ALGEBRAIC COMPUTATION OF SPIN COEFFICIENTS IN NEWMAN-PENROSE FORMALISM USING MATHEMATICA

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Abstract. Newman-Penrose (NP)formalism [5, 13], in general relativity is a tetrad formalism in which various geometrical quantities are projected on a chosen null tetrad basis. Spin coefficients in NP formalism replace connection coefficients in geometry. Applications of spin coefficients are discussed by many authors, in diverse fields of general relativity and cosmology, like Ahsan [1], Ahsan et al [2], Hasmani and Ahsan [8], Hasmani and Katkar [10] and others. The computations involved are very lengthy and complicated. Motivated by such a large number of applications and looking to the complexity involved, we have exploited some features of computer algebra system Mathematica to carry out algebraic computation of spin coefficients in Newman-Penrose formalism for given metric and null tetrad. In this paper the required Mathematica program is also included.

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1. INTRODUCTION

The physical space of general relativity is a four dimensional Riemannian space characterized by a metric

$$ds^2 = g_{ij}dx^i dx^j,$$

where g_{ij} are components of metric tensor, which is a symmetric second rank tensor and x^i (i =1 to 4) denote the space-time coordinates. The components of metric are functions of space-time coordinates. Here and in what follows, we have used Einstein's summation convention, according to which a repeated index in product is summed over the entire range. As we are discussing four dimensional space-times, the indices will take values 1 to 4. Various geometrical quantities are defined as derivatives and their combinations of the metric tensor.

Newman-Penrose formalism was proposed by Newman and Penrose[13] in which a null basis comprising of a pair of real null vectors l^i and n^i and a pair of complex conjugate null vectors m^i and $\bar{m}^i$ is chosen so that

$$l^i m_i \; = \; l^i \bar{m}_i \; = \; n^i m_i \; = \; n^i \bar{m}_i \; = \; 0$$

$$l^i l_i \; = \; n^i n_i \; = \; m^i m_i \; = \; \bar{m}^i \bar{m}_i \; = \; 0$$

and the normalization condition

$$l^i n_i \; = \; 1, \; m^i \bar{m}_i \; = \; -1$$

The basis vectors, regarded as directional derivatives , can be denoted as

$$D \; = \; l^i \, \frac{\partial}{\partial x^i}, \triangle \; = \; n^i \, \frac{\partial}{\partial x^i}$$

$$\delta \; = \; m^i \, \frac{\partial}{\partial x^i}, \; \bar{\delta} \; = \; \bar{m}^i \, \frac{\partial}{\partial x^i}$$

Once a field of complex null tetrad is assigned in space-time, the tetrad formalism can then be used for describing geometric objects.

The contravariant components of the metric tensor in terms of null tetrad are

$$g^{ij} = l^i n^j + n^i l^j - m^i \bar{m}^j - \bar{m}^i m^j$$

while

$$g_{ij} = l_i n_j + n_i l_j - m_i \bar{m}_j - \bar{m}_i m_j$$

The Ricci rotation coefficients lead to the following spin-coefficients [5]

$$\kappa = \gamma_{311} = l_{i;j} m^i l^j, \; \tau = \gamma_{312} = l_{i;j} m^i n^j$$

$$\sigma = \gamma_{313} = l_{i;j} m^i m^j, \ \rho = \gamma_{314} = l_{i;j} m^i \bar{m}^j$$

$$\pi = \gamma_{241} = -n_{i;j} \bar{m}^i l^j, \ \nu = \gamma_{242} = -n_{i;j} \bar{m}^i n^j$$

$$\mu = \gamma_{243} = -n_{i;j} \bar{m}^i m^j, \ \lambda = \gamma_{244} = -n_{i;j} \bar{m}^i \bar{m}^j$$

$$\epsilon = \frac{1}{2}(\gamma_{211} + \gamma_{341}) = \frac{1}{2}(l_{i;j} n^i l^j - m_{i;j} \bar{m}^i l^j)$$

$$\gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}) = \frac{1}{2}(l_{i;j} n^i n^j - m_{i;j} \bar{m}^i n^j)$$

$$\beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}) = \frac{1}{2}(l_{i;j} n^i m^j - m_{i;j} \bar{m}^i m^j)$$

$$\alpha = \frac{1}{2}(\gamma_{214} + \gamma_{344}) = \frac{1}{2}(l_{i;j} n^i \bar{m}^j - m_{i;j} \bar{m}^i \bar{m}^j),$$

where a semicolon followed by an index (say j) denotes covariant derivative with respect to the corresponding space-time coordinate (x^j). It is defined by

$$u_{a;b} = \frac{\partial u_a}{\partial x^b} - \Gamma^h_{ab} u_h,$$

further Γ^i_{jk} are called Christoffel symbols of second kind and they are defined by

$$\Gamma^i_{jk} = \frac{1}{2} g^{ih} \left(\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right)$$

The spin coefficients defined above may be complex; and complex conjugate of any quantity can be obtained by replacing the index 3, wherever it occurs, by the index 4, and conversely.

Applications of Newman-Penrose formalism are discussed by many authors, in diversified fields of general relativity and cosmology, to quote a few, Ahsan et al [2] have expressed various kinematical quantities like acceleration, twist, expansion etc in terms of spin coefficients, Hasmani and Ahsan [8] have expressed Raychaudhari's equation in terms of spin coefficients, Hasmani and Katkar [10] have used spin coefficients to study electric like behavior of Weyl tensor, Hasmani and Joshi [9] have expressed Electric and Magnetic parts of Weyl conformal tensor, Ahsan et al [3, 4] have studied Weyl-Lanczos equation and solutions are obtained using Newman-Penrose formalism, and analyzed gravitational fields. Further importance of this formalism is discussed in [12].

2. MATHEMATICA NOTEBOOK

It is known that computations in general relativity are very lengthy and complicated, hence there is a high risk of errors and also a lot of time goes in getting required equations. These difficulties can be overcome if computer is used to carry out computations. Also, this saves a lot of time and also ensures error-free results. It is need-less to say that the time saved can be fruitfully used to analyze the results. Mathematica is a Computer Algebra System with many features and easy to use commands which can be used to carry out various computations useful in relativity. Earlier, Hasmani and Rathva [11] had used Mathematica to carry out algebraic computation of components of Ricci tensor. More recently, Hasmani [7] has used Mathematica to compute complex scalars representing Weyl tensor in Newman-Penrose formalism. Here, we have used Mathematica to carry out algebraic computation of spin coefficients in Newman-Penrose formalism.

In the program written for this purpose simple commands for derivative of a function and matrix manipulation are used [14]. Also, some logical commands are used for the output to avoid repetition due to symmetry properties of quantities under consideration. The file consisting of the commands is called a MATHEMATICA Notebook. The coordinates, metric components and the chosen tetrad are required input. Using the input the program computes spin coefficients, complex scalars representing Weyl conformal tensor. The output is designed so that coordinates, non-zero metric components, chosen tetrad and spin coefficients are displayed.

The Mathematica Notebook for the computation is pasted below. On running this notebook Mathematica prompts for the input in terms of coordinates, metric components and the null tetrad chosen, then the computations are done by the software and the output is generated.

MATHEMATICA NOTEBOOK

```
t1 = SessionTime[];
n1 = 4;
For[i = 1, i <= n1, i++, msg = "Enter Co-ordinate - " <> ToString[i];
 c[i] = Input[msg]]
```

```
coor = Table[c[i], {i, n1}];
o1 = 4;
For[i = 1, i <= o1, i++,
 For[j = 1, j <= o1, j++,
  msg = "Metric Element - Row-" <> ToString[i] <> " Col-" <>
    ToString[j];  m[i, j] = Input[msg]]]
metric = Table[m[i, j], {i, o1}, {j, o1}];
c1 = 4;
For[i = 1, i <= c1, i++, msg = "Enter NPL element - " <> ToString[i];
 np1[i] = Input[msg];]
npl = Table[np1[i], {i, c1}];
For[i = 1, i <= c1, i++, msg = "Enter NPN element - " <> ToString[i];
 np2[i] = Input[msg];]
npn = Table[np2[i], {i, c1}];
For[i = 1, i <= c1, i++, msg = "Enter NPM element - " <> ToString[i];
 np3[i] = Input[msg];]
npm = Table[np3[i], {i, c1}];
For[i = 1, i <= c1, i++,
 msg = "Enter NPMBAR element - " <> ToString[i]; np4[i] = Input[msg];]
npmbar = Table[np4[i], {i, c1}];
For[i = 1, i <= 4, i++, xi[i] = coor[[i]]]
For[i = 1, i <= 4, i++,
 For[j = 1, j <= 4, j++, glowij[i, j] = metric[[i, j]]]]
gup = Inverse[metric];
For[i = 1, i <= 4, i++,
  For[j = 1, j <= 4, j++, gupij[i, j] = gup[[i, j]]]];
gama = Table[(1/2) (D[metric[[i, k]], xi[j]] +
      D[metric[[j, k]], xi[i]] - D[metric[[i, j]], xi[k]]), {i,
    4}, {j, 4}, {k, 4}];
For[i = 1, i <= 4, i++,
  For[j = 1, j <= 4, j++,
   For[k = 1, k <= 4, k++, gamaijcomak[i, j, k] = gama[[i, j, k]]]]];
gamaup =
```

```
  Table[Sum[gupij[h, k] gamaijcomak[i, j, k], {k, 4}], {h, 4}, {i,
    4}, {j, 4}];
For[h = 1, h <= 4, h++,
  For[i = 1, i <= 4, i++,
   For[j = 1, j <= 4, j++, gamahij[h, i, j] = gamaup[[h, i, j]]]]];
Print ["The coordinates chosen are"]
For[i = 1, i <= 4, i++, Print[" x", i, "  =  ", coor[[i]]]]
Print["The non-zero components of metric are"]
For[i = 1, i <= 4, i++,
 For[j = 1, j <= 4, j++,
  If[metric[[i, j]] =!= 0, Print["g", i, j, "  =  ", metric[[i, j]]]]]]
Print["The tetrad chosen is"]
Print["\!\(\*SuperscriptBox[\"l\", \"i\"]\) = ", npl]
Print["\!\(\*SuperscriptBox[\"n\", \"i\"]\) = ", npn]
Print["\!\(\*SuperscriptBox[\"m\", \"i\"]\) = ", npm]
Print["\!\(\*OverscriptBox[SuperscriptBox[\"m\", \"i\"], \"_\"]\) = \
", npmbar]
npllow = metric.npl;
npnlow = metric.npn;
npmlow = metric.npm;
npmbarlow = metric.npmbar;
nplisemij =
  Table[D[npllow[[i]], xi[j]] -
    Sum[gamahij[h, i, j] npllow[[h]], {h, 4}], {j, 4}, {i, 4}];
npnisemij =
  Table[D[npnlow[[i]], xi[j]] -
    Sum[gamahij[h, i, j] npnlow[[h]], {h, 4}], {j, 4}, {i, 4}];
npmisemij =
  Table[D[npmlow[[i]], xi[j]] -
    Sum[gamahij[h, i, j] npmlow[[h]], {h, 4}], {j, 4}, {i, 4}];
npmbarisemij =
  Table[D[npmbarlow[[i]], xi[j]] -
    Sum[gamahij[h, i, j] npmbarlow[[h]], {h, 4}], {j, 4}, {i, 4}];
```

```
kappa = FullSimplify[npm.nplisemij.npl];
tau = FullSimplify[npm.nplisemij.npn];
sigma = FullSimplify[npm.nplisemij.npm];
rho = FullSimplify[npm.nplisemij.npmbar];
pi = FullSimplify[-npmbar.npnisemij.npl];
nu = FullSimplify[-npmbar.npnisemij.npn];
lambda = FullSimplify[-npmbar.npnisemij.npmbar];
mu = FullSimplify[-npmbar.npnisemij.npm];
epsilon =
  FullSimplify[(1/2) (npn.nplisemij.npl - npmbar.npmisemij.npl)];
gamma = FullSimplify[(1/2) (npn.nplisemij.npn - npmbar.npmisemij.npn)];
beta = FullSimplify[(1/2) (npn.nplisemij.npm - npmbar.npmisemij.npm)];
alpha = FullSimplify[(1/2) (npn.nplisemij.npmbar -
      npmbar.npmisemij.npmbar)] ;
Print["\[Kappa]= ", kappa]
Print["\[Tau] = ", tau]
Print["\[Sigma] = ", sigma]
Print["\[Rho] = ", rho]
Print["\[Pi] = ", pi]
Print["\[Nu] = ", nu]
Print["\[Mu] = ", mu]
Print["\[Lambda] = ", lambda]
Print["\[Epsilon]  = ", epsilon]
Print["\[Gamma] = ", gamma]
Print["\[Beta] = ", beta]
Print["\[Alpha] = ", alpha]
t2 = SessionTime[];
Print["Time taken = ", t2 - t1, "seconds"]
```

3. Computation for the Gödel Universe

The geometry of the Gödel universe is described by the metric

$$ds^2 \;=\; dt^2 \;-\; dx^2 - dy^2 \;+\; \frac{1}{2}e^{2qy}dz^2 \;+\; 2e^{qy}dz\;dt,$$

where the parameter q is related to the vorticity of the fluid. The input for Gödel metric is as follows,

```
coor = {x, y, z, t};
metric = {{-1, 0, 0, 0}, {0, -1, 0, 0},
{0, 0, (1/2)Exp[2q y],Exp[q  y]}, {0, 0, Exp[q  y], 1}};
npl = {1/Sqrt[2], 0, 0, 1/Sqrt[2]};
npn = {-1/Sqrt[2], 0, 0, 1/Sqrt[2]};
npm = {0, 1/Sqrt[2], -I  Exp[-q y], I };
npmbar = {0, 1/Sqrt[2], I  Exp[-q y], -I };
```

Using these values as input when prompted the following output is generated ,

The coordinates chosen are
x1 = x
x2 = y
x3 = z
x4 = t

The non-zero components of metric are
g11 = -1
g22 = -1
g33 = $\frac{1}{2}e^{2qy}$
g34 = e^{qy}
g43 = e^{qy}
g44 = 1

The tetrad chosen is
$l^i = \left\{\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right\}$
$n^i = \left\{-\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right\}$
$m^i = \left\{0, \frac{1}{\sqrt{2}}, -ie^{-qy}, i\right\}$
$\overline{m^i} = \left\{0, \frac{1}{\sqrt{2}}, ie^{-qy}, -i\right\}$

The spin coefficients are

$\kappa = 0$

$\tau = 0$

$\sigma = 0$

$\rho = -\frac{iq}{2}$

$\pi = 0$

$\nu = 0$

$\mu = -\frac{iq}{2}$

$\lambda = 0$

$\epsilon = -\frac{iq}{4}$

$\gamma = -\frac{iq}{4}$

$\beta = \frac{q}{2\sqrt{2}}$

$\alpha = -\frac{q}{2\sqrt{2}}$

4. Conclusion

We have written a program for computing spin coefficients in NP formalism using simple commands of computer algebra system Mathematica. The program is tested for many metrics. The results presented here are for the Gödel metric. It takes around 90 seconds to compute these expressions, the time taken includes the time for input also.The results presented here are checked with the results obtained by Cohen et al [6] . It is hoped that the program presented here will be useful to researchers in general relativity; also it is hoped that some useful programs will be written in this direction. We have applied the results to Gödel metric and we have shown that the metric is of Petrov type-D. Further it is shown that the parameter q characterizing the rotation enters in the expression of Ψ_2, this can be regarded as the radiation is due to rotation.

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