GENERAL BOOSTS IN LORENTZIAN PLANE E_1^2

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Abstract. In this paper we obtained the equations of boosts (rotations in the Lorentzian plane E_1^2) about an arbitrary point (h, k) and showed that the set of all boosts is not a group.

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1. Introduction

Gans gave the equations of general rotations about an arbitrary point (h, k) and the theorems about resultant of translations and rotations in the Euclidean plane (see [1]).

We have generalized Gans' work to Lorentzian plane E_1^2 and prove the corresponding theorems therein.

Note that the Lorentzian plane is the Metric space $E_1^2 = \left(\mathbb{R}^2, <,>\right)$, where the metric is given by

$$< X, Y >= x_1y_1 - x_2y_2 \ , \qquad X = (x_1, x_2) \, , \; Y = (y_1, y_2).$$

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2. Equations of General Boosts

There are four kinds of isometries in two-dimensional Lorentz-Minkowski space E_1^2. They are the following :

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \quad , \quad \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ -\sinh\alpha & -\cosh\alpha \end{pmatrix}$$

$$\begin{pmatrix} -\cosh\alpha & \sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \quad , \quad \begin{pmatrix} -\cosh\alpha & \sinh\alpha \\ \sinh\alpha & -\cosh\alpha \end{pmatrix}.$$

The ones whose determinants are $+1$ are rotations (boosts), the ones whose determinants are -1 are reflections (see [2]). Hence the boosts about the origin are given by the matrices

$$R_\alpha = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \quad , \quad \begin{pmatrix} -\cosh\alpha & \sinh\alpha \\ \sinh\alpha & -\cosh\alpha \end{pmatrix} = B_\alpha.$$

Let $O^{'} = (h, k)$ be an arbitrary point, α be the angle of rotation, and let $P' = \left(x^{'}, y^{'}\right)$, denote the image of $P = (x, y)$.

We express the rotation in terms of an auxiliary $\xi, \eta-$coordinate system, with origin $O^{'}$, whose axis are parallel to x, y axis and similarly directed.

If the coordinats of P and $P^{'}$ in the auxiliary system are (ξ, η) and $\left(\xi^{'}, \eta^{'}\right)$, then we have

$$\left(\xi^{'}, \eta^{'}\right) = (\xi\cosh\alpha + \eta\sinh\alpha, \xi\sinh\alpha + \eta\cosh\alpha)\,. \tag{1}$$

Denote this boost by

$$\begin{array}{cccccc} f: & E_1^2 & \to & E_1^2 & & \\ & (\xi, \eta) & \to & f(\xi, \eta) & = & \left(\xi^{'}, \eta^{'}\right) \end{array}$$

and (respectively denote the second boosts by

$$\begin{array}{cccccc} g: & E_1^2 & \to & E_1^2 & & \\ & (\xi, \eta) & \to & g(\xi, \eta) & = & \left(\xi^{'}, \eta^{'}\right) \end{array}$$

where in this case

$$\left(\xi^{'}, \eta^{'}\right) = (-\xi\cosh\alpha + \eta\sinh\alpha, \xi\sinh\alpha - \eta\cosh\alpha)\,). \tag{$1^{'}$}$$

The relations between the original and the auxiliary coordinates of P, $P^{'}$ are

$$(x, y) = (\xi + h, \eta + k) \text{ and } \left(x^{'}, y^{'}\right) = \left(\xi^{'} + h, \eta^{'} + k\right) \tag{2}$$

substituting the values of ξ, η, ξ', η' into (1) (respectively into (1')) we obtain the equations we have been seeking. Thus the boosts through angle α about the point (h,k) has the equations

(3)
$$\left(x' - h, y' - k\right) = ((x-h)\cosh\alpha + (y-k)\sinh\alpha, (x-h)\sinh\alpha + (y-k)\cosh\alpha)$$

(respectively

$$(3') \qquad \left(x' - h, y' - k\right) = (-(x-h)\cosh\alpha + (y-k)\sinh\alpha, (x-h)\sinh\alpha - (y-k)\cosh\alpha)).$$

Hence

$$(4) \qquad \begin{cases} \left(x', y'\right) = (x\cosh\alpha + y\sinh\alpha + a, x\sinh\alpha + y\cosh\alpha + b) \\ \text{where} \quad a = h(1-\cosh\alpha) - k\sinh\alpha \\ \qquad\qquad b = k(1-\cosh\alpha) - h\sinh\alpha \end{cases}$$

(respectively from (3')

$$(4') \qquad \begin{cases} \left(x', y'\right) = (-x\cosh\alpha + y\sinh\alpha + c, x\sinh\alpha - y\cosh\alpha + d) \\ \text{where} \quad c = h(1+\cosh\alpha) - k\sinh\alpha \\ \qquad\qquad d = k(1+\cosh\alpha) - h\sinh\alpha \end{cases} .$$

Equation (4) (respectively equation (4') are seen to be the resultant TR_α (respectively SB_α) of the following boost. R_α about O and translation T (respectively following boost B_α about O and translation S)

$$\begin{aligned} \left(x', y'\right) &= R_\alpha(x,y) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x\cosh\alpha + y\sinh\alpha, x\sinh\alpha + y\cosh\alpha) \end{aligned}$$

$$\left(x'', y''\right) = T\left(x', y'\right) = \left(x' + a, y' + b\right)$$

(respectively

$$\begin{aligned} \left(x', y'\right) &= B_\alpha(x,y) = \begin{pmatrix} -\cosh\alpha & \sinh\alpha \\ \sinh\alpha & -\cosh\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (-x\cosh\alpha + y\sinh\alpha, x\sinh\alpha - y\cosh\alpha) \end{aligned}$$

$$\left(x'', y''\right) = S\left(x', y'\right) = \left(x' + c, y' + d\right) \;).$$

Therefore the boost represented by (3) is equal to TR_α (respectively the boost represented by $(3^{'})$ is equal to SB_α).

Theorem 1. *The equations of the boost about an arbitrary point through an angle α is given by the equation in (4) (respectively in $(4^{'})$).*

Proof.

$$\begin{array}{llll} f: & E_1^2 & \to & E_1^2 \\ & (x,y) & \to & f(x,y) = \begin{array}{l} (x\cosh\alpha + y\sinh\alpha + h(1-\cosh\alpha) - k\sinh\alpha, \\ x\sinh\alpha + y\cosh\alpha + k(1-\cosh\alpha) - h\sinh\alpha) \end{array} \end{array}$$

$$\begin{array}{llll} g: & E_1^2 & \to & E_1^2 \\ & (x,y) & \to & g(x,y) = \begin{array}{l} (-x\cosh\alpha + y\sinh\alpha + h(1+\cosh\alpha) - k\sinh\alpha, \\ x\sinh\alpha - y\cosh\alpha + k(1+\cosh\alpha) - h\sinh\alpha) \end{array} \end{array}$$

Note that for f and g we have

$$d\left(f(P), f(Q)\right) = d(P,Q), \quad d(g(P), g(Q)) = d(P,Q),$$

where

$$d(P,Q) = \left\| \overrightarrow{PQ} \right\| = \sqrt{< \overrightarrow{PQ}, \overrightarrow{PQ} >}.$$

Hence f and g are isometries. Moreover

$$f(h,k) = (h,k) \text{ and } g(h,k) = (h,k).$$

Thus f and g are the boosts about the point (h,k) and through an angle α. □

Using this same R (respectively B), we can also find a translation $T^{'}$ (respectively $S^{'}$) such that the boost (3) (respectively boost $(3^{'})$) is equal to $RT^{'}$ (respectively $BS^{'}$). To prove this we write

$$\left(x^{'}, y^{'}\right) = T^{'}(x,y) = \left(x + a^{'}, y + b^{'}\right)$$

and

$$\left(x^{''}, y^{''}\right) = R\left(x^{'}, y^{'}\right) = \left(x^{'}\cosh\alpha + y^{'}\sinh\alpha, x^{'}\sinh\alpha + y^{'}\cosh\alpha\right)$$

(respectively

$$\left(x^{'}, y^{'}\right) = S^{'}(x,y) = \left(x + c^{'}, y + d^{'}\right)$$

and

$$\left(x^{''}, y^{''}\right) = B\left(x^{'}, y^{'}\right) = \left(-x^{'}\cosh\alpha + y^{'}\sinh\alpha, x^{'}\sinh\alpha - y^{'}\cosh\alpha\right) \quad).$$

Hence we get

$$\begin{aligned}\left(x^{''},y^{''}\right) &= RT^{'}(x,y)\\ &= \left((x+a^{'})\cosh\alpha+(y+b^{'})\sinh\alpha,(x+a^{'})\sinh\alpha+(y+b^{'})\cosh\alpha\right)\end{aligned}$$

(respectively

$$\begin{aligned}\left(x^{''},y^{''}\right) &= BS^{'}(x,y)\\ &= \left(-(x+c^{'})\cosh\alpha+(y+d^{'})\sinh\alpha,(x+c^{'})\sinh\alpha-(y+d^{'})\cosh\alpha\right))\end{aligned}$$

or

$$\begin{aligned}\left(x^{''},y^{''}\right) = RT^{'}(x,y) = (&x\cosh\alpha+y\sinh\alpha+(a^{'}\cosh\alpha+b^{'}\sinh\alpha),\\ &x\cosh\alpha+y\sinh\alpha+(a^{'}\sinh\alpha+b^{'}\cosh\alpha))\end{aligned}\tag{5}$$

(respectively

$$\begin{aligned}\left(x^{''},y^{''}\right) = BS^{'}(x,y) = (&-x\cosh\alpha+y\sinh\alpha+(d^{'}\sinh\alpha-c^{'}\cosh\alpha),\\ &x\sinh\alpha-y\cosh\alpha+(c^{'}\sinh\alpha-d^{'}\cosh\alpha))\ ).\end{aligned}\tag{5$^{'}$}$$

Now we find $a^{'}$, $b^{'}$ (respectively $c^{'}$, $d^{'}$) so that (5) (respectively ($5^{'}$)) is the same transformation as (4) (respectively ($4^{'}$)) i.e. from $TR = RT^{'}$ (respectively $SB = BS^{'}$) we have

$$\begin{aligned}a^{'} &= -k\sinh\alpha+h(\cosh\alpha-1)\\ b^{'} &= -h\sinh\alpha+k(\cosh\alpha-1)\end{aligned}$$

(respectively

$$c^{'} = -k\sinh\alpha-h(\cosh\alpha+1)$$

$$d^{'} = -h\sinh\alpha-k(\cosh\alpha+1)\ ).$$

It is readily seen that $T \neq T^{'}$ and $S \neq S^{'}$. Thus we have proved the following:

Theorem 2. *Any given boost not the identity, through an angle about a point other than the origin is the resultant of a boost R_α (respectively B_α) through α about the origin followed by a translation T, or if a translation $T^{'}$ followed by R_α (respectively B_α). The two translations are distinct, uniquely determined, and neither translation is I.*

Corollary 1. *If T is a translation and R_α (respectively B_α) is a boost through α about the origin and neither transformation is I, then $R_\alpha T$ (respectively $B_\alpha T$) and TR_α (respectively TB_α) are distinct boosts through α about points other than the origin.*

3. Resultants of Translations and Boosts

It is readily checked that the resultant of two translations is a translation and that the resultant of two boosts about origin $O = (0,0)$ is a boost about that point i.e.

$$R_\alpha R_\beta = R_{\alpha+\beta},\ B_\alpha B_\beta = R^{-1}_{\alpha+\beta} = R_{-(\alpha+\beta)}$$

$$R_\alpha B_\beta = B_{\beta-\alpha}, B_\beta R_\alpha = B_{\beta-\alpha}$$

(i) Let R_α and R_β be two boosts about the same point (h,k) through α, and β respectively. Clearly

$$R_\alpha = TR^{'}_\alpha \text{ and } R_\beta = R^{''}_\beta T^{'}$$

where $R^{'}_\alpha$, $R^{''}_\beta$ are boosts about $O = (0,0)$ through α, and β respectively. T and $T^{'}$ are translations.

Hence $R_\alpha R_\beta = TR^{'}_\alpha R^{''}_\beta T^{'}$. From the above discussion $R^{'''}_{\alpha+\beta} = R^{'}_\alpha R^{''}_\beta$ is a boost about $O = (0,0)$ through $\alpha+\beta$. Therefore $R_\alpha R_\beta = TR^{'''}_{\alpha+\beta}T^{'}$.

(ii) Let R_α and R_β be two boosts about different points (h,k) and $\left(h^{'},k^{'}\right)$. Again it is easy to see that

$$R_\alpha R_\beta = TR^{'''}_{\alpha+\beta}T^{'},$$

where $R^{'''}_{\alpha+\beta}$ is a boost about $O = (0,0)$.

(iii) Let R_α and B_β are two boosts about different points (h,k) and $\left(h^{'},k^{'}\right)$. Hence we have

$$R_\alpha = TR^{'}_\alpha \text{ and } B_\beta = B^{'}_\beta S^{'},$$

where $R^{'}_\alpha$, $B^{'}_\beta$ are boosts about $O = (0,0)$ and T and $S^{'}$ are translations.

Therefore $R_\alpha B_\beta = TR^{'}_\alpha B^{'}_\beta S^{'}$. Note that $R^{'}_\alpha B^{'}_\beta = B^{''}_{\beta-\alpha}$ is a boost about $O = (0,0)$. For

$$B_\beta R_\alpha = (S^{''}B^{'}_\beta)(R^{''}_\alpha T^{'}),$$

where $B^{'}_\beta R^{''}_\alpha = B^{''}_{\beta-\alpha}$ is boost about $O = (0,0)$.

(iv) Let B_α and B_β are two boosts about different points.

Clearly

$$B_\alpha = SB_\alpha^{'},\ B_\beta = B_\beta^{''}S^{'},$$

where $B_\alpha^{'}$, $B_\beta^{''}$ are boosts about $O=(0,0)$. Hence

$$B_\alpha B_\beta = SB_\alpha^{'}B_\beta^{''}S^{'} = SR_{\alpha+\beta}^{-1}S^{'} = SR_{-(\alpha+\beta)}S^{'}.$$

$R_{-(\alpha+\beta)}$ is a boost through $-(\alpha+\beta)$ about $O=(0,0)$.

Thus in all cases problem reduces to

$$TR_{\alpha+\beta}^{'''}T^{'},\ S^{''}B_{\beta-\alpha}^{''}T^{'},\ TB_{\beta-\alpha}^{''}S^{''} \text{ and } SR_{-(\alpha+\beta)}S^{'},$$

where $R_{\alpha+\beta}^{'''}$, $B_{\beta-\alpha}^{''}$, $R_{-(\alpha+\beta)}$ are boosts about $O=(0,0)$, S, T, $S^{'}$, $T^{'}$, $S^{''}$ are translations. From Corollary 1 we know that the resultant of a translation and a boosts about $O=(0,0)$ (when neither is the identify) is a boost about a point not O. Hence it remains to determine the nature of the resultant of (a) a translation and a boost about a point not O, and (b) two boosts about any points. In doing this we assume that none of these transformations is the identity.

(a) Let T (respectively S) be a translation, R (respectively B) a boost about a point not O, and consider TR (respectively SB). By theorem 2, we know that $R=T^{'}R^{'}$ (respectively $B=S^{'}B^{'}$) where $T^{'}$ (respectively $S^{'}$) is a translation and $R^{'}$ (respectively $B^{'}$) is a boost about O. Hence using the associative property of transformations, we have that

$$TR = T(T^{'}R^{'}) = (TT^{'})R^{'} = T_1R^{'}$$

(respectively

$$SB = S(S^{'}B^{'}) = (SS^{'})B^{'} = S_1B^{'}\quad)$$

where T_1 (respectively S_1) is a translation. Note that $T_1R^{'}$ (respectively $S_1B^{'}$) is simply the boost $R^{'}$ (respectively $B^{'}$) if $T_1=I$ (respectively $S_1=I$), and by corollary 1 it is a boost about a point not O if $T_1\neq I$ (respectively $S_1\neq I$). Thus, TR (respectively SB) is always a boost. In the same way RT (respectively BS) can be shown to be a boost.

(b) Let R, $R^{'}$ (respectively B, $B^{'}$ or respectively R, B) be boosts about any points. If neither point is O, then by theorem 2, $R=TR_1$ and $R^{'}=R_1^{'}T^{'}$ (respectively $B=SB_1$ and $B^{'}=B_1^{'}S^{'}$ or respectively $R=TR_1$ and $B=B_1^{'}S^{'}$) where R_1, $R_1^{'}$, B_1, $B_1^{'}$ are boosts about O and T, $T^{'}$, S, $S^{'}$ are translations. These equations are still true if one of the given boosts, say R (respectively B), is about O, except that then $T = I$ (respectively $S = I$). In any case then, we have

$$RR^{'} = (TR_1)(R_1^{'}T^{'}) = T(R_1R_1^{'})T^{'} = TR_2T^{'} = (TR_2)T^{'}$$

(respectively

$$BB^{'} = (SB_1)(B_1^{'}S^{'}) = S(B_1B_1^{'})S^{'} = SR_3S^{'},$$

respectively

$$RB = (TR_1B_1^{''}S^{'}) = T(R_1B_1^{''})S^{''} = (TB_2)S^{''},$$

where R_2, (respectively R_3) (respectively B_2) is a boost about O. If $R_2 = I$ (respectively $R_3 = I$ and respectively $B_2 = I$) which occurs if R_1, $R_1^{'}$ (respectively B_1, $B_1^{'}$, respectively R_1, $B_1^{''}$) are mutually inverse then $RR^{'}$ (respectively $BB^{'}$ and RB) is clearly a translation. If R_2 (respectively R_3, respectively B_2) $\neq I$ then TR_2 (respectively SR_3, respectively TB_2) is a boost whose center is not O. Unless, $T = I$ (respectively $S = I$ or respectively $T = I$) using the discussion above it follows that $TR_2T^{'}$ (respectively $SR_3S^{'}$ respectively $TB_2S^{''}$), and hence $RR^{'}$ (respectively $BB^{'}$ and respectively RB) is a boost. Same argument works for BR.

Thus we have proved:

Theorem 3. *The resultant in either order of a translation ($\neq I$) and a boost ($\neq I$) is a boost. The resultant of two boosts about any point is a boost or a translation.*

Considering the fact that the inverses of translations and boosts are translations and boosts respectively, it is clear that we have also proved:

Theorem 4. *The set of all translations and boosts is a group. The set off all boosts is not a group.*

Remark 1. *Note that set of all R_α's about $O = (0, 0)$ is a group. Yet set off all B_α's about O is not a group. Note also that a translation and a boost are generally not commutative. This was already apperant from corollary 1.*

The same is true of two boosts,

Example 1. *Let $R_{\pi/2}$, $B_{\pi/2}$ be boosts through $\pi/2$ about $(0, 0)$, $(0, 1)$ respectively it is easy to see that $R_{\pi/2}B_{\pi/2} \neq B_{\pi/2}R_{\pi/2}$.*

References

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