

ON INVARIANT SUBMANIFOLDS IN GENERALIZED SASAKIAN SPACE FORMS

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ABSTRACT. In this article, we establish some inequalities for invariant submanifolds involving totally real sectional curvature and the scalar curvature. The equality cases are also discussed.

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1. INTRODUCTION

Perhaps one of the most significant aspects of submanifold theory is that which deals with the relations between the main extrinsic invariants and the main intrinsic invariants of a submanifold. B.Y. Chen [5] introduced a series of invariants on a Kaehler manifold and proved several general inequalities involving these invariants for Kaehler submanifolds in complex space forms. In [11] the authors established similar inequalities for invariant submanifolds in locally conformal almost cosymplectic manifolds. In [1] the authors introduced the notion of a generalized Sasakian

space form as an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ whose curvature tensor is given by

$$(1.1) \quad \begin{aligned} \bar{R}(X, Y)Z = & f_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned}$$

where f_1, f_2, f_3 are differential functions on $\bar{M}$.

Sasakian space forms are natural examples of generalized Sasakian space forms with constant functions. It is known that generalized Sasakian space forms exist with non-constant functions and arbitrary dimensions. In [2] the authors studied trans-Sasakian generalized Sasakian space forms with examples of such spaces. In the present paper, we study invariant submanifolds of a generalized Sasakian space form.

The paper is organized as follows. In section 2, necessary details about submanifolds and generalized Sasakian space forms are reviewed. In section 3, we derive the relations involving totally real sectional curvature and the scalar curvature for invariant submanifolds of a generalized Sasakian space form. The inequalities involving invariant $\delta_k^r(p)$ [5], are established for invariant submanifolds in the last section.

2. PRELIMINARIES

A $(2m+1)$ -dimensional Riemannian manifold $(\bar{M}, g)$ is said to be an almost contact metric manifold if there exist on $\bar{M}$, a $(1,1)$ tensor field ϕ , a vector field ξ and 1-form η such that

$$(2.1) \quad \eta(\xi) = 1, \phi^2 X = -X + \eta(X)\xi \text{ and}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on $\bar{M}$. Then, $\phi(\xi) = 0$ and $\eta o \phi = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is called the fundamental 2-form of $\bar{M}$. The almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\bar{M} \times IR$ defined by

$$(2.2) \quad J\left(X, \lambda \frac{d}{dt}\right) = \left(\phi X - \lambda \xi, \eta(X) \frac{d}{dt}\right),$$

is integrable, where X is tangent to $\bar{M}$, t is the coordinate on IR and λ is a smooth function on $\bar{M} \times IR$.

The condition for (ϕ, ξ, η, g) being normal is equivalent to the vanishing of the torsion tensor

$$(2.3) \quad [\phi, \phi] + 2d\eta \otimes \xi,$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ .

An almost contact metric manifold is called Sasakian manifold if

$$(2.4) \quad (\bar{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad \bar{\nabla}_X \xi = -\phi X,$$

for any vector fields X, Y on $\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection of g .

In [13], J.A. Oubina introduced the notion of a trans-Sasakian structure. An almost contact metric manifold $\bar{M}$ is a trans-Sasakian manifold if there exist two smooth functions α and β on $\bar{M}$ such that

$$(2.5) \quad (\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y, \phi X)),$$

for any vector fields X, Y on $\bar{M}$. In this case we say that the trans-Sasakian structure is of type (α, β) . From (2.5) it follows that

$$(2.6) \quad \bar{\nabla}_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi),$$

$$(2.7) \quad (\bar{\nabla}_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

Let $\bar{M}(f_1, f_2, f_3)$ be a $(2m+1)$ -dimensional generalized Sasakian space form. For each plane section $\pi \subset T_p \bar{M}, p \in \bar{M}$, we denote by $\bar{K}(\pi)$, the sectional curvature of the plane section π .

Let $\{e_1, e_2, \dots, e_n, \dots, e_{2m+1}\}$ be an orthonormal basis of the tangent space $T_p \bar{M}(f_1, f_2, f_3)$. Then the scalar curvature τ of $\bar{M}$ is defined by

$$(2.8) \quad \tau = \sum_{i < j} \bar{K}(e_i, e_j), \quad i, j = 1, \dots, n, \dots, 2m+1,$$

where $\bar{K}(e_i, e_j)$ is the sectional curvature of the section spanned by e_i and e_j .

A plane section $\pi \subset T_p \bar{M}$ is called totally real if $\phi\pi$ is perpendicular to π and is denoted by π^r . A plane section $\pi \subset T_p \bar{M}$ is called a ϕ -section if it is spanned by X and ϕX , where X is a unit tangent vector in $T_p \bar{M}$ orthogonal to the structure vector field ξ . In this case, the sectional curvature $\bar{K}(X, \phi X)$ is called the ϕ -sectional curvature of π and is denoted by $H(X)$.

Let M be an $(n+1)$ -dimensional submanifold of $\bar{M}(f_1, f_2, f_3)$ and denote by h , ∇ and $\nabla^\perp$ the second fundamental form of M , the induced connection on M and the normal bundle $T^\perp M$. Then the Gauss and Weingarten formulae are given respectively by

$$(2.9) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.10) \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

for all vector fields X, Y tangent to M and a vector field N normal to M , where A_N is the shape operator in the direction of N . The second fundamental form and

the shape operator are related by

$$(2.11) \quad g(h(X, Y), N) = g(A_N X, Y).$$

Let R be a Riemannian curvature tensor of M , then the Gauss equation is given by

$$(2.12) \quad \begin{aligned} \bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) \\ - g(h(X, Z), h(Y, W)), \text{ for all } X, Y, Z, W \in TM. \end{aligned}$$

Let $\{e_1, e_2, \dots, e_{n+1}\}$ be an orthonormal basis of $T_p M$, $p \in M$. Then the mean curvature vector $H(p)$ and the scalar curvature $\tau(p)$ are given respectively by

$$(2.13) \quad H(p) = \frac{1}{n+1} \sum_{i=1}^{n+1} h(e_i, e_i), \quad \tau(p) = \sum_{i < j} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j .

The submanifold M is totally geodesic in $\bar{M}$ if $h = 0$, and minimal if $H = 0$.

We put

$$(2.14) \quad h_{ij}^r = g(h(e_i, e_j), e_r) \text{ and}$$

$$||h||^2 = \sum_{i,j=1}^{n+1} g(h(e_i, e_j), h(e_i, e_j)).$$

A submanifold M tangent to the structure vector field ξ is said to be an invariant submanifold of $\bar{M}$ if there exists a ϕ -invariant distribution D (i.e. $\phi(D) = D$) on M such that

$TM = D \oplus IR \cdot \xi$ [4]. We assume that $\dim M = n+1$, $\dim D = 2d$, where $n = 2d$.

3. TOTALLY REAL SECTIONAL CURVATURE FOR INVARIANT SUBMANIFOLDS

Let $M(n \geq 2)$ be an invariant submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Then, we have orthogonal decomposition $T_p M = D(p) \oplus IR \cdot \xi$, $p \in$

M . We choose an orthonormal basis $\{e_1, \dots, e_d, \bar{e}_1 = \phi e_1, \dots, \bar{e}_d = \phi e_d, e_{2d+1} = \xi\}$ for $T_p M$ and an orthonormal basis $\{\alpha_1, \dots, \alpha_{m-d}, \bar{\alpha}_1 = \phi \alpha_1, \dots, \bar{\alpha}_{m-d} = \phi \alpha_{m-d}\}$ for $T_p^\perp M$. Then, with respect to such an orthonormal basis, the (1,1) tensor field ϕ on M is given by

$$(3.1) \quad \phi = \begin{pmatrix} 0 & -I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where I_d denotes an identity matrix of degree d . If $\bar{M}$ is (α, β) trans-Sasakian manifold, then by (2.5) and (2.9), we have

$$(3.2) \quad \phi h(X, Y) = h(\phi X, Y), \quad h(\phi X, \phi Y) = -h(X, Y),$$

for all $X, Y \in TM$.

For each $k \in IR$, $p \in M$, we define an invariant δ_k^r by

$$(3.3) \quad \delta_k^r(p) = \tau(p) - k \inf K^r(p),$$

where $\inf K^r(p) = \inf_{\pi^r} \{K(\pi^r)\}$ and π^r runs over all totally real plane sections in $D_p \subset T_p M$.

Lemma 3.1. *Let $M(n \geq 2)$ be an invariant submanifold of a trans-Sasakian generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Then, we have*

$$(3.4) \quad \begin{aligned} & K(X, Y) + K(X, \phi Y) - \frac{3}{2}(f_1 - f_2) \\ &= \frac{1}{4}\{H(X + \phi Y) + H(X - \phi Y) + H(X + Y) + H(X - Y) - H(X) - H(Y)\}, \end{aligned}$$

for all orthonormal vector fields X, Y orthogonal to the structure vector field ξ , with $g(X, \phi Y) = 0$.

Proof. For each point $p \in M$ and any orthonormal vectors X, Y orthogonal to the structure vector field ξ , with $g(X, \phi Y) = 0$, from (1.1), (2.12) and (3.2), we have

$$(3.5) \quad K(X, Y) + K(X, \phi Y) = 2(f_1 - \|h(X, Y)\|^2),$$

$$(3.6) \quad H(X) + H(Y) = 2(f_1 + 3f_2 - \|h(X, X)\|^2 - \|h(Y, Y)\|^2).$$

The lemma 3.1 follows from (3.5) and (3.6).

Theorem 3.2. *Let $M(n \geq 2)$ be an invariant submanifold of a trans-Sasakian generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Then, we have*

$$(3.7) \quad \inf K^r(p) \leq f_1, \quad p \in M.$$

The equality in (3.7) holds at $p \in M$ if and only if p is a totally geodesic point.

Proof. We put

$$(3.8) \quad D^1(p) = \{X \in D(p) : g(X, X) = 1\}$$

and $W_p = \{(X, Y) : X, Y \in D^1(p) \text{ such that } g(X, Y) = g(X, \phi Y) = 0\}.$

Then W_p is a closed subset of $D^1(p) \times D^1(p)$. It is easy to verify that if $\{X, Y\}$ spans a totally real plane section, then both $\{X + \phi Y, X - \phi Y\}$ and $\{X + Y, X - Y\}$ also span totally real plane sections. We define a function $\hat{H} : W_p \rightarrow \mathbb{R}$ by

$$(3.9) \quad \hat{H}(X, Y) = H(X) + H(Y), \quad (X, Y) \in W_p,$$

then there exists $(\bar{X}, \bar{Y}) \in W_p$, such that $\hat{H}(X, Y)$ attains an absolute maximum value. From (3.4), we have

$$(3.10) \quad K(\bar{X}, \bar{Y}) + K(\bar{X}, \phi \bar{Y}) - \frac{3}{2}(f_1 - f_2) \leq \frac{1}{4}\hat{H}(\bar{X}, \bar{Y}).$$

On the other hand, for a unit tangent vector $X \in D^1(p)$, it is easy to see that

$$(3.11) \quad H(X) \leq f_1 + 3f_2.$$

Thus, we obtain from (3.10) and (3.11) that

$$(3.12) \quad K(\bar{X}, \bar{Y}) + K(\bar{X}, \phi\bar{Y}) \leq 2f_1,$$

which implies (3.7).

If the equality in (3.7) holds at p , then

$$(3.13) \quad K(X, Y) + K(X, \phi Y) \geq 2 \inf K^r = 2f_1.$$

Therefore, from (3.5) and (3.13) we have

$$(3.14) \quad h(X, Y) = 0,$$

for all $X, Y \in D^1(p)$ with $g(X, Y) = g(X, \phi Y) = 0$.

In view of (3.2) and (3.14), we have

$$(3.15) \quad h(X + \phi Y, X - \phi Y) = 0, \quad h(X + Y, X - Y) = 0,$$

that is, $h(X, X) = 0$. Since every tangent vector of $D(p)$ must lie in a totally real plane section of $D(p)$, so we have,

$$(3.16) \quad h(X, Y) = 0, \text{ for all } X, Y \in D(p).$$

The equations (3.16) and (2.6) show that p must be a totally geodesic point.

The converse is straight forward.

4. MINIMAL INVARIANT SUBMANIFOLDS

The maximum Ricci curvature function on a Riemannian manifold $\bar{M}$ denoted by $\hat{Ric}$ is defined as [6]

$$(4.1) \quad \hat{Ric}(p) = \max\{Ric(X) : X \in T_p^1 \bar{M}\}, \text{ where } T_p^1 \bar{M} = \{X \in T_p \bar{M} : g(X, X) = 1\}.$$

If $n = 3$, $\hat{Ric}$ is the Chen first invariant δ_M [7].

For $n > 3$, $\hat{Ric}$ is the Chen invariant $\delta_{(n-1)}$ [8]. Now we have

Theorem 4.1. *Let M be an n -dimensional submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ such that the structure vector field ξ is tangent to M . Then for each unit vector $X \in T_p M$, orthogonal to ξ , we have*

$$(4.2) \quad Ric(X) \leq \frac{1}{4}n^2\|H\|^2 + (n-1)f_1 + 3f_2 - f_3.$$

An unit tangent vector X orthogonal to ξ at p satisfies the equality case of (4.2) if and only if

$$(4.3) \quad h(X, Y) = 0, \forall Y \in T_p M \text{ orthogonal to } X,$$

$$2h(X, Y) = nH(p).$$

The proof follows from theorem (5.1) of [10].

Theorem 4.2. *Let M be an $(n+1)$ -dimensional invariant submanifold of a $(2m+1)$ -dimensional generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Then*

$$(4.4) \quad \hat{Ric} \leq \frac{1}{4}(n+1)^2\|H\|^2 + nf_1 + 3f_2 - f_3,$$

where $\hat{Ric}$ is the maximum Ricci curvature function on M .

Moreover, if M satisfies the equality case of (4.4) identically and the shape operator satisfies

$$(4.5) \quad A_{\phi X}Y = A_{\phi Y}X, \quad X, Y \in TM,$$

then M is a minimal submanifold.

Proof. The inequality (4.4) is an immediate consequence of the inequality (4.2). We assume that M is an invariant submanifold of $\bar{M}(f_1, f_2, f_3)$, which satisfies the equality case of (4.4) at a point $p \in M$.

Without loss of generality we may choose an orthonormal basis $\{e_0 = \xi, e_1, \dots, e_n\}$ of $T_p M$ such that $\hat{Ric}(p) = Ric(e_n)$.

Then, by (4.3), it follows that,

$$(4.6) \quad h(e_i, e_n) = 0, \quad \forall i \in (0, 1, n-1); \quad h(e_n, e_n) = \sum_{i=1}^{n-1} h(e_i, e_i),$$

where $\{e_0 = \xi, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ is the orthonormal basis of $T_p \bar{M}(f_1, f_2, f_3)$ and e_{n+1} parallel to the mean curvature vector $H(p)$.

From equation (4.5), we have

$$(4.7) \quad h_{ij}^{n+k} = h_{ik}^{n+j} = h_{jk}^{n+i} \text{ for any } i, j, k \in (1, \dots, n).$$

Thus, from (4.6) and (4.7), we have

$$(n+1)H(p) = \sum_{i=1}^n h_{ii}^{n+1} e_{n+1} = 2h_{nn}^{n+1} e_{n+1} = 2h_{1n}^{2n} e_{n+1} = 0.$$

Therefore M is a minimal submanifold.

Theorem 4.3. Let M be an $(n+1)$ -dimensional ($n \geq 4$) invariant submanifold in a $(2m+1)$ -dimensional trans-Sasakian generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$.

If M is minimal, then for each $k \in (-\infty, 4]$, $\delta_k^r(p)$ satisfies

$$(4.8) \quad \delta_k^r(p) \leq \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - kf_1, \quad p \in M$$

$$(4.9) \quad \delta_k^r(p) = \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - kf_1, \quad p \in M \text{ holds,}$$

for some $k \in (-\infty, 4)$ if and only if p is a totally geodesic point.

$$(4.10) \quad \delta_4^r(p) = \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - 4f_1, \quad p \in M,$$

holds if and only if there exists an orthonormal basis

$\{e_1, e_d, e_{\bar{1}} = \phi e_1, \dots, e_{\bar{d}} = \phi e_d, e_{2d+1} = \xi\}$ for $T_p M$ and an orthonormal basis $\{\alpha_1, \alpha_{m-d}, \alpha_{\bar{1}} = \phi \alpha_1, \dots, \alpha_{\overline{m-d}} = \phi \alpha_{m-d}\}$ for $T_p^\perp M$ such that the shape operators of M take the following forms:

$$(4.11) \quad A_{\alpha_p} = \begin{pmatrix} A'_{\alpha_p} & A''_{\alpha_p} & 0 \\ A''_{\alpha_p} & -A'_{\alpha_p} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\bar{\alpha}_p} = \begin{pmatrix} -A''_{\alpha_p} & A'_{\alpha_p} & 0 \\ A'_{\alpha_p} & A''_{\alpha_p} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A'_{\alpha_p} = \begin{pmatrix} h_{11}^p & h_{12}^p & 0 \\ h_{12}^p & -h_{11}^p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A''_{\alpha_p} = \begin{pmatrix} -h_{11}^{\bar{p}} & -h_{12}^{\bar{p}} & 0 \\ -h_{12}^{\bar{p}} & h_{11}^{\bar{p}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $p \in (1, 2, m-d)$.

Proof. From (1.1) and (2.12), we get

$$(4.12) \quad \begin{aligned} R(X, Y, Z, W) = & g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \\ & + f_1 \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ & + f_2 \{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) + 2g(X, \phi Y)g(\phi Z, W)\} \\ & + f_3 \{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + g(X, Z)\eta(Y)\eta(W) - \\ & g(Y, Z)\eta(X)\eta(W)\}. \end{aligned}$$

The equation (4.12) implies that the scalar curvature τ and the second fundamental form h of M at p satisfy

$$(4.13) \quad 2\tau(p) = n(n+1)f_1 + 3nf_2 - 2nf_3 - \|h\|^2,$$

which implies

$$(4.14) \quad \tau(p) \leq \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3$$

with the equality holding if and only if p is a totally geodesic point.

Now, we suppose that $\pi \subset T_p M$ is a given totally real plane section. We choose an orthonormal basis $\{e_1, \dots, e_d, e_{\bar{1}} = \phi e_{\bar{1}}, \dots, e_{\bar{d}} = \phi e_d, e_{2d+1} = \xi\}$ for $T_p M$ and an orthonormal basis $\{\alpha_1, \dots, \alpha_{m-d}, \alpha_{\bar{1}} = \phi \alpha_1, \dots, \alpha_{\overline{m-d}} = \phi \alpha_{m-d}\}$, for $T_p^\perp M$ such that a given totally real plane section, $\pi = \text{span} \{e_1, e_2\}$, $\pi \subset D(p)$.

With respect to such a basis, we have

$$(4.15) \quad A_{\alpha_p} = \begin{pmatrix} A'_\alpha & A''_\alpha & 0 \\ A''_\alpha & -A'_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\alpha_{\bar{p}}} = \begin{pmatrix} -A''_\alpha & A'_\alpha & 0 \\ A'_\alpha & A''_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha = 1, 2, \dots, p,$$

where A'_α and A''_α are $d \times d$ matrices.

From (4.13) and (4.15), we have

$$\begin{aligned} -2\tau(p) + n(n+1)f_1 + 3nf_2 - 2nf_3 &= 4 \sum_{p=1}^{m-d} \{\|A'_\alpha\|^2 + \|A''_\alpha\|^2\} \\ &\geq 4 \sum_{p=1}^{m-d} \{(h_{11}^p)^2 + (h_{22}^p)^2 + 2(h_{12}^p)^2 + (h_{11}^{\bar{p}})^2 + (h_{22}^{\bar{p}})^2 + 2(h_{12}^{\bar{p}})^2\} \\ &\geq -8 \sum_{p=1}^{m-d} \{h_{11}^p h_{22}^p - (h_{12}^p)^2 + h_{11}^{\bar{p}} h_{22}^{\bar{p}} - (h_{12}^{\bar{p}})^2\} \\ &= -8(K(\pi) - f_1) \\ &= -8K(\pi) + 8f_1 \end{aligned}$$

$$\text{i.e. } \tau(p) - \frac{1}{2}n(n+1)f_1 - \frac{3}{2}nf_2 + nf_3 \leq 4K(\pi) - 4f_1.$$

Thus we get,

$$(4.16) \quad \tau(p) - 4K(\pi) \leq \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - 4f_1,$$

with equality holding if and only if

$$(4.17) \quad h_{11}^p + h_{22}^p = 0, \quad h_{1j}^p = h_{2j}^p \dots = h_{ij}^p = 0, \quad p \in \{1, \dots, m-d, \bar{1}, \dots, \overline{m-d}\},$$

$$i, j \in \{3, \dots, n+1\}.$$

Since inequality (4.16) holds for any totally real plane section, we get

$$(4.18) \quad \tau - 4 \inf K^r(p) \leq \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - 4f_1.$$

For $\lambda \in (0, \infty)$, from (4.14) we get

$$(4.19) \quad \lambda\tau(p) \leq \frac{\lambda}{2}(n(n+1)f_1 + 3nf_2) - \lambda nf_3,$$

which together with (4.18) proves the inequality

$$(4.20) \quad \delta_k^r \leq \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - kf_1, \text{ for any } k \in (0, 4).$$

Combining (4.14), (4.18) and (4.20), we obtain inequality (4.8) for $k \in [0, 4]$. The inequality (4.8) with $k \in (-\infty, 0)$ follows from (4.14) and theorem 3.2.

To prove (4.9), we assume M is an invariant submanifold of $\bar{M}(f_1, f_2, f_3)$ which is also minimal and satisfies $\delta_k^r = \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - kf_1$ identically for some $k \in (-\infty, 4)$. Then, we have following three cases :

A. $k = 0$, which gives

$$\delta_0^r(p) = \tau(p) = \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3,$$

then (4.14) implies that p is a totally geodesic point.

B. $k \in (0, 4)$, then applying (4.18) and the definition of $\delta_k^r(p)$, we have

$$\begin{aligned} \delta_k^r(p) &= \tau(p) - k \inf K^r(p) = \left(1 - \frac{k}{4}\right) \delta_0^r(p) + \frac{k}{4} \delta_4^r(p) \\ &\leq \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - kf_1, \end{aligned}$$

which implies, in particular that

$$\tau(p) = \delta_0^r(p) = \frac{1}{2}n(n+1)f_1 + 3nf_2 - nf_3.$$

Therefore p is a totally geodesic point.

C. $k \in (-\infty, 0)$, then (4.14), together with the definition of $\delta_k^r(p)$ and theorem 3.2 yield

$$\begin{aligned} \delta_k^r(p) &= \tau(p) - k \inf K^r(p) \\ &\leq \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - kf_1. \end{aligned}$$

In particular, this gives $\delta_0^r = \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3$. Hence, p must be a totally geodesic point. Conversely, if p is a totally geodesic point, then applying theorem 3.2 and (4.14), we have (4.9).

Now, we assume M is an invariant submanifold of $\bar{M}(f_1, f_2, f_3)$ which is minimal and satisfies (4.10). Then, the inequality (4.16) becomes equality which yields (4.17). From this, we conclude that the shape operator of M takes the form as in (4.11) with respect to some orthonormal basis $\{e_1, \dots, e_d, \phi e_1, \dots, \phi e_d, e_{2d+1} = \xi, \alpha_1, \dots, \alpha_{m-d}, \phi \alpha_1, \dots, \phi \alpha_{m-d}\}$ of $T_p \bar{M}(f_1, f_2, f_3)$.

Conversely, suppose that the shape operator of M at p takes the form as in (4.11) with respect to some orthonormal basis $\{e_1, \dots, e_d, \phi e_1, \dots, \phi e_d, e_{2d+1} =$

$\xi, \alpha_1, \dots, \alpha_{m-d}, \phi\alpha_1, \dots, \phi\alpha_{m-d}\}$ for $T_p\bar{M}(f_1, f_2, f_3)$, then the inequality (4.16) becomes an equality, which together with (4.8) gives

$$\frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - 4f_1 \geq \delta_4^r(p) \geq \tau(p) - 4K(\pi) = \frac{1}{2}n(n+1)f_1 + \frac{3}{2}nf_2 - nf_3 - 4f_1.$$

This completes the proof.

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REFERENCES

- [1] Alegre, P., Blair, D.E. and Carriazo, A., Generalized Sasakian space forms, Israel J. Math., 141 157-183 (2004).
- [2] Alegre, P., and Carriazo, A., Structures on generalized Sasakian space forms, Differential Geometry and its applications, 26 no.6, 656-666 (2008).
- [3] Blair, D.E., Contact manifolds in Riemannian geometry, Lecture notes in Mathematics Vol-509, Springer Verlag, Berlin (1976).
- [4] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M. and Fernandez, M., Semi-slant submanifolds of a Sasakian manifold, Geom. Dedicata, 78 183-199 (1999).
- [5] Chen, B.Y., A series of Kaehlerian invariants and their applications to Kaehlerian Geometry, Beitrage zur Algebra Geom., 42 no.1, 165-178 (2001).
- [6] Chen, B.Y., On Ricci curvature of isotropic and Lagrangian submanifolds in complex space forms, Arch. Math. (Basel), 74 154-160 (2000).
- [7] Chen, B.Y., Some pinching and classification theorems for minimal submanifolds, Arch. Math., 60 568-578 (1993).
- [8] Chen, B.Y., Some new obstructions to minimal and Lagrangian isometric immersions, Japan J. Math., 26 1-17 (2000).
- [9] Ghefari, R. Al., Al-Solamy, F.R. and Shahid, M.H., Contact CR-warped product submanifolds in generalized Sasakian space forms, Balkan J. Geom. Appl., 11 no.1, 1-10 (2006).
- [10] Hong, S. and Tripathi, M.M., On Ricci curvature of submanifolds of generalized Sasakian space forms, Int. J. Pure and Appl. Math. Sci., 2 no.2, 173-201 (2005).
- [11] Li, X., Huang, G. and Xu, J., Some inequalities for submanifolds in locally conformal almost cosymplectic manifolds, Soochow J. Math., 31 no.3, 309-319 (2005).
- [12] Mihai, I. and Ghisoeiu, V., Minimality of certain contact slant submanifolds in Sasakian space forms, Int. J. Pure and Appl. Math. Sci., 1 95-99 (2004).

- [13] Oubina, J.A., New Classes of almost contact metric structure, Publ. Math. Debrecen, 32 no. 3-4, 187-193 (1985).
- [14] Yano, K. and Kon, M., Structures on Manifolds, Series in Pure Mathematics, World Scientific Publishing Co. Singapore, (1984).