

ACCESSIBILITY AND EVERYWHERE CHAOS

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ABSTRACT. A TDS (X, f) is called N -accessible if for any $\varepsilon > 0$ and any N nonempty open subsets $U_1, U_2, \dots, U_N$ of X there are N points $x_1 \in U_1, x_2 \in U_2, \dots, x_N \in U_N$ and a positive integer n such that $\max\{d(f^n(x_i), f^n(x_j)) : i, j \in \{1, 2, \dots, N\}\} < \varepsilon$. A TDS (X, f) is called N -sensitive if there is a positive number τ such that in every nonempty open subset U of X there are N distinct points $x_1, x_2, \dots, x_N$ of U and a positive integer n with $\min\{d(f^n(x_i), f^n(x_j)) : i, j \in \{1, 2, \dots, N\}, i \neq j\} > \tau$. A TDS (X, f) is called (N_1, N_2) -everywhere chaotic if it is N_1 -sensitive and N_2 -accessible. We point out that there is a system which is (N_1, N_2) -everywhere chaotic, but it is not $(N_1 + 1, N_2)$ -everywhere chaotic for any positive integers N_1 and N_2 .

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1. INTRODUCTION

Throughout this paper a topological dynamical system (TDS for short) is a pair (X, f) , where X is a metric space with a metric d and $f : X \rightarrow X$ is a continuous surjective map. The complexity of a TDS is a centrum topic of the research since the notion of chaos is introduced by Li and Yorke [3] in 1975. There have been several

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attempts to give new description of chaos. The concept of everywhere chaos was introduced by Hisao Kato [2] in 1996. A TDS (X, f) is called everywhere chaotic if it satisfies the following conditions.

(1) (X, f) is sensitive [1]. That is, there exists a positive number τ such that in every nonempty open subset U of X there are points x_1, x_2 of U and a positive integer n with $d(f^n(x_1), f^n(x_2)) > \tau$.

(2) (X, f) is accessible [2]. That is, for any $\varepsilon > 0$ and any nonempty open subsets U_1, U_2 of X , there are points $x_1 \in U_1$ and $x_2 \in U_2$ and a positive integer n such that $d(f^n(x_1), f^n(x_2)) < \varepsilon$.

Recently, the notion of N -sensitivity ($N \geq 2$) is introduced [6]. A TDS (X, f) is called N -sensitive if there is a positive number τ such that in every nonempty open subset U of X there are N distinct points $x_1, x_2, \dots, x_N$ of U and a positive integer n with

$$\min\{d(f^n(x_i), f^n(x_j)) : i, j \in \{1, 2, \dots, N\}, i \neq j\} > \tau.$$

Such a τ is called an N -sensitivity constant of f . When $N = 2$, (X, f) is usually sensitivity.

There is TDS which is N -sensitive but not $(N + 1)$ -sensitive for any positive integer $N \geq 2$ ([5],[8]).

Inspired by these idea, we define the notions of N -accessibility ($N \geq 2$), (N_1, N_2) -everywhere chaos and N -Ruelle-Takens chaos in this article.

A TDS (X, f) is called N -accessible if for any $\varepsilon > 0$ and any N nonempty open subsets $U_1, U_2, \dots, U_N$ of X there are N points $x_1 \in U_1, x_2 \in U_2, \dots, x_N \in U_N$ and a positive integer n such that

$$\max\{d(f^n(x_i), f^n(x_j)) : i, j \in \{1, 2, \dots, N\}\} < \varepsilon.$$

When $N = 2$, (X, f) is usually accessibility [2].

A TDS (X, f) is called (N_1, N_2) -everywhere chaotic if it is N_1 -sensitive and N_2 -accessible. When $N_1 = N_2 = 2$, (X, f) is usually everywhere chaos [2].

A TDS (X, f) is called *N-Ruelle-Takens chaotic* if (X, f) is topologically transitive and N -sensitive. When $N = 2$, (X, f) is usually Ruelle-Takens chaos [4].

This paper is organized as follows. In Section 2, we study N -accessibility. Finally, we investigate the (N_1, N_2) -everywhere chaos, and point out that there is a system which is (N_1, N_2) -everywhere chaotic, but it is not $(N_1 + 1, N_2)$ -everywhere chaotic for any positive integers N_1 and N_2 .

In this article, positive integers, nonnegative integers and positive rational numbers are denoted by $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{Q}_+$ respectively. For a TDS (X, f) let $\text{orb}(x, f) = \{f^n(x) : n = 0, 1, 2, \dots\}$. Write N -th product system of (X, f) is TDS $(X^N, f^{(N)})$ for an integer $N \geq 2$, where X^N is the cartesian product of X with itself N times and $f^{(N)}$ represents the simultaneous action of f in each coordinate of X^N .

2. N -ACCESSIBILITY

An N -tuple $(x_1, x_2, \dots, x_N) \in X^N$ ($N \geq 2$) is N -proximal [8] if there exists an increasing sequence of positive integers $\{n_k\}_{k=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} d(f^{n_k}(x_i), f^{n_k}(x_j)) = 0 \text{ for any } i, j \in \{1, 2, \dots, N\}.$$

The set consisting of all N -proximal tuples is denoted by $P_N(X, f)$. When $N = 2$, $P_2(X, f)$ is the proximal relation of X . It is easy to see that

$$P_N(X, f) = \bigcap_{k=1}^\infty \bigcup_{j=1}^\infty (f \times f \times \dots \times f)^{-j} \Delta_N(1/k),$$

where $\Delta_N(1/k) = \{(x_1, x_2, \dots, x_N) : d(x_i, x_j) < 1/k, i, j \in \{1, 2, \dots, N\}\}$. Let

$$Q_N(X, f) = \bigcap_{k=1}^\infty \overline{\bigcup_{j=1}^\infty (f \times f \times \dots \times f)^{-j} \Delta_N(1/k)}.$$

When $N = 2$, $Q_2(X, f)$ is the regionally proximal relation on X . An N -tuple $(x_1, x_2, \dots, x_N) \in X^N$ is N -regionally proximal [8] if $(x_1, x_2, \dots, x_N) \in Q_N(X, f)$. It is easy to see that $(x_1, x_2, \dots, x_N) \in Q_N(X, f)$ if and only if for any $\varepsilon > 0$ and any neighborhood U_i of x_i there are $x'_i \in U_i$ and a positive integer k such that $d(f^k(x'_i), f^k(x'_j)) < \varepsilon$ for any $i, j \in \{1, 2, \dots, N\}$.

Proposition 2.1 Suppose that (X, d) is a metric space. Let (X, f) be a TDS.

- (1) (X, f) is N -accessible;
- (2) $P_N(X, f)$ is dense in X^N ;
- (3) $Q_N(X, f) = X^N$.

Then (2) $\Rightarrow$ (3) and (3) $\Leftrightarrow$ (1). If X is complete, then above all assertions are equivalent.

Proof. (2) $\Rightarrow$ (3). Since $P_N(X, f) \subset Q_N(X, f)$ and $Q_N(X, f)$ is closed, then $\overline{P_N(X, f)} \subset Q_N(X, f)$. Thus $Q_N(X, f) = X^N$ by (2).

(3) $\Rightarrow$ (1). Suppose that $Q_N(X, f) = X^N$. For any $\varepsilon > 0$ and any N nonempty open subsets $U_1, U_2, \dots, U_N$ of X , choose $x_1 \in U_1, x_2 \in U_2, \dots, x_N \in U_N$, then $(x_1, x_2, \dots, x_N) \in Q_N(X, f)$. So there are N points $x'_1 \in U_1, x'_2 \in U_2, \dots, x'_N \in U_N$ and a positive integer k such that $d(f^k(x'_i), f^k(x'_j)) < \varepsilon$ for any $i, j \in \{1, 2, \dots, N\}$. Thus $\max_{i,j \in \{1,2,\dots,N\}} \{d(f^k(x'_i), f^k(x'_j))\} < \varepsilon$. Hence (X, f) is N -accessible.

(1) $\Rightarrow$ (3). Assume that (1) holds. Let $(x_1, x_2, \dots, x_N) \in X^N$. For any $\varepsilon > 0$ and any neighborhood U_i of x_i , by (1) then there are $x'_i \in U_i$ and a positive integer k such that $\max\{d(f^k(x'_i), f^k(x'_j)) : i, j \in \{1, 2, \dots, N\}\} < \varepsilon$. This implies $(x_1, x_2, \dots, x_N) \in Q_N(X, f)$. Thus $Q_N(X, f) = X^N$.

Suppose that X is complete. Next we show that (1) $\Rightarrow$ (2).

For each $m \in \mathbb{N}$ and each $\gamma \in \mathbb{Q}_+$, consider the set

$$P_{m,\gamma} = \{(x_1, x_2, \dots, x_N) \in X^N : \exists k > m, \max_{i,j \in \{1,2,\dots,N\}} \{d(f^k(x_i), f^k(x_j))\} < \gamma\}.$$

Now we prove that $P_{m,\gamma}$ is a dense open subset of X^N . It is obviously that $P_{m,\gamma}$ is open in X^N . Suppose that $U_1, U_2, \dots, U_N$ are N nonempty open subsets of X . Choose $x_1 \in U_1, x_2 \in U_2, \dots, x_N \in U_N$. If $\max_{i,j \in \{1,2,\dots,N\}} d(f^m(x_i), f^m(x_j)) = 0$, then $(x_1, x_2, \dots, x_N) \in P_{m,\gamma}$. If $\max_{i,j \in \{1,2,\dots,N\}} d(f^m(x_i), f^m(x_j)) > 0$, put $\varepsilon = \frac{1}{2} \min\{\max_{i,j \in \{1,2,\dots,N\}} d(f^k(x_i), f^k(x_j)) : 0 \leq k \leq m\}$. Since (X, f) is N -accessible, then there exist $x'_1 \in U_1, x'_2 \in U_2, \dots, x'_N \in U_N$ and a positive integer t such that $\max_{i,j \in \{1,2,\dots,N\}} d(f^t(x'_i), f^t(x'_j)) < \min\{\gamma, \varepsilon\}$. If we choose x'_i such that x'_i is sufficiently close to x_i for $i = 1, 2, \dots, N$, we may assume that $t > m$. This implies that $(x'_1, x'_2, \dots, x'_N) \in P_{m,\gamma}$. Thus $P_{m,\gamma}$ is dense in X^N . Since

$P_N(X, f) = \bigcap_{m=1}^{\infty} \bigcap_{\gamma \in \mathbb{Q}_+} P_{m, \gamma}$, then $P_N(X, f)$ is dense in X^N .

A factor map $\pi : (X, f) \rightarrow (Y, g)$ between dynamical systems is a continuous surjective map $\pi : X \rightarrow Y$ such that $g \circ \pi = \pi \circ f$.

Proposition 2.2 Let $\pi : (X, f) \rightarrow (Y, g)$ be a factor map between systems. Suppose that π is uniformly continuous. If (X, f) is N -accessible then so is (Y, g) .

Proof. Let $\varepsilon > 0$. Suppose that $U_1, U_2, \dots, U_N$ are N nonempty open subsets of Y . Then every $\pi^{-1}(U_i) (1 \leq i \leq N)$ is a nonempty open subset of X . As π is uniformly continuous, then there is $\delta > 0$ such that $d_2(\pi(x), \pi(y)) < \varepsilon$ when $d_1(x, y) < \delta$. Since (X, f) is N -accessible, then there are N points $x_1 \in \pi^{-1}(U_1), x_2 \in \pi^{-1}(U_2), \dots, x_N \in \pi^{-1}(U_N)$ and a positive integer n such that

$$\max\{d_1(f^n(x_i), f^n(x_j)) : i, j \in \{1, 2, \dots, N\}\} < \delta.$$

So

$$\max\{d_2(\pi(f^n(x_i)), \pi(f^n(x_j))) : i, j \in \{1, 2, \dots, N\}\} < \varepsilon.$$

As $\pi(f^n(x_i)) = g^n(\pi(x_i))$, then

$$\max\{d_2(g^n(\pi(x_i)), g^n(\pi(x_j))) : i, j \in \{1, 2, \dots, N\}\} < \varepsilon.$$

Hence, (Y, g) is N -accessible.

Next we recall some notions about inverse limit spaces.

Let (X, d) be a compact metric space. Suppose that f is continuous surjective map. Put

$$\tilde{X} = \{\bar{x} = (x_0, x_1, x_2, \dots) : f(x_{i+1}) = x_i, x_i \in X \text{ for any } i \in \mathbb{Z}_+\}.$$

The space $\tilde{X}$ is called the inverse limit space of (X, f) . The f is called the bonding map. The $\tilde{X}$ is a compact metric space with a metric D : $D(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}$ for any $\bar{x} = (x_0, x_1, x_2, \dots), \bar{y} = (y_0, y_1, y_2, \dots) \in \tilde{X}$.

The shift map $\sigma : \tilde{X} \rightarrow \tilde{X}$ is defined by

$$\sigma((x_0, x_1, x_2, \dots)) = (f(x_0), x_0, x_1, x_2, \dots) \text{ for any } \bar{x} = (x_0, x_1, x_2, \dots) \in \tilde{X}.$$

It is easy to see that the shift map σ is a homeomorphism. For any $i \in \mathbb{Z}_+$ and for any $\bar{x} = (x_0, x_1, x_2, \dots) \in \tilde{X}$ the projection $\pi_i : \tilde{X} \rightarrow X$ is defined by

$\pi_i((x_0, x_1, x_2, \dots)) = x_i$. Every $\pi_i : \tilde{X} \rightarrow X$ is a continuous and open surjective map. In particular, $\pi_i : (\tilde{X}, \sigma) \rightarrow (X, f)$ is a factor map.

Theorem 2.3 Let (X, d) be a compact metric space and $f : X \rightarrow X$ be continuous surjective map. Then the shift map $\sigma : \tilde{X} \rightarrow \tilde{X}$ is N -accessible if and only if $f : X \rightarrow X$ is N -accessible.

Proof. Suppose that f is N -accessible. Put $\text{diam}(X) = M$. Then for any $\varepsilon > 0$ there exists a positive integer m such that $\sum_{i=m+1}^{\infty} \frac{M}{2^i} < \varepsilon/4$. Because f is uniformly continuous, then there is $\delta > 0$ such that $d(f^k(a), f^k(b)) < \varepsilon/4$ for $k = 0, 1, 2, \dots, m$ when $d(a, b) < \delta$. Let $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_N$ be N nonempty open subsets of $\tilde{X}$. Then for above given $\delta > 0$ there exist a positive integer l (choose $l > m$) and $x_m^1 \in \pi_m(\tilde{U}_1), x_m^2 \in \pi_m(\tilde{U}_2), \dots, x_m^N \in \pi_m(\tilde{U}_N)$ such that $\max\{d(f^l(x_m^i), f^l(x_m^j)) : i, j \in \{1, 2, \dots, N\}\} < \delta$. So $\max\{d(f^{l+k}(x_m^i), f^{l+k}(x_m^j)) : i, j \in \{1, 2, \dots, N\}\} < \varepsilon/4$ for $k = 0, 1, 2, \dots, m$. Put $\overline{x^i} = (f^m(x_m^i), f^{m-1}(x_m^i), \dots, f(x_m^i), x_m^i, x_{m+1}^i, \dots)$ for $i = 1, 2, \dots, N$. Thus

$$\sigma^l(\overline{x^i}) = (f^{m+l}(x_m^i), f^{m+l-1}(x_m^i), \dots, f(x_m^i), x_m^i, x_{m+1}^i, \dots).$$

Hence one has $D(\sigma^l(\overline{x^i}), \sigma^l(\overline{x^j})) \leq \varepsilon/4(1 + 1/2 + \dots + 1/2^m) + \sum_{i=m+1}^{\infty} \frac{M}{2^i} < \varepsilon$ for any $i, j \in \{1, 2, \dots, N\}$. Thus $\max\{D(\sigma^l(\overline{x^i}), \sigma^l(\overline{x^j})) : i, j \in \{1, 2, \dots, N\}\} < \varepsilon$. i.e., σ is N -accessible.

The converse assertion is trivial by Proposition 2.2.

3. (N_1, N_2) -EVERYWHERE CHAOS

Proposition 3.1 If a TDS (X, f) is topologically weak-mixing, then (X, f) is (N_1, N_2) -everywhere chaotic for any positive integers N_1 and N_2 .

Proof. (1) Let $\varepsilon > 0$. We choose N_1 nonempty open subset $V_1, V_2, \dots, V_{N_1}$ of X such that

$$\min\{d(V_i, V_j) : i, j \in \{1, 2, \dots, N_1\}, i \neq j\} > \varepsilon.$$

Suppose that U is a nonempty open subset of X . Since (X, f) is topologically weak-mixing, then $(X^{N_1}, f^{(N_1)})$ is topologically weak-mixing [7]. So $(X^{N_1}, f^{(N_1)})$ is transitive. Then there is a positive integer k such that $f^{(k)}(U \times U \times \cdots \times U) \cap (V_1 \times V_2 \times \cdots \times V_{N_1}) \neq \emptyset$, i.e. there exist N_1 distinct points $x_1, x_2, \dots, x_{N_1} \in U$ such that $f^k(x_1) \in V_1, f^k(x_2) \in V_2, \dots, f^k(x_{N_1}) \in V_{N_1}$. Hence $\min_{i,j \in \{1,2,\dots,N_1\}, i \neq j} d(f^k(x_i), f^k(x_j)) > \varepsilon$, i.e. (X, f) is N_1 -sensitive.

(2) For any $\delta > 0$, choose a nonempty open subset V of X satisfying $\text{diam}(V) < \delta$. Since $(X^{N_2}, f^{(N_2)})$ is transitive, then for any N_2 nonempty open subsets $U_1, U_2, \dots, U_{N_2}$ of X , there is a positive integer t such that $f^{(t)}(U_1 \times U_2 \times \cdots \times U_{N_2}) \cap (V \times V \times \cdots \times V) \neq \emptyset$. This implies that there is $x_i \in U_i$ such that $f^t(x_i) \in V$ for any $i \in \{1, 2, \dots, N_2\}$. So $\max_{i,j \in \{1,2,\dots,N_2\}} d(f^t(x_i), f^t(x_j)) < \delta$. Thus (X, f) is N_2 -accessible.

Hence (X, f) is (N_1, N_2) -everywhere chaotic by (1) and (2).

The following example comes from [2]. The example shows that the converse of Proposition 3.1 is not true.

Example 3.2 Let $[0, 1] \rightarrow [0, 1]$ be a map defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \in [0, 1/2] \\ -2x + 2, & \text{if } x \in [1/2, 3/4] \\ 2x - 1, & \text{if } x \in [3/4, 1] \end{cases}$$

Note that for any nonempty open subset U of $[0, 1]$ there is $n \in \mathbb{Z}_+$ such that $f^n(U)$ contains an open set of $[1/2, 1]$, and $f|_{[1/2, 1]} : [1/2, 1] \rightarrow [1/2, 1]$ is topologically mixing. By using these facts, we see that f is (N_1, N_2) -everywhere chaotic for any positive integers N_1 and N_2 . But f is not topologically weak-mixing.

The following lemma can be as a corollary of Proposition 3.5 in [6]. We give a direct proof for completeness here.

Lemma 3.3 Suppose that a TDS (X, f) has a transitive point. If (X, f) has a fixed point, then for any positive integer N there is a dense set $L_N \subset X^N$ satisfying for any point $(x_1, x_2, \dots, x_N)$ in L_N , we have

$$\liminf_{n \rightarrow \infty} \max\{d(f^n(x_i), f^n(x_j)) : i, j \in \{1, 2, \dots, N\}\} = 0.$$

Hence (X, f) is N -accessible for any positive integer N .

Proof. Let x be a fixed point of (X, f) and let ω be a transitive point of (X, f) . Now we show that for every point $z = (z_1, z_2, \dots, z_N) \in (\text{orb}(\omega, f))^N$ one has

$$\liminf_{n \rightarrow \infty} \max\{d(f^n(z_i), f^n(z_j)) : i, j \in \{1, 2, \dots, N\}\} = 0.$$

For any $i \in \{1, 2, \dots, N\}$ there is a positive integer k_i such that $z_i = f^{k_i}(\omega)$. Since ω is a transitive point, then there exists an increasing sequence of positive integers $\{n_t\}$ such that $\lim_{t \rightarrow \infty} f^{n_t}(\omega) = x$. Then we have $\lim_{t \rightarrow \infty} f^{n_t}(z_i) = \lim_{t \rightarrow \infty} f^{n_t}(f^{k_i}(\omega)) = \lim_{t \rightarrow \infty} f^{k_i}(f^{n_t}(\omega)) = f^{k_i}(x) = x$ for any $i \in \{1, 2, \dots, N\}$. Hence one has $\lim_{t \rightarrow \infty} d(f^{n_t}(z_i), f^{n_t}(z_j)) = 0$ for any $i, j \in \{1, 2, \dots, N\}$, this implies that

$$\liminf_{n \rightarrow \infty} \max\{d(f^n(z_i), f^n(z_j)) : i, j \in \{1, 2, \dots, N\}\} = 0.$$

Since $(\text{orb}(\omega, f))^N$ is dense in X^N . Put $L_N = (\text{orb}(\omega, f))^N$. Then the set L_N satisfies the desired condition. By Proposition 2.1, (X, f) is N -accessible.

The following example comes from [8]. The example shows that there is a system which is (N_1, N_2) -everywhere chaotic, but it is not $(N_1 + 1, N_2)$ -everywhere chaotic for any positive integers N_1 and N_2 .

Example 3.4 Firstly we construct a recurrent point $x \in \{0, 1\}^{\mathbb{Z}_+}$ as follows: Let $A_1 = 1010 \dots 10, k_1 = |A_1| > N_1$ and

$$A_2 = A_1 0^{k_1} A_1 0^{k_1} 10^{k_1-1} 0^{k_1} A_1 0^{k_1} 110^{k_1-2} 0^{k_1} \dots A_1 0^{k_1} 1^{N_1-2} 0^{k_1-N_1+2} 0^{k_1} A_1 0^{k_1}.$$

Inductively, when A_n is defined, let $k_n = |A_n|$ and

$$A_{n+1} =$$

$$A_n 0^{k_n} A_n 0^{k_n} 10^{k_n-1} 0^{k_n} A_n 0^{k_n} 110^{k_n-2} 0^{k_n} \dots A_n 0^{k_n} 1^{N_1-2} 0^{k_n-N_1+2} 0^{k_n} A_n 0^{k_n}.$$

It is clear that $x = \lim_{n \rightarrow \infty} A_n$ is a recurrent point. Let $(X, \sigma) = (\overline{\text{orb}(x, \sigma)}, \sigma)$. Then (X, σ) is transitive, not minimal as $0 = (0, 0, \dots) \in X$. Since (X, σ) is transitive and has a fixed point 0. By Lemma 3.3 for any positive integer N_2 we have (X, σ) is N_2 -accessible. By [8] (X, σ) is N_1 -sensitive, not $N_1 + 1$ -sensitive. So

(X, σ) is (N_1, N_2) -everywhere chaotic, but it is not $(N_1 + 1, N_2)$ -everywhere chaotic.

A map is semi-open if the image of a nonempty open subset contains a nonempty open subset.

We need following lemma which comes from [5].

Lemma 3.5 Let $\pi : (X, f) \rightarrow (Y, g)$ be a factor map between systems. If π is semi-open and (Y, g) is N -sensitive, then so is (X, f) .

Theorem 3.6 Let (X, d) be a compact metric space and $f : X \rightarrow X$ be a continuous surjective map. Then the shift map $\sigma : \tilde{X} \rightarrow \tilde{X}$ is N -sensitive if and only if f is N -sensitive.

Proof. Suppose that $\sigma : \tilde{X} \rightarrow \tilde{X}$ is N -sensitive with an N -sensitivity constant $\delta > 0$. Put $\text{diam}(X) = M$. Then there exists a positive integer m such that $\sum_{i=m+1}^{\infty} \frac{M}{2^i} < \delta/4$. Because f is uniformly continuous, then there exists $\gamma > 0$ (choose $\gamma < \delta/4$) such that $d(f^k(a), f^k(b)) < \delta/4$ for $k = 0, 1, 2, \dots, m$, when $d(a, b) < \gamma$.

Next we show that (X, f) is N -sensitive with an N -sensitivity constant γ .

If this is not the case, then there exists a nonempty open subset U of X , for any N distinct points $x_m^1, x_m^2, \dots, x_m^N$ and any positive integer l one has $\min\{d(f^l(x_m^i), f^l(x_m^j)) : i, j \in \{1, 2, \dots, N\}, i \neq j\} \leq \gamma$. Since $(\tilde{X}, \sigma)$ is N -sensitive with an N -sensitivity constant δ , then there exist $\overline{y^1}, \overline{y^2}, \dots, \overline{y^N} \in \pi_m^{-1}(U)$ and a positive integer t such that

$$\min\{D(\sigma^t(\overline{y^i}), \sigma^t(\overline{y^j})) : i, j \in \{1, 2, \dots, N\}, i \neq j\} > \delta (*)$$

Because $\pi_m(\overline{y^1}) = y_m^1, \pi_m(\overline{y^2}) = y_m^2, \dots, \pi_m(\overline{y^N}) = y_m^N \in U$. Without loss of generality, we assume that $d(f^t(\pi_m(\overline{y^1})), f^t(\pi_m(\overline{y^2}))) \leq \gamma$.

Since

$$\begin{aligned} \sigma^t(\overline{y^1}) &= (f^{t+m}(y_m^1), f^{t+m-1}(y_m^1), \dots, f^m(y_m^1), \dots, f(y_m^1), y_m^1, y_{m+1}^1, \dots) \\ &= (f^m(f^t(y_m^1)), f^{m-1}(f^t(y_m^1)), \dots, f^t(y_m^1), \dots, f(y_m^1), y_m^1, y_{m+1}^1, \dots) \end{aligned}$$

and

$$\begin{aligned}\sigma^t(\overline{y^2}) &= (f^{t+m}(y_m^2), f^{t+m-1}(y_m^2), \dots, f^m(y_m^2), \dots, f(y_m^2), y_m^2, y_{m+1}^2, \dots) \\ &= (f^m(f^t(y_m^2)), f^{m-1}(f^t(y_m^2)), \dots, f^t(y_m^2), \dots, f(y_m^2), y_m^2, y_{m+1}^2, \dots)\end{aligned}$$

Then $D(\sigma^t(\overline{y^1}), \sigma^t(\overline{y^2})) < \delta/4(1+1/2+\dots+1/2^m) + \sum_{i=m+1}^{\infty} \frac{M}{2^i} < \delta$. This contradicts to (*).

The converse assertion is trivial by Lemma 3.5.

By Theorem 2.3 and Theorem 3.6 we have the following theorem.

Theorem 3.7 Let (X, f) be a TDS. Suppose that X is compact metric space. Then $(\tilde{X}, \sigma)$ is (N_1, N_2) -everywhere chaotic if and only if so is (X, f) .

Theorem 3.8 Suppose that (X, f) has a fixed point. If (X, f) is N_1 -Ruelle-Takens chaotic, then (X, f) is (N_1, N_2) -everywhere chaotic for any positive integer N_2 .

Proof. If (X, f) has a fixed point, then it is N_2 -accessible by Lemma 3.3. Hence (X, f) is (N_1, N_2) -everywhere chaotic.

Ye and Zhang show that the following lemma [8].

Lemma 3.9 Suppose that X is a compact metric space. A Devaney chaotic system (X, f) is N -sensitive for any positive integer N .

By Theorem 3.8 and Lemma 3.9, following corollary holds.

Corollary 3.10 Let X be compact metric space. Suppose that (X, f) is Devaney chaotic. Then for any positive integer N_1 , (X, f) is N_1 -Ruelle-Takens chaotic, and (X, f) is (N_1, N_2) -everywhere chaotic for any positive integer N_2 if it has a fixed point.

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