the system (7) at $\epsilon_1 = 0.001$, $\epsilon_2 = 0.003$, $a = b = 1$, $c = 2$, $t \in [0, 6.38]$ with initial value $(x_0, y_0) = (1, 0)$

Also we have

$$N_{\epsilon_1} = -\omega_0 a \frac{m}{n} r^2 \pi, \quad M_{\epsilon_1} = 0$$

$$M_{\epsilon_2} = 0 \quad \text{if } \omega \neq 1, \quad \text{and} \quad M_{\epsilon_2} = \omega_0 \frac{m}{n} r^2 \sin \theta \quad \text{if } \omega = 1$$

$$N_{\epsilon_2} = 0 \quad \text{if } \omega \neq 1 \quad \text{and} \quad N_{\epsilon_2} = 2\pi \frac{m}{n} \cos \theta \quad \text{at } \omega = 1$$

Theorem 4.1. Effect of external forced and nonlinear damping on the free oscillator system can persist the limit cycle for some values of θ . In fact for a $(\epsilon_1, \epsilon_2) \neq (0, 0)$ and for sufficient small $p > 0$, the system (7) has a 2π periodic limit cycle. Also the detuning K might create a circle which bifurcate from some values of the θ .

Proof. The bifurcation matrix is: $\frac{1}{r^2}(A, B)$ where $A = (N_{\epsilon_1}, M_{\epsilon_1})^T$, and $B = (N_{\epsilon_2} + kr^2, M_{\epsilon_2})^T$. In order to have at least one-dimension ker for the matrix, we put $\sin \theta = 0$. So $\theta = n\pi$, $n = 0, \pm 1, \pm 2, \dots$. Hence at these points the limit cycle can persist, i.e., at 1:1 resonant. Note that the elements of the matrix are at $\omega = 1$. detuning k can satisfied the following equation

$$kr + \pi = 0 \quad \text{or} \quad \pi - kr = 0.$$

Then we have a circle, for each n , bifurcate from the initial values at $\theta = 0$ or $\theta = n\pi$. In fact we have a family of periodic orbits as $\epsilon = (\epsilon_1, \epsilon_2)$ varies in the space (ϵ_1, ϵ_2) . In this last case we have shown the effect of detuning on the system.

Note that in the above result if $a = 0$, i.e. the linear term of damping is equal to zero, then one column of the bifurcation matrix is equal to zero. Hence in this case we have high degenerate.

5. FORCED RAYLEIGH OSCILLATOR

In this section we consider the forced Rayleigh oscillator. The equation is

$$\ddot{x} - \epsilon(1 - \dot{x}^2)\dot{x} + x = B \sin \omega t. \quad (8)$$

This differential equation is basic system describing oscillator. If we put the $\epsilon = 0$ and $B = 0$, then the equation $\ddot{x} + x = 0$ has only periodic solutions. For $\epsilon \neq 0$ and $B \neq 0$, we consider the effect of nonlinear damping and the effect of forcing on this equation. The method is as in the section 4. In fact the system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \epsilon(1 - y^2)y + B \sin \omega t. \end{aligned} \quad (9)$$

Therefore here

$$f = (y, -x), \quad g_1(0, (1 - y^2)y), \quad g_2 = (0, \sin \omega t)$$

The periodic solutions at $\epsilon = B = 0$ are

$$(r \cos(t + \theta), r \sin(t + \theta))$$

with initial value $(r \cos \theta, r \sin \theta)$. For this system the N_ϵ , M_ϵ , N_B and M_B are:

$$\begin{aligned} N_\epsilon &= 0, \quad M_\epsilon = r^2(1 - \frac{3}{4}r^2) \\ N_B &= r\pi \sin \theta + \kappa r \quad \omega = 1, \quad M_B = -r\pi \cos \theta \end{aligned}$$

, The bifurcation matrix for the forced Rayleigh oscillator is $P^1 = \frac{1}{r^2}(A_1, B_1)^T$ where $A_1 = (N_\epsilon, M_\epsilon)$ and $B_1 = (N_B + r^2\kappa, M_B)$. Here κ is the detuning.

Therefore the following results are now easy to check:

Lemma 3.

1. If $\pi \sin \theta + \kappa r \neq 0$, then P^1 has no zero except when $r^2 = \frac{4}{3}$ in which case $\ker P^1$ is spanned by vector $(1, 0)$;
2. If $\pi \sin \theta + \kappa r = 0$, then $\ker P^1$ is spanned by $(-r\pi \cos \theta, r^2(1 - \frac{3}{4}r^2))$.

Note that when $r^2 = \frac{4}{3}$, then the matrix P^1 is in the degenerate case. But this result is the same as in [11], see Fig. 2 as in [11]. *Note that the result in [11]*

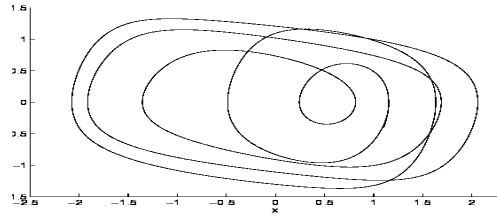


FIGURE 2. Attractor for the system (8) at $\epsilon = 1$, $\omega = 0.588$,
 $B = 0.870$, $t \in [0, 38]$, $(x_0^2 + y_0^2 = \frac{4}{3})$

is by numerical, but our method is analytic. In fact this result shows the effect of harmonic entrainment and forced on the equation.

Now we consider the case 2 in the lemma 3 . In this case $\pi \sin \theta + \kappa r = 0$, which implies the circle $C : r = -\frac{\pi}{\kappa} \sin \theta = a \sin \theta$.

The parameterized of this circle is

$$(x, y) = (\frac{a}{2} \cos \alpha, \frac{a}{2}(1 + \sin \alpha)),$$

where $\alpha \in R \mod 2\pi$. Now we define the map

$$\beta : \xi \longrightarrow \ker P^1(\xi).$$

This map represent the closed curves ,such as A , which are given by the parametrization

$$\varphi : \alpha \longrightarrow (\frac{a}{2} \cos \alpha, \frac{a}{2}(1 + \sin \alpha)(1 - \frac{3}{4}(1 + \sin \alpha)))$$

Thus for $\xi \in C$ and (ϵ, B) -space , we have $(\epsilon, B) \in \ker P^1$ precisely when the line $\epsilon = bB$ intersect the curve A at some value α corresponding to $\xi \in C$. This intersection should be transversal; this implies the fixed points of the Poincare map, see theorem 5.1.

Theorem 5.1. Suppose that $(\xi_0, B_0) \neq (0, 0) \in R^2$ and the line $R(\epsilon_0, B_0)$ intersect the curve A transversally at the point $\varphi(\alpha_0)$ where α_0 represents the point $\xi_0 \in C$. Then for sufficiently small $p \geq 0$ and $(\epsilon, B) = p(\epsilon_0, B_0)$, the forced

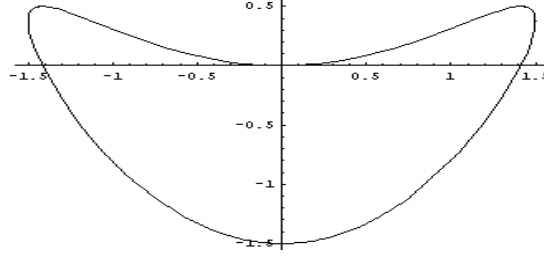


FIGURE 3. This figure drawn for $a = 3$ and $\alpha \in [1, 2\pi]$

Rayleigh oscillator has a 2π -periodic orbit closed to ξ_0 in R^2 .

Proof. As in section 3, we put $\epsilon^- = (\epsilon, B) = ps$. Now from (4.1) the solutions $\delta(\epsilon, B) = 0$ with $p \geq 0$ are given by

$$Q(\xi, s, p) = pP_1(\xi)s + O(p^2) = 0$$

If $s_0 \in \ker P^1(\xi_0)$, then $Q(\xi_0, s_0, 0) = 0$. The transversally assumption means that $Q_s(\xi_0, s_0, 0) \neq 0$. So Implicit function theorem implies that there exists a unique C^∞ solution $s = S(\xi, p)$ with ξ close to ξ_0 and for small p and with $s_0 = S(\xi_0, p)$. Therefore we have *periodic orbits*.

This result shows *how the forced Raleigh oscillator equation can changes in the two parameter space (ϵ, B)* .

The following figure (Fig. 3) shows one of the closed curve A . It is obviously that the line $\epsilon = bB$, for some b , can transversally intersect the curve A . Therefore this line provide us with the intersection point(s) which are the fixed point(s) of the Poincare map or the zero(es) of the displacement map $\delta(\xi, \epsilon, B)$. Hence as the parameter b varies and we draw the different closed curves, we will have a family of the periodic orbits.

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