TIME-LIKE SMARANDACHE CURVES DERIVED FROM A SPACE-LIKE HELIX

AHMAD TAWFIK ALI
PRESENT ADDRESS: KING ABDUL AZIZ UNIVERSITY
FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS
P.O. BOX 80203, JEDDAH, 21589, SAUDI ARABIA
PERMANENT ADDRESS: MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE, AL-AZHAR UNIVERSITY
NASR CITY, 11448, CAIRO, EGYPT
ATALI71@YAHOO.COM

Abstract. A regular curve in Minkowski space-time E_1^4, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve. We define a special case of such curves and call it Smarandache $\mathbf{TB}_2$ curves in the Minkowski space-time E_1^4. Moreover, we compute formulas of its Frenet apparatus according to base curve via the method expressed in [5, 6]. By this way, we obtain another orthonormal frame of a such curve in E_1^4.

AMS Classification: 53C40, 53C50

Keywords: Frenet apparatus, Time-like curves, Minkowski space-time, Smarandache curve.

1. Introduction

It is well-known that: if a curve is differentiable in an open interval, then at each point, a set of mutually orthogonal unit vectors can be constructed. These vectors are called Frenet frame or moving frame vectors. The rates of change of these frame

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 8, NUMBER 1 (2010) 93-100.

vectors along the curve define the curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve, is called *Frenet apparatus* of the curves [4].

At the beginning of the twentieth century, Einstein's theory opened a door to use of degenerate sub-manifolds. Then, the researchers treated some of classical differential geometry topics and extended to Lorentz manifolds. For instance, in [7], the author introduces a method to calculate Frenet apparatus of space-like curves in Minkowski 4-space according to signature $(+,+,+,-)$. Thereafter, in [5, 6], in an analogous way, mentioned method is adapted to space-like and time-like curves in Minkowski space-time (with respect to signature $(-,+,+,+)$, e.g.).

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve is called a Smarandache curve. We deal with a special Smarandache curves which is defined by the tangent vector and second binormal vector fields. We call such curves as Smarandache $\mathbf{TB}_2$ curves.

In this paper, we compute Frenet apparatus of this kind of curves using the method expressed as in [5, 6]. Moreover, we hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented (A more complete elementary treatment can be found in [3]).

Minkowski four-dimensional space-time E_1^4 is the real vector space R_1^4 endowed with the standard flat metric given by:

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \tag{1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4. Since g is an indefinite metric, recall that a vector $v \in \mathrm{E}_1^4$ can have one of the three causal characters; it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null (light-like) if $g(v,v)$=0 and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally a space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v,w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. Thus, a space-like or a time-like curve α is said to be parameterized by arclength function s, if $g(\alpha', \alpha') = \pm 1$.

Denote by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then the vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are, respectively, the tangent, the principal normal, the binormal and the trinormal vector fields.

Firstly, let $\alpha(s)$ be a time-like unit speed curve in E_1^4, parameterized by arclength function s. Then for the curve α the following Frenet equations are given in [3, 1]:

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}_1'(s) \\ \mathbf{B}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}_1(s) \\ \mathbf{B}_2(s) \end{bmatrix} \tag{2}$$

where $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are mutually orthogonal vectors satisfying equations:

$$g(\mathbf{T}, \mathbf{T}) = -1, \quad g(\mathbf{N}, \mathbf{N}) = g(\mathbf{B}_1, \mathbf{B}_1) = g(\mathbf{B}_2, \mathbf{B}_2) = 1.$$

Secondly, let $\alpha(s)$ be a space-like unit speed curve has a time-like principal normal vector in E_1^4. Then the following Frenet equations are given also in [3, 1]:

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}_1'(s) \\ \mathbf{B}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}_1(s) \\ \mathbf{B}_2(s) \end{bmatrix} \tag{3}$$

where $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are mutually orthogonal vectors satisfying equations:

$$g(\mathbf{N}, \mathbf{N}) = -1, \quad g(\mathbf{T}, \mathbf{T}) = g(\mathbf{B}_1, \mathbf{B}_1) = g(\mathbf{B}_2, \mathbf{B}_2) = 1.$$

Here κ, τ and σ are first, second and third curvature of curve α, respectively.

In the same space, in [5, 6] authors used a vector product and gave a method to calculate the Frenet frame for an arbitrary space-like and time-like curves by the following definition and theorem:

Definition 2.1 Let $\mathbf{a} = (a_1, a_2, a_3, a_4)$, $\mathbf{b} = (b_1, b_2, b_3, b_4)$ and $\mathbf{c} = (c_1, c_2, c_3, c_4)$ be vectors in E_1^4. The vector product in Minkowski space-time E_1^4 is defined by the determinant

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = - \begin{vmatrix} -\mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 & \mathrm{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{4}$$

where e_1, e_2, e_3 and e_4 are coordinate direction vectors satisfying

$$\mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3 = \mathrm{e}_4, \;\; \mathrm{e}_2 \wedge \mathrm{e}_3 \wedge \mathrm{e}_4 = \mathrm{e}_1, \;\; \mathrm{e}_3 \wedge \mathrm{e}_4 \wedge \mathrm{e}_1 = \mathrm{e}_2, \;\; \mathrm{e}_4 \wedge \mathrm{e}_1 \wedge \mathrm{e}_2 = -\mathrm{e}_3.$$

Theorem 2.2 Let $\alpha = \alpha(s)$ be an arbitrary time-like curve in Minkowski space-time E_1^4 . The Frenet apparatus of α can be written as follows:

$$\mathbf{T} = \frac{\alpha'}{\|\alpha'\|}, \tag{5}$$

$$\mathbf{N} = \frac{\|\alpha'\|^2 \alpha' + g(\alpha', \alpha'')\alpha'}{\left\| \|\alpha'\|^2 \alpha' + g(\alpha', \alpha'')\alpha' \right\|}, \tag{6}$$

$$\mathbf{B}_1 = \mu \, \mathbf{N} \wedge \mathbf{T} \wedge \mathbf{B}_2, \tag{7}$$

$$\mathbf{B}_2 = \mu \, \frac{\mathbf{T} \wedge \mathbf{N} \wedge \alpha'''}{\|\mathbf{T} \wedge \mathbf{N} \wedge \alpha'''\|}, \tag{8}$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 \alpha'' + g(\alpha', \alpha'')\alpha' \right\|}{\|\alpha'\|^4}, \tag{9}$$

$$\tau = \frac{\|\mathbf{T} \wedge \mathbf{N} \wedge \alpha'''\| \, \|\alpha'\|}{\left\| \|\alpha'\|^2 \alpha'' + g(\alpha', \alpha'')\alpha' \right\|}, \tag{10}$$

and

$$\sigma = \frac{g(\alpha^{(IV)}, \mathbf{B}_2)}{\|\mathbf{T} \wedge \mathbf{N} \wedge \alpha'''\| \, \|\alpha'\|}. \tag{11}$$

Here μ is taken ± 1 to make $+1$ of determinant of $[\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2]$ matrix. By this way, Frenet frame provides positively oriented.

Recall that an arbitrary curve is a $W-$curve or only a *helix* if it has constant Frenet curvatures. From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvatures κ, τ and σ [2].

3. Time-like Smarandache curves derived from space like $W-$curve

Definition 3.1 A regular curve in E_1^4, whose position vector is obtained by Frenet frame vectors on another regular curve, is called a Smarandache curve [8].

Remark 3.2 Formulas of all Smarandache curves Frenet apparatus can be determined by the expressed method.

Now, let us define a special form of Definition 3.1.

Definition 3.3 Let $\xi = \xi(s$ be an unit space-like curve has a time-like principal normal and non-zero constant curvatures κ, τ and σ. Suppose $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ be moving frame on it. Smarandache $\mathbf{TB}_2$ curves derived with the curve $\xi)$ is defined by:

$$\mathbf{X} = \mathbf{X}(s_{\mathbf{X}}) = \frac{\mathbf{T}(s) + \mathbf{B}_2(s)}{\sqrt{\kappa^2 - \sigma^2}}. \tag{12}$$

Theorem 3.4 Let $\xi = \xi(s)$ be an unit space-like curve has a time-like principal normal and non-zero constant curvatures κ, τ and σ. Let $\mathbf{X} = \mathbf{X}(s_{\mathbf{X}})$ be Smarandache $\mathbf{TB}_2$ curves defined by frame vectors of a curve ξ. Then:

(i): The curve $\mathbf{X} = \mathbf{X}(s_{\mathbf{X}})$ is a time-like curve.

(ii): Frenet apparatus $\{\mathbf{T}_{\mathbf{X}}, \mathbf{N}_{\mathbf{X}}, \mathbf{B}_{1\mathbf{X}}, \mathbf{B}_{2\mathbf{X}}, \kappa_{\mathbf{X}}, \tau_{\mathbf{X}}, \sigma_{\mathbf{X}}\}$ of Smarandache $\mathbf{TB}_2$ curve $\mathbf{X} = \mathbf{X}(s_{\mathbf{X}})$ can be formed by Frenet apparatus $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2, \kappa, \tau, \sigma\}$ of a curve $\xi = \xi(s)$.

Proof: Let $\mathbf{X} = \mathbf{X}(s_{\mathbf{X}})$ be a Smarandache $\mathbf{TB}_2$ curve defined by an above statement. Differentiating both sides of (12), we easily have

$$\frac{d\mathbf{X}}{ds_{\mathbf{X}}} \frac{ds_{\mathbf{X}}}{ds} = \frac{\kappa \, \mathbf{N}(s) - \sigma \, \mathbf{B}_2(s)}{\sqrt{\kappa^2 - \sigma^2}}. \tag{13}$$

The inner product $g(\mathbf{X}', \mathbf{X}') = -1$, leads to

$$\frac{d\mathbf{X}_s}{ds} = \|\mathbf{X}'(s_{\mathbf{X}})\| = \sqrt{|g(\mathbf{X}', \mathbf{X}')|} = 1, \tag{14}$$

where $'$ denotes derivative according to $s_{\mathbf{X}} = s$. So, the Smarandache $\mathbf{TB}_2$ curve $\mathbf{X} = \mathbf{X}(s_{\mathbf{X}})$ is a time-like curve. Thus, the tangent vector is obtained as

$$\mathbf{T}_{\mathbf{X}} = \frac{\kappa\,\mathbf{N}(s) - \sigma\,\mathbf{B}_2(s)}{\sqrt{\kappa^2 - \sigma^2}}. \tag{15}$$

In order to calculate principal normal, we differentiating the above equation (left and right hand side) with respect to the variable s using Frenet frame equations (2) and (3), respectively, we have

$$\kappa_{\mathbf{X}}\,\mathbf{N}_{\mathbf{X}} = \frac{\kappa^2\,\mathbf{T} - \tau\sigma\mathbf{N} + \kappa\tau\,\mathbf{B}_1 - \sigma^2\,\mathbf{B}_2}{\sqrt{\kappa^2 - \sigma^2}}. \tag{16}$$

Thus, the principal normal vector and the first curvature, respectively, take the following form:

$$\mathbf{N}_{\mathbf{X}} = \frac{\kappa^2\,\mathbf{T} - \tau\sigma\,\mathbf{N} + \kappa\tau\,\mathbf{B}_1 - \sigma^2\,\mathbf{B}_2}{\sqrt{\kappa^4 + \tau^2(\kappa^2 - \sigma^2) + \sigma^4}}, \tag{17}$$

$$\kappa_{\mathbf{X}} = \sqrt{\frac{\kappa^4 + \tau^2(\kappa^2 - \sigma^2) + \sigma^4}{\kappa^2 - \sigma^2}}, \tag{18}$$

Now, we form the following differentiations with respect to s

$$\begin{cases} \mathbf{X}' = \dfrac{1}{A_1}\Big(\kappa\,\mathbf{N} - \sigma\,\mathbf{B}_2\Big), \\ \mathbf{X}'' = \dfrac{1}{A_1}\Big(\kappa^2\,\mathbf{T} - \tau\sigma\,\mathbf{N} + \kappa\tau\,\mathbf{B}_1 - \sigma^2\,\mathbf{B}_2\Big), \\ \mathbf{X}''' = \dfrac{1}{A_1}\Big(- \kappa\tau\sigma\,\mathbf{T} + \kappa(\kappa^2 + \tau^2)\mathbf{N} + \sigma(\sigma^2 - \tau^2)\mathbf{B}_1 + \kappa\tau\sigma\,\mathbf{B}_2\Big), \\ \mathbf{X}^{IV} = \dfrac{1}{A_1}\Big(\kappa^2(\kappa^2 + \tau^2)\mathbf{T}(s) - (\kappa^2 + \tau^2 - \sigma^2)(\tau\sigma\,\mathbf{N} - \kappa\tau\mathbf{B}_1) + \sigma^2(\sigma^2 - \tau^2)\mathbf{B}_2\Big), \end{cases} \tag{19}$$

where $A_1 = \sqrt{\kappa^2 - \sigma^2}$. We express the following vector product to determine the second binormal vector field of the curve $\mathbf{X}(s_{\mathbf{X}})$

$$\mathbf{T}_{\mathbf{X}} \wedge \mathbf{N}_{\mathbf{X}} \wedge \mathbf{X}''' = -\frac{1}{A_1\,A_2} \begin{vmatrix} \mathbf{T} & -\mathbf{N} & \mathbf{B}_1 & \mathbf{B}_2 \\ 0 & \kappa & -\sigma & 0 \\ \kappa^2 & -\tau\sigma & \kappa\tau & -\sigma^2 \\ l_1 & l_2 & l_3 & l_4 \end{vmatrix}, \tag{20}$$

where $A_2 = \sqrt{\kappa^4 + \tau^2(\kappa^2 - \sigma^2) + \sigma^4}$ and $l_i\,(i = 1, 2, 3, 4)$ are the components of the differentiable vector function $\mathbf{X}'''$. From the above equation, we obtain

$$\mathbf{T_X} \wedge \mathbf{N_X} \wedge \mathbf{X}''' = \frac{\kappa\sigma}{A_1^2\, A_2}\Big(m_1\,\mathbf{T} + m_2\,\mathbf{N} + m_3\,\mathbf{B}_1 + m_4\,\mathbf{B}_2\Big), \tag{21}$$

where

$$\begin{cases} m_1 = \sigma^2(\sigma^2 - \tau^2) + \kappa^2(\sigma^2 + \tau^2), \\ m_2 = \tau\sigma(\kappa^2 - \sigma^2), \\ m_3 = \kappa\tau(\sigma^2 - \kappa^2), \\ m_4 = \tau^2(\kappa^2 - \sigma^2) + \kappa^2(\kappa^2 + \sigma^2). \end{cases} \tag{22}$$

By means of equations (2) and (21), we write

$$\mathbf{B}_{2\mathbf{X}} = -\frac{1}{A_2 A_3}\Big(m_1\,\mathbf{T} + m_2\,\mathbf{N} + m_3\,\mathbf{B}_1 + m_4\,\mathbf{B}_2\Big), \tag{23}$$

where $A_3 = \sqrt{\kappa^4 + 2(\kappa^2\tau^2 - \tau^2\sigma^2 + \sigma^2\kappa^2) + \sigma^4}$. Considering equations (10) and (11), we have the second and the third curvatures of $\mathbf{X}(s_\mathbf{X})$ as follows:

$$\tau_\mathbf{X} = \frac{\kappa\,\sigma\,\sqrt{\kappa^4 + 2(\kappa^2\tau^2 - \tau^2\sigma^2 + \sigma^2\kappa^2) + \sigma^4}}{\sqrt{\kappa^2 - \sigma^2}\sqrt{\kappa^4 + \tau^2(\kappa^2 - \sigma^2) + \sigma^4}}, \tag{24}$$

$$\sigma_\mathbf{X} = \frac{\kappa\,\sigma\,\sqrt{\kappa^2 - \sigma^2}}{\sqrt{\kappa^4 + \tau^2(\kappa^2 - \sigma^2) + \sigma^4}}. \tag{25}$$

We know that: the first binormal vector can be founded by the vector product $\mathbf{B}_{1\mathbf{X}} = \mu\,\mathbf{N_X} \wedge \mathbf{T_X} \wedge \mathbf{B}_{2\mathbf{X}}$. So, we get:

$$\mathbf{B}_{1\mathbf{X}} = -\frac{1}{A_1^2\, A_3}\Big(n_1\,\mathbf{T} + n_2\,\mathbf{N} + n_3\,\mathbf{B}_1 + n_4\,\mathbf{B}_2\Big), \tag{26}$$

where

$$\begin{cases} n_1 = \tau(\kappa^2 - \sigma^2), \\ n_2 = \sigma(\kappa^2 + \sigma^2), \\ n_3 = \kappa(\kappa^2 + \sigma^2), \\ n_4 = \tau(\sigma^2 - \kappa^2). \end{cases} \tag{27}$$

Consequently, we determined Frenet apparatus of a time-like Smarandache curve derived from a space-like $W-$curve.

Corollary 3.5 The frame $\{\mathbf{T_X}, \mathbf{N_X}, \mathbf{B}_{1\mathbf{X}}, \mathbf{B}_{2\mathbf{X}}\}$ is another orthonormal frame of a time-like curve in $\mathbf{E}_1^4$, where $\mu = 1$ to make $+1$ determinant $\Big[\mathbf{T_X}, \mathbf{N_X}, \mathbf{B}_{1\mathbf{X}}, \mathbf{B}_{2\mathbf{X}}\Big]$ matrix.

References

[1] Camci, C., Ilarslan, K. and Sucurovic, E. On pseudohyperbolical curves in Minkowski space-time, Turk. J. Math. Vol 27, 315–328, (2003).

[2] İlarslan, K. and Boyacıoğlu, Ö. Position vectors of a timelike and a null helix in Minkowski 3-space, Chaos, Solitons and Fractals, Vol 38, 1383–1389, (2008).

[3] O'Neill B, Semi-Riemannian Geometry with Applications to Relativity, New York, 1983 (Academic Press).

[4] Ozyilmaz, E. and Yilmaz, S. Involute-evolute curve couples in the Euclidean 4-space, Int. J. Open Problems Compt. Math. Vol 2, No.2, 168–174, (2009).

[5] Yilmaz, S. and Turgut, M. On the differential geometry of the curves in Minkowski space-time I, Int. J. Contemp. Math. Sci. Vol 3, No.27, 1343–1349, (2008).

[6] Yilmaz, S., Ozyilmaz, E. and Turgut, M. On the differential geometry of the curves in Minkowski space-time II, Int. J. Comput. Math. Sci. Vol 3, No.2, 53–55, (2009).

[7] Yilmaz, S, Spherical indicators ocurves and characterizations of some speial curves in four dimensional Lotrenzian space L^4, Ph.D. dissertation, Dept. Math., Dokuz Eylul Univ., Izmir, Turkey, 2001.

[8] Turgut, M. and Yilmaz, S. Smarandache Curves in Minkowski space-time, Int. J. Math. Comb. Vol 3, 51-55, (2008).