

**MULTIFRACTAL DECOMPOSITION OF THE ENTROPIES
SPECTRUM FOR COUNTABLE
SUBSHIFTS IN FINER LEVELS SETS**

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ABSTRACT. In a previous article we considered a finer decomposition of the spectrum of local entropies. In that case the dynamics are taken on compact spaces. Here we continue with this analysis but now for systems with non-compact phase space. Specifically, we consider here subshifts with countable alphabet.

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1. INTRODUCTION

The Multifractal Analysis is a very important subfield in the area of Dimension Theory of Dynamical Systems. One of the main motivations to introduce it is the study of fractal sets by invariants. To do this decompositions in level sets with some scaling property, which are called multifractal decompositions, have been considered. The first works in this area appeared in the physical literature [13], [14],[2],[1].

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The mathematical rigorization and the general formulation of the concept was presented in [3]. In this last contribution, the multifractal decomposition in level sets for a given map $g : X \rightarrow [-\infty, +\infty]$ is given by

$$(1) \quad K_\alpha = K_\alpha(g) = \{x \in X : g(x) = \alpha\},$$

and so the space X is decomposed as $X = \left(\bigcup_\alpha K_\alpha\right) \cup Y$, where Y is the set in which g is not defined and which is negligible in some sense.

Within this context, if G is a function defined on sets, and $F(\alpha) = G(K_\alpha)$. then the map F is called *the multifractal spectrum* specified by the pair (g, G) .

The complexity of the multifractal decomposition is analyzed by adequate characteristic dimensions like the Hausdorff dimension and the topological entropy. Accordingly interesting examples of multifractal spectra are:

- The *dimension multifractal spectrum*: here $g(x) = D_\mu(x)$, the dimension of the measure μ and $F(\alpha) = \dim_H K_\alpha$ the Hausdorff dimension of the set K_α . This spectrum was previously studied for particular cases in [9] and [14].

- The *multifractal spectrum of local entropies*: in this case g is taken as the local entropy of a dynamical map f and $F(\alpha) = h_{top}(f, K_\alpha)$, the Bowen topological entropy (for non-compact sets).

Indeed the Hausdorff dimension and the topological entropy are special cases of Carathéodory dimensions in metric spaces, so there is a close relationship between the fields of multifractal analysis and dimension theory of dynamical systems.

To obtain a complete description of multifractal spectra one must introduce families of measures $\{\mu_\alpha\}_{\alpha \in \mathbf{R}}$ such that $\mu_\alpha(K_\alpha) = 1$. To do this is necessary to get a parameterization $\alpha(q)$ of the level sets with $\mu_q(K_{\alpha(q)}) = 1$ and $\mu_q(K_\alpha) = 0$ if $\alpha \neq \alpha(q)$.

Among relevant works in multifractal analysis we quote [23],[3],[26],[4],[27] and [29]. In [17],[18],[19],[20] we have used the formalism developed by Takens and Verbitski[26] to describe multifractal spectra as well as to connect it with other fields, namely the study of growth rates of Kleinian action groups on the hyperbolic plane and the setting of formal thermodynamic relationships. More recently, Pesin and Zhang[24] developed a formalism for continuous maps on compact spaces which

admits the so called inducing schemes. In [5], [22], [11] were treated issues related with the study of the “dimensionless” of subsets of the multifractal sets.

For expansive homeomorphisms $f : X \rightarrow X$ with the specification property and with X a compact metric space, we have considered [21] a decomposition of the the entropy spectrum level sets $K_\alpha = \{x \in X : h_\mu(f, x) = \alpha\}$ of the form

$$(2) \quad K_\alpha = \bigcup_{V \in \mathbf{R}^+} (K_{\alpha,V} \cup \mathcal{I}).$$

The entropy $h_\mu(f, x)$ there considered is the usual local entropy [8] whereas the sets $K_{\alpha,V}$ are special sublevel sets and $\mathcal{I}$ is the irregular part of the decomposition. It was established that $h_{top}(f, K_{\alpha,V}) = h_{top}(f, K_\alpha)$ for a set of values of V . That work extended results of Jordan and Pollicott [15] obtained taking a finer multifractal decomposition for the dimension spectrum of C^2 unidimensional maps.

The aim of present note is to obtain a similar result but for systems with non-compact phase space. The dynamical systems to be considered are now Markov subshifts

$$(3) \quad \Sigma_{\mathcal{A},B}^\infty = \{\omega = (\omega_n)_{n \in \mathbf{Z}} : \omega_n \in \mathcal{A}, B(\omega_i, \omega_{i+1}) = 1, \text{ for } i \in \mathbf{Z}\},$$

with $\mathcal{A}$ an infinite countable set and $B : \mathcal{A} \times \mathcal{A} \rightarrow \{0, 1\}$ its incidence matrix. This symbolic space is endowed with a dynamical structure considering the Bernoulli shift $\sigma : \Sigma_{\mathcal{A},B}^\infty \rightarrow \Sigma_{\mathcal{A},B}^\infty$, $(\sigma\omega)_n = \omega_{n+1}$.

We shall prove the following result:

Theorem: Let $\psi : \Sigma_{\mathcal{A},B}^\infty \rightarrow \mathbf{R}$ be a map with an equilibrium state μ_ψ which is assumed to be different from the measure of the maximal entropy μ_{max} and let $\mathcal{E}(\alpha) = h_{top}(\sigma, K_\alpha)$ be the spectrum of local entropies. Then, for a parameterization of levels $\alpha(q)$ there exists a finer decomposition $K_{\alpha(q),V(q)}$ such that $\mathcal{E}(\alpha(q)) = \mathcal{E}(\alpha(q), V(q)) := h_{top}(\sigma, K_{\alpha(q),V(q)})$ for $V(q)$ large enough.

One interesting geometric aspect of the countable alphabet subshifts is the modelling of flows on non-compact surfaces. In particular the geodesic flow on the modular surface $M = H^2/SL_2(\mathbf{Z})$ is symbolically represented by a countable Markov subshift. Moreover, it was conjectured by Sinai that geodesic flows on non-compact manifolds can be always modelled by countable alphabet subshifts. An important

contribution to approve this conjecture is the work of Gurevich and Katok[12] who obtained a coding for geodesics in the surface $H^2/SL_2(\mathbf{Z})$.

Of important aid to prove the theorem will be the thermodynamic formalism for countable alphabet subshifts introduced by Mauldin and Urbanski[16], which we have also used to describe the multifractal spectrum of local entropies for Markov systems with infinite alphabet.

2. PRELIMINARY DEFINITIONS AND FACTS

We begin by recalling the definition of topological entropy for non-compact sets. Let $f : X \rightarrow X$ be a continuous map and (X, d) a compact metrizable space. Let $\mathcal{U} = \{U_1, \dots, U_N\}$ be finite covering of X and a string is defined as a sequence

$L = (\ell_0, \ell_1, \dots, \ell_{n-1})$ such that $\{U_{\ell_0}, \dots, U_{\ell_{n-1}}\} \subset \mathcal{U}$, $\ell_i \in \{1, 2, \dots, N\}$. The length of the string

$L = (\ell_0, \ell_1, \dots, \ell_{n-1})$ is denoted by $n(L) = n$. Let us call $W_n(\mathcal{U})$ the set of the strings L with length n for the covering $\mathcal{U}$.

Let

$$X(L) = U_{\ell_0} \cap f^{-1}(U_{\ell_1}) \cap \dots \cap f^{-n+1}(U_{\ell_{n-1}}),$$

if $Z \subset X$ it says that $\Pi = \{L = (\ell_0, \ell_1, \dots, \ell_{n-1})\}$ covers Z if

$$Z \subset \bigcup_{L \in \Pi} X(L).$$

For any real number s :

$$M(Z, \mathcal{U}, s, n) = \inf_{\Pi} \sum_{L \in \Pi} \exp(-sn(L)),$$

where the infimum is taken over all the collections of strings $\Pi \subset W_n(\mathcal{U})$ which cover Z .

Let

$$M(\mathcal{U}, Z, s) = \lim_{n \rightarrow \infty} M(\mathcal{U}, Z, s, n).$$

There is an unique number $\bar{s}$ such that $M(\mathcal{U}, Z, \cdot)$ jumps from $+\infty$ to 0, now let

$$h_{top}(f, Z, \mathcal{U}) = \bar{s} = \sup \{s : M(\mathcal{U}, Z, s) = +\infty\} = \inf \{s : M(\mathcal{U}, Z, s) = 0\}.$$

Finally the number

$$(4) \quad h_{top}(f, Z) = \lim_{\Delta(\mathcal{U}) \rightarrow 0} h_{top}(f, Z, \mathcal{U}), \quad \Delta(\mathcal{U}) = \text{diameter of } \mathcal{U}$$

is the *topological entropy* of f restricted to Z .

When X is not necessarily compact then we set

$\bar{h}_{top}(f, Z) := \sup \{h_{top}(f, Z \cap K) : K \subset X \text{ and } K \text{ compact}\}$, it holds that if $Z \subset K \subset X$ and K is compact then

$$\bar{h}_{top}(f, Z) = h_{top}(f, Z), \text{ indeed}$$

$$h_{top}(f, Z) = h_{top}(f, Z \cap K) = \bar{h}_{top}(f, Z \cap K) = \bar{h}_{top}(f, Z).$$

Now we revise the main results of the Mauldin and Urbanski thermodynamic formalism which is a main ingredient to be used. Let us consider subshifts like those defined in Eq. (3). The space $\Sigma_{\mathcal{A}, B}^\infty$ is endowed with the metric $d_t(\omega, \eta) = \exp(-t(N-1))$, where N is the length of the maximal initial common of symbols of sequences ω and η . By Ω^* is denoted the space of finite B -admissible sequences and by Ω_n^* the space of admissible sequences of length n , i.e. the elements of Ω_n^* will be strings $\omega_1\omega_2\ldots\omega_n$ with $B(\omega_i, \omega_{i+1}) = 1$, for any $i = 1, 2, \ldots, n$. The length of a finite string ω is denoted by $|\omega|$. For a sequence $\omega \in \Sigma_{\mathcal{A}, B}^\infty$ the expression $\Pi_n(\omega)$ means its truncation to the first n symbols.

An incidence matrix B is *irreducible* if for any $i, j \in \mathcal{A}$ there exists some sequence in Ω_n^* whose first symbol is i and the last one is j . A matrix B is *finitely irreducible* if there exists a set $\Lambda \subset \Omega^*$ such that for any $i, j \in \mathcal{A}$ there is a sequence belonging to Λ that links i with j .

A matrix B is *primitive* if there exists n such that for any $i, j \in \mathcal{A}$ there is a sequence of length n , $\omega_1\omega_2\ldots\omega_n \in \Omega_n^*$ with $\omega_1 = i$ and $\omega_n = j$ and is *finitely primitive* if there exists a set $\Lambda \subset \Omega^*$ formed by strings of the same length such that for any $i, j \in \mathcal{A}$ there is a sequence $\omega_1\omega_2\ldots\omega_n \in \Lambda$ such that $i\omega_1\omega_2\ldots\omega_nj \in \Omega^*$.

If $\omega \in \Sigma_{\mathcal{A}, B}^\infty$, then the *cylinder of length n* which contains ω is the set

$$(5) \quad C^n(\omega) = \{\eta \in \Sigma_{\mathcal{A}, B}^\infty : \Pi_n(\omega) = \Pi_n(\eta)\}$$

and by C^n is denoted the set of all cylinders of length n .

For a potential $\psi : \Sigma_{\mathcal{A}, B}^\infty \rightarrow \mathbf{R}$, the *topological pressure* of ψ is the number

$$(6) \quad P(\psi, \sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[\sum_{\omega: \Pi_n(\omega) \in \Omega_n^*} \exp \left(\sup_{\eta \in C_n(\omega)} \{(S_n(\psi))(\eta)\} \right) \right],$$

where $S_n(\psi)(x) := \sum_{i=0}^{n-1} \psi(\sigma^i(x))$ (*statistical sum*). Let us denote $Z_n(\psi) = \sum_{\omega: \Pi_n(\omega) \in \Omega_n^*} \exp \left(\sup_{\eta \in C_n(\omega)} \{(S_n(\psi))(\eta)\} \right)$.

A potential ψ is called *acceptable* if it is uniformly continuous with respect to the metric d_t and

$$\text{osc}(\psi) := \sup_i \{ \sup \varphi|_{C_i} - \inf \varphi|_{C_i} \} < \infty, \text{ where } C_i = \{\omega : \Pi_0(\omega) = i\}.$$

Let F a finite subset of the alphabet $\mathcal{A}$, the topological pressure restricted to F is

$$(7) \quad P_F(\psi, \sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[\sum_{\omega: \Pi_n(\omega) \in \Omega_n^* \cap F^n} \exp \left(\sup_{\eta \in C_n(\omega \cap F)} \{(S_n(\psi))(\eta)\} \right) \right],$$

$$\text{where } C^n(\omega \cap F) = \{\eta \in F^{\mathcal{A}} : \Pi_n(\omega) = \Pi_n(\eta)\}.$$

Theorem (*Approximation theorem*[16]): If $\psi : \Sigma_{\mathcal{A}, B}^\infty \rightarrow \mathbf{R}$ is acceptable and the incidence matrix B is finitely reducible then

$$P(\psi, \sigma) = \sup \{P_F(\psi, \sigma) : F \subset \mathcal{A} \text{ finite}\}$$

Theorem (*Variational principle* [16]): Let the symbolic dynamical system $(\Sigma_{\mathcal{A}, B}^\infty, \sigma)$ where the incidence matrix B is finitely irreducible, and let $\psi : \Sigma_{\mathcal{A}, B}^\mathbf{Z} \rightarrow \mathbf{R}$ be acceptable. Then

$$P(\psi, \sigma) = \sup \left\{ h_\mu(\sigma) + \int \psi d\mu \right\},$$

where the supremum is taken over all the σ -invariant measures with $\int \psi d\mu > -\infty$ and $h_\mu(\sigma)$ is the usual measure theoretic entropy.

If B is finitely primitive then $P(\psi, \sigma) < \infty$ if and only if $Z_1(\psi) < \infty$.

An *equilibrium state* for the potential ψ is a measure μ_ψ such that $P(\psi, \sigma) = h_{\mu_\psi}(\sigma) + \int \psi d\mu_\psi$.

A *Gibbs state* associated to a potential ψ is a measure μ_ψ such that for any positive integer n and for any $\omega \in \Sigma_{\mathcal{A},B}^\infty$ with $|\omega| = n$, there are constants $A_1, A_2 > 0$ such that

$$(8) \quad A_1 \leq \frac{\mu_\psi(C^n(\omega))}{\exp(-nP(\psi, \sigma)) \exp(\sup\{S_n(\psi|_{C^n(\omega)})\})} \leq A_2.$$

To ensure existence and uniqueness of Gibbs states the following special class of potentials must be considered:

Let us call $\mathcal{S}$ to the class constituted by the potentials $\psi : \Sigma_{\mathcal{A},B}^\infty \rightarrow \mathbf{R}$ satisfying the next conditions:

- 1) $V(\psi) := \sum_{n=1}^{\infty} \left[\sup_{\omega, \eta: \Pi_n(\omega) = \Pi_n(\eta)} \{|\psi(\omega) - \psi(\eta)|\} \right] < \infty$.
- 2) $\sum_{n=1}^{\infty} \exp(\sup \varphi|_{C_n}) < \infty$, where $C_n = \{\omega : \Pi_0(\omega) = n\}$.

If the incidence matrix is finitely irreducible then the Gibbs states are eigenmeasures of the conjugate of the Ruelle operator[25]

$$(9) \quad \begin{aligned} \mathcal{L} &= \mathcal{L}_\varphi : C(\Sigma_{\mathcal{A},B}^\infty) \rightarrow C(\Sigma_{\mathcal{A},B}^\infty) \\ \mathcal{L}_\psi(\phi)(\omega) &= \sum_{n: B(n, \omega_1)=1} \exp \psi(\langle n, \omega \rangle) \phi(\langle n, \omega \rangle), \end{aligned}$$

where $\langle n, \omega \rangle$ denotes the sequence $n\omega_1\omega_2\dots$. To be really an eigenmeasure of the conjugate $\mathcal{L}_\psi^*(\phi)(\psi) = \mu(\mathcal{L}_\psi(\phi))$ a Gibbs state should be restricted the action of the operator to a special class of functions. Notice that if $\mathcal{L}_\psi$ acts on the class $\mathcal{S}_t$ then $\|\mathcal{L}_\psi\|_0 < \infty$. Thus if $\psi \in \mathcal{S}_t$ and the matrix is finitely irreducible then there exists an ergodic measure μ_ψ such that $\mathcal{L}_\psi^*(\mu_\psi) = \lambda\mu_\psi$, where $\lambda = \exp(P(\psi, \sigma))$ [16]. For the uniqueness of Gibbs states the incidence matrix should be finitely primitive.

The analytical properties of the pressure are established in the following way[16]:

Let the space

$\mathcal{H}_0 = \{\psi : I^\infty \rightarrow \mathbf{C} \text{ bounded and continuous}\}$, here we take $I = \mathbf{N}$. and let

$\mathcal{H}_t = \{\psi \in \mathcal{H}_0 : M_t(\psi) < \infty\}$, where

$M_t(\psi) := \inf \{L : |\psi(\omega) - \psi(\eta)| \leq L d_t(\omega, \eta), \omega_1 = \eta_1\} < \infty$ ($M_t(\psi) < \infty$ implies ψ acceptable).

The space $\mathcal{H}_t$ is endowed with the norm

$\|\psi\|_t := \|\psi\|_0 + M_t(\psi)$. Thus this space is Banach.

If $\psi \in \mathcal{H}_t \cap \mathcal{S}$, for some $t > 0$, and the incidence matrix is irreducible then there exist an eigenmeasure of the conjugate operator $\mathcal{L}_\varphi^*$ which is a Gibbs state for φ .

Now we consider a renormalized operator $\mathcal{L}_0 = \exp(P(\psi, \sigma))\mathcal{L}_\psi$, the number $\exp(P(\psi, \sigma))$ is precisely the spectral radius of $\mathcal{L}_\psi$ [16]. Also is valid the following estimation [16] for $\mathcal{L}_0 : \mathcal{H}_t \rightarrow \mathcal{H}_t$

$$\left\| \mathcal{L}_0^n(\phi) - \int \psi d\mu_\psi \right\|_t \leq K \gamma^n \|\phi\|_t,$$

for some constants K, γ ($0 < \gamma < 1$) and μ_ψ the Gibbs state associated to the potential $\psi \in \mathcal{H}_t \cap \mathcal{S}_t$. From this estimation deduces that the spectrum of $\mathcal{L}_0$ consists in an isolated simple eigenvalue $\lambda = \exp(P(\psi, \sigma))$ and a set contained in the disc $\{|z| < \gamma\lambda\}$.

Let the operator $\mathcal{L}_{q,\psi} \in \mathbf{L}(\mathcal{H}_t \cap \mathcal{S})$ given by

$$\mathcal{L}_{q,\psi}(\phi) = \sum_{n: B(n, \omega_1)=1} \exp[\psi(\langle n, \omega \rangle) + q\tau(\langle n, \omega \rangle)] \phi(\langle n, \omega \rangle),$$

then[16] the map $q \mapsto \mathcal{L}_{q,\psi}$ is differentiable for $q > \theta(\psi, \tau)$, where $\theta(\psi, \tau) := \max \{\inf \{q : \sup \{q\tau\} < \infty\}, \inf \{q : \sup \{\psi\} + \sup \{q\tau\} < \infty\}\}$.

By the perturbation theory of analytic dependence of an isolated eigenvalue, holds that the map

$(q, s) \mapsto P(q\psi + s\bar{\tau}, \sigma)$ is real analytic in both the two variables for $q > \theta(\psi) := \inf \{q : P(q\psi, \sigma) < \infty\}$, $s > \theta(\psi, \tau)$

The derivative with respect to q of the pressure function is given by

$$(10) \quad \frac{dP(q\psi + s\bar{\tau}, \sigma)}{dq} \Big|_{q=q_0} = \int \psi d\mu_{q_0\psi + s\bar{\tau}},$$

for $q > \theta(\psi)$ and where $\mu_{q_0\psi+s\bar{\tau}}$ is the Gibbs state for the potential $q_0\psi + s\bar{\tau} \in \mathcal{H}_t \cap \mathcal{S}$. Also the maps $\psi, \bar{\tau}$ must verify that $\int (|\psi| + |\bar{\tau}|) d\mu_{q_0\psi-s\bar{\tau}} < \infty$.

Let the “free energy” $T(q) = T_\psi(q) := P(q\psi, \sigma) - qP(\psi, \sigma)$, now if the potentials are in the class $\mathcal{H}_t \cap \mathcal{S}$ then the free energy is real analytic for $q > \theta(\psi)$ and

$T(q) = \int \psi d\mu_q - P(\psi, \sigma)$, with μ_q the Gibbs state associated to the potential $q\psi - P(\psi, \sigma)$.

Recall the topological entropy for non-compact phase space,

$$\bar{h}_{top}(\sigma, Z) := \sup \{h_{top}(\sigma, Z \cap K) : K \subset \Sigma_{\mathcal{A}, B}^\infty \text{ with } K \text{ compact and } \sigma - \text{invariant}\},$$

in this case compact means a finite alphabet subshift, and let (approximation theorem and variational principle)

$$\begin{aligned} h(\sigma) &= P(0, \sigma) = \sup \{h(\sigma|_K) : K \subset \Sigma_{\mathcal{A}, B}^\infty \text{ with } K \text{ compact and } \sigma - \text{invariant}\} \\ &= \sup \{h_\mu(\sigma) : \mu \text{ is } \sigma - \text{invariant}\}, \end{aligned}$$

thus

$$\begin{aligned} h(\sigma, Z) &= \sup \{h(\sigma|_K) : K \subset Z \text{ with } K \text{ compact and } \sigma - \text{invariant}\} \\ &= \sup \{h_\mu(\sigma|_K) : \mu \text{ is } \sigma - \text{invariant}\}. \end{aligned}$$

We have

$$h(\sigma, K) = h(\sigma, Z \cap K) = h_{top}(\sigma, Z \cap K), \text{ so that } h(\sigma, Z) \leq h_{top}(\sigma, Z).$$

The other inequality also holds, in fact

$h_{top}(\sigma, Z \cap K) = h(\sigma, Z \cap K) = \sup \{h_\mu(\sigma) : \mu(Z \cap K) = 1\} \leq h(\sigma, Z)$, then $h(\sigma, Z) \geq h_{top}(\sigma, Z)$. In this way we have,

$$\bar{h}_{top}(\sigma, Z) = h_{top}(\sigma, Z) = h(\sigma, Z \cap K).$$

In view of this we shall denote the topological entropy in $\Sigma_{\mathcal{A}, B}^\infty$, also by h_{top} .

The multifractal description for the spectrum of local entropies in the case of countable alphabet subshifts was done as:

Theorem[17] : Let $\Sigma_{\mathcal{A}, B}^\infty = \{\omega = (\omega_n)_{n \in \mathbf{Z}} : \omega_n \in \mathcal{A}, B(\omega_i, \omega_{i+1}) = 1, \text{ for } i \in \mathbf{Z}\}$ be a subshift of finite type, whose incidence matrix is finitely primitive and $\psi : \Sigma_{\mathcal{A}, B}^\infty \rightarrow \mathbf{R}$ belonging to the class $\mathcal{S}$, or $\mathcal{H}_t \cap \mathcal{S}$ according to the differentiability or analyticity properties of the free energy desired, and with $\int \psi d\mu > -\infty$ for any σ -invariant measure μ . Let $\mathcal{E}(\alpha) := h_{top}(\sigma, K_\alpha)$, with $K_\alpha = \{\omega : h_\mu(\sigma, \omega) = \alpha\}$, $\alpha \in (\alpha_i, \alpha_s)$, now if $\alpha(q) = -T'(q)$ then $\mathcal{E}(\alpha(q)) = T(q) + q\alpha(q)$. Besides the free energy $T(q)$ is strictly convex.

The free energy $T(q)$ is real analytic for any $q \in I_1 \subset \{q : P(q\psi, \sigma) < \infty\}$, so the spectrum $\mathcal{E}(\alpha)$ is analytic in the interval for $\alpha \in (\alpha_i, \alpha_s)$, such that $T(q) \in (\alpha_i, \alpha_s)$ when $q \in I_1$. Notice that in the case of countable subshifts the spectrum is not in general bounded, it may be unbounded for values $q_0 < \inf \left\{ q : P(q\hat{\psi}, \sigma) < \infty \right\}$.

Thus $T(q)$ has a Legendre transform $L(\alpha) = \inf \{T(q) + q\alpha : q \in \mathbf{R}\}$ which is strictly convex. We have that $L'(\alpha) = q$ if $\alpha = -T'(q)$, with the values of q, α for which the free energy and the legendre transform are analytic. So it can be deduced that the Legendre transform of $T(q)$ is precisely $\mathcal{E}(\alpha)$ and $(T, \mathcal{E})$ form a pair of Legendre, i.e.

$\mathcal{E}(\alpha) = \inf \{T(q) + q\alpha : q \in \mathbf{R}\}$ and $T(q) = \sup \{\mathcal{E}(\alpha) - q\alpha : \alpha \in (\alpha_i, \alpha_s)\}$. If q is interpreted as inverse of the temperature in the free energy function $T(q)$, then the parameter α may be the internal energy per volume and $\mathcal{E}(\alpha)$ the internal energy. In this way the spectrum is described by the family of measures $\{\mu_q\}$ and the Legendre pair $(T, \mathcal{E})$.

3. FINER STRUCTURE OF THE SPECTRUM OF LOCAL ENTROPIES FOR COUNTABLE ALPHABET SUBSHIFTS

We shall consider symbolic spaces

$\Sigma_{\mathcal{A}, B}^\infty = \{\omega = (\omega_n)_{n \in \mathbf{Z}} : \omega_n \in \mathcal{A}, B(\omega_i, \omega_{i+1}) = 1, \text{ for } i \in \mathbf{Z}\}$, with incidence matrix finitely primitive and we work with potentials in the class $\mathcal{H}_t \cap \mathcal{S}$. These conditions ensure regular properties of the free energy. The law of the iterated logarithm presented in [10] is formulated in a dynamical context as:

Let $\psi, \phi \in \mathcal{H}_t \cap \mathcal{S}$ with $\int \phi d\mu_\psi = 0$ and such that the variance $Var(\phi, \mu_\psi) := \int \psi^2 d\mu_\psi$ is strictly positive then

$$\lim_{n \rightarrow \infty} \frac{S_n(\phi)(\omega)}{\sqrt{n \log \log n}} = \sqrt{2Var(\phi, \mu_\psi)} \text{ for } \mu - a.e. \omega \in \Sigma_{\mathcal{A}, B}^\infty.$$

The condition $Var(\phi, \mu_\psi) > 0$ is verified for the potentials $\mu_\phi \neq \mu_{\max}$, $\mu_\psi \neq \mu_{\max}$, where $\mu_{\max}$ is the measure of maximal entropy, i.e. the measure associated to the potential $\psi \equiv 0$. This is due the classification of Gibbs states given by the Mauldin-Urbanski formalism. The condition $\mu_\psi \neq \mu_{\max}$ is equivalent to ψ not being cohomologous to a constant function.

Let $\mathcal{C}^n(\omega)$ be the cylinder of length n containing ω (cf. eq. 5); the *upper and lower measure-theoretic local entropies* can be defined (the partition by cylinders is

a generator) by

$$\overline{h_\mu}(\sigma, \omega) = \lim_{n \rightarrow \infty} \sup -\frac{1}{n} \log \mu(\mathcal{C}^n(\omega))$$

$$\underline{h_\mu}(\sigma, \omega) = \lim_{n \rightarrow \infty} \inf -\frac{1}{n} \log \mu(\mathcal{C}^n(\omega)).$$

By the Brin-Katok theorem[8], the local entropy does exist, i.e. $\overline{h_\mu}(\sigma, \omega) = \underline{h_\mu}(\sigma, \omega) := h_\mu(\sigma, \omega)$ for $\mu - a.e.$ $\omega \in \Sigma_{\mathcal{A}, B}^\infty$.

Consider now the multifractal sets

$$(11) \quad K_\alpha = \{\omega \in \Sigma_{\mathcal{A}, B}^\infty : h_\mu(\sigma, \omega) = \alpha\},$$

with a finer decomposition

$$(12) \quad K_{\alpha, V} = \left\{ \omega \in \Sigma_{\mathcal{A}, B}^\infty : \limsup_{n \rightarrow \infty} \frac{\mu(\mathcal{C}^n(\omega)) - n\alpha}{\sqrt{n \log \log n}} = V \right\}$$

Now we reformulate the result of this note:

Theorem : Let ψ be a potential in the class $\mathcal{H}_t \cap \mathcal{S}$ with $\mu_\psi \neq \mu_{\max}$, or equivalently ψ not cohomologous to a constant, if $\alpha(q) = -T'(q) = P(\psi, \sigma) - \int \psi d\mu_q$ and $\mathcal{E}(\alpha) := h_{top}(\sigma, K_\alpha)$ then $\mathcal{E}(\alpha(q)) = \mathcal{E}(\alpha(q), V(q))$, for $q \in I_1 \subset \{q : P(q\psi, \sigma) < \infty\}$ and for enough large $V(q)$.

Proof: Let $\psi_q := \varphi - P(\psi, \sigma) - \alpha(q) = \psi - \int \psi d\mu_q$, we have $\int \psi_q d\mu_q = 0$, where μ_q is the Gibbs state associated to the potential $q\psi - P(q\psi, \sigma)$, or what is the same to $q\psi$ because the addition of a constant gives the same Gibbs states By the law of the iterated logarithm

$$(13) \quad \limsup_{n \rightarrow \infty} \frac{S_n(\psi_q)(\omega)}{\sqrt{n \log \log n}} = \sqrt{2Var(\psi_q, \mu_q)}.$$

On the other hand $\limsup_{n \rightarrow \infty} \frac{S_n(\psi_q)(\omega)}{\sqrt{n \log \log n}} = \limsup_{n \rightarrow \infty} \frac{S_n(\psi)(\omega) - n(P(\psi, \sigma) - \alpha(q))}{\sqrt{n \log \log n}} = \limsup_{n \rightarrow \infty} \frac{\mu(\mathcal{C}^n(\omega)) - n\alpha(q)}{\sqrt{n \log \log n}}$. So that the claim of the theorem is verified for the special value $V(q) = Var(\psi_q, \mu_q)$, with law of the iterated logarithm. Hereafter we will try to find values greater than $V(q)$.

By the mentioned classification of Gibbs states if $\mu_\psi \neq \mu_{\max}$ then $S_n(\psi)(\omega) \neq 0$ for some $\omega \in P_n(\sigma) = \{\omega : \sigma^n(\omega) = \omega\}$, therefore, in particular, there exist sequences $\omega, \omega' \in \Sigma_{\mathcal{A},B}^\infty$ which are fixed points of σ^n with $\psi(\omega) \neq 0, \psi(\omega') \neq 0$. Let $\Pi_n(\omega)$ and $\Pi_n(\omega')$ be the respective truncations of ω and ω' to the n -first symbols. Each cylinder $\mathcal{C}^n(\omega)$ can be represented by the image of $\Sigma_{\mathcal{A},B}^\infty$ under contractile transformations $\{C_i : i \in \mathbf{Z}\}$, and let us denote $C_{\Pi_n(\omega)} = C_{\omega_0\omega_1\ldots\omega_{n-1}} = C_{\omega_{n-1}} \circ \ldots \circ C_{\omega_0}$. Given the word $\tau_0\tau_1\ldots\tau_{n-1}$ let $\eta = C_{\Pi_n(\tau)}(\omega)$ and let us define the number $\gamma_{\Pi_n(\tau)}(\omega)$ by the condition

$$\begin{aligned} S_n(\psi)(\eta) &= S_n(\psi)(C_{\tau_0\tau_1\ldots\tau_{n-1}}(\omega)) = \\ &\gamma_{\Pi_n(\tau)}(\omega)S_n(\psi)(C_{\Pi_n(\omega)}(\omega)) + (1 - \gamma_{\Pi_n(\tau)}(\omega))S_n(\psi)(C_{\Pi_n(\omega')}(\omega)). \end{aligned}$$

Since the partition by cylinders is generating

$$\lim_{|\omega| \rightarrow \infty} C_{\Pi_n(\omega)}(\omega) = \omega \text{ and } \lim_{|\omega| \rightarrow \infty} C_{\Pi_n(\omega')}(\omega) = \omega'.$$

For a potential ψ let

$U_{n,\varepsilon} = \{\mathcal{C}^n(\omega) : \mathcal{C}^n(\omega) \cap \{\eta : |S_n(\psi)(\eta)| < \varepsilon\} \neq \emptyset\}$, then for any sequence $\tau = \tau_0\tau_1\ldots\tau_{n-1}\ldots$ let us introduce perturbations $S_{n,\lambda}(\phi)(\omega)$, $\lambda \in (0, 1)$, of the statistical sum $S_n(\phi)(\omega)$ defined, following [15], by

$$\exp((S_{n,\lambda}(\psi)(C_{\Pi_n(\tau)}(\omega)))) = (1 - \lambda) \exp(S_n(\psi)(C_{\Pi_n(\tau)}(\omega))) \text{ if } \mathcal{C}^n(\tau) \in U_{n,\varepsilon}$$

$$\begin{aligned} \exp((S_{n,\lambda}(\psi)(C_{\Pi_n(\omega)}(\omega)))) &= \exp((S_n(\psi)(C_{\Pi_n(\omega)}(\omega)))) + \\ \lambda \sum_{\mathcal{C}^n(\tau) \in U_{n,\varepsilon}} \gamma_{\Pi_n(\tau)}(\omega) \exp((S_n(\psi)(C_{\Pi_n(\omega)}(\omega)))) & \end{aligned}$$

$$\begin{aligned} \exp((S_{n,\lambda}(\psi)(C_{\Pi_n(\omega')}(\omega)))) &= \exp((S_n(\psi)(C_{\Pi_n(\omega')}(\omega)))) + \\ \lambda \sum_{\mathcal{C}^n(\tau) \in U_{n,\varepsilon}} (1 - \gamma_{\Pi_n(\tau)}(\omega)) \exp((S_n(\psi)(C_{\Pi_n(\omega)}(\omega)))) & \end{aligned}$$

On cylinders corresponding to other contractions the statistical sums are not modified.

Let us consider the Ruelle operator[25]

$$(14) \quad \mathcal{L}_\psi(\phi) = \sum_{n:B(n,\omega_1)=1} \exp[\psi(\langle n, \omega \rangle)] \phi(\langle n, \omega \rangle)$$

where recall that $\langle n, \omega \rangle$ is the sequence $n\omega_1\omega_2\ldots$.

Similarly to [21] can be established that $\mathcal{L}_{S_n, \lambda(\psi)}(1) = \mathcal{L}_{S_n(\psi)}(1)$ and also $\mathcal{L}_{S_n, \lambda(\psi)}S_n(\phi) = \mathcal{L}_{S_n(\psi)}S_n(\phi)$. The only precaution is the narrower class of potential considered due the lack of compactness of the phase space.

Like in [21] can be assumed, without losing generality, that any pair of potentials ψ, ϕ satisfy $\mathcal{L}_\psi \psi = 0$ and $\mathcal{L}_\psi 1 = 1$. This is done by considering adjusted potentials $\bar{\psi} = \psi - P(\psi, \sigma)$, since the spectral radius of $\mathcal{L}_\psi$ is $\exp(-P(\psi))$ [25] we get $\mathcal{L}_{\bar{\psi}}(1) = \exp(-P(\bar{\psi})) = 1$ (recall that the Gibbs states are not affected by adding constants to the potentials). We have that $\|\mathcal{L}_\psi^n \psi\|_t \leq C\beta^n \|\psi\|_t$, for some $\beta \in (0, 1)$. Thus if $\phi = \sum_{n=1}^{\infty} \mathcal{L}_\psi^n \psi$ then this series converges in $(\mathcal{H}_t \cap \mathcal{S}, \|\cdot\|_t)$.

Let $\tilde{\psi} = \psi - C_\sigma \phi + \phi$, where C_σ is the composition operator $C_f \phi = \phi \circ f$, thus $\mathcal{L}_\psi(C_\sigma \phi) = \phi$ and

$\mathcal{L}_\psi \tilde{\psi} = \mathcal{L}_\psi \psi - \phi + \mathcal{L}_\psi \phi = \mathcal{L}_\psi \psi - \mathcal{L}_\psi \psi = 0$. Therefore can be assumed that $\mathcal{L}_{S_n(\psi)}(1) = 1$, $\mathcal{L}_{S_n, \lambda(\psi)}S_n(\phi) = 0$ and consequently, $\mathcal{L}_{S_n, \lambda(\psi)}(1) = 1$, $\mathcal{L}_{S_n, \lambda(\psi)}S_n(\phi) = 0$. Since $\int \psi d\mu_\varphi = \int \psi d\mathcal{L}_\psi^*(\mu_\varphi) = \int \mathcal{L}_\psi \psi d\mu_\varphi = 0$ we have $\mathcal{L}_\psi \psi = 0$ then $\int \psi d\mu_\varphi = 0$.

It holds that

$$\lim_{n \rightarrow \infty} \frac{1}{n} (S_n(\psi)(C_{\Pi_n(\omega)}(\omega))) = \lim_{n \rightarrow \infty} \frac{1}{n} (\psi(C_{\Pi_n(\omega)}(\omega)) + (n-1)\psi(\omega)) = \delta := \psi(\omega)$$

and similarly

$$\lim_{n \rightarrow \infty} \frac{1}{n} (S_n(\varphi)(C_{\Pi_n(\omega')}(\omega))) = \delta' := \psi(\omega').$$

For the class of potentials considered is verified a bounded distortion property in the following way:

$$\begin{aligned} & |S_n(\psi)(C_{\Pi_n(\omega)}(\omega)) - S_n\psi(C_{\Pi_n(\omega)}(\omega'))| \\ & \leq \sum_{i=0}^{n-1} |\psi(\sigma^i(C_{\Pi_n(\omega)}(\omega))) - \psi(\sigma^i(C_{\Pi_n(\omega)}(\omega')))| \\ & \leq \sum_{i=0}^{n-1} M_t(\psi) d_t(\sigma^i(C_{\Pi_n(\omega)}(\omega)), \sigma^i(C_{\Pi_n(\omega)}(\omega'))) \\ & \leq M_t(\psi) \frac{e^{-t}}{1 - e^{-t}} d_t(\sigma^i(C_{\Pi_n(\omega)}(\omega)), \sigma^i(C_{\Pi_n(\omega)}(\omega'))), \end{aligned}$$

thus for ω, ω' in the same cylinder

$$S_n\psi(C_{\Pi_n(\omega)}(\omega')) - K \leq S_n(\psi)(C_{\Pi_n(\omega)}(\omega)) \leq K + S_n\psi(C_{\Pi_n(\omega)}(\omega'))$$

$$S_n(\psi)(C_{\Pi_n(\omega')}(\omega')) - K \leq S_n(\psi)(C_{\Pi_n(\omega')}(\omega)) \leq K + S_n(\psi)(C_{\Pi_n(\omega')}(\omega')),$$

for some constant K , and so that

$$\begin{aligned} S_n(\psi) (C_{\Pi_n(\omega)}(\omega))^2 &\geq (K + n\delta)^2 \\ S_n(\psi) (C_{\Pi_n(\omega')}(\omega))^2 &\geq (K - n\delta')^2. \end{aligned}$$

Let μ_λ be the Gibbs state for the potential $S_{n,\lambda}(\psi)$,

with the assumption $\mathcal{L}_{S_{n,\lambda}(\psi)} S_n(\phi) = 0$, the above bounds lead to the estimation

$$(15) \quad \int S_n^2(\phi) d\mu_\lambda \geq (K + n\delta)^2 \mu_\lambda(C_{\Pi_n(\omega)}(\omega)) + (K - n\delta')^2 \mu_\lambda(C_{\Pi_n(\omega')}(\omega)).$$

Let $\mathcal{E}(\alpha(q))$ the entropy spectrum for the level parameterization $\alpha(q) = -T'(q) = P(\psi, \sigma) - \int \psi d\mu_q$ and $\psi_q := \varphi - P(\psi, \sigma) - \alpha(q)$, being $\int \psi_q d\mu_q = 0$, where μ_q is the Gibbs state for the potential $q\psi - qP(\psi, \sigma)$. If $\mu_{\lambda,q}$ is the Gibbs state associated to the potential $S_{n,\lambda}(q\psi - qP(\psi, \sigma))$ then $\int \psi_q d\mu_{\lambda,q} = 0$, since can be assumed $\mathcal{L}_{S_{n,\lambda}(q\psi - qP(\psi, \sigma))} S_n(\psi_q) = 0$.

Let $V(q) := \text{Var}(\psi_q, \mu_q)$ and let us consider the map

$\lambda \mapsto \text{Var}(S_n(\psi_q), \mu_{\lambda,q})$, by the Mauldin-Urbanski thermodynamic formalism this assignation is continuous, because $\mu_{\lambda,q}$ is the eigenvector corresponding to the eigenvalue $\exp(-P(S_{n,\lambda}(q\psi - qP(\psi, \sigma))))$ and the map $\lambda \mapsto P(S_{n,\lambda}(\varphi_q))$ is continuously differentiable.

If $\bar{V}(q) \geq \text{Var}(S_n(\psi_q), \mu_{0,q})$ then, by the estimations of eq.(13),

$\text{Var}(S_n(\psi_q), \mu_{\lambda,q}) \geq \bar{V}(q)$ and so there is a $\bar{\lambda} \in (0, 1)$ such that $\text{Var}(S_n(\psi_q), \mu_{\bar{\lambda},q}) = \bar{V}(q)$. Therefore, by the law of the iterated logarithm

$$\limsup_{m \rightarrow \infty} \frac{S_{nm}(\psi_q)(\omega)}{\sqrt{m \log \log m}} = \sqrt{2n\bar{V}(q)}.$$

Let $r = nm + k$, then

$$\frac{S_r(\psi_q)(\omega)}{\sqrt{r \log \log r}} = \frac{S_{nm}(\psi_q)(\omega) + S_k(\psi_q)(\sigma^{nk}(\phi))}{\sqrt{(nm + k) \log \log (nm + k)}},$$

so that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{S_r(\psi_q)(\omega)}{\sqrt{r \log \log r}} &= \limsup_{m \rightarrow \infty} \frac{S_{nm}(\psi_q)(\omega)}{\sqrt{nm \log \log (nm + k)}} \\ &= \sqrt{\frac{1}{n}} \limsup_{m \rightarrow \infty} \frac{S_{nm}(\psi_q)(\omega)}{\sqrt{m \log \log m}} = \sqrt{2\bar{V}(q)}. \end{aligned}$$

Thus $\mathcal{E}(\alpha(q)) = \mathcal{E}(\alpha(q), \bar{V}(q))$, and the spectrum $\mathcal{E}(\alpha(q), \bar{V}(q))$ is described by the family of measures $\{\mu_{\bar{\lambda}, q}\}$. ■

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