

LAGRANGIAN AND HAMILTONIAN FORMULATIONS OF THE DAMPED HARMONIC OSCILLATOR USING CAPUTO FRACTIONAL DERIVATIVE

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ABSTRACT. In this work, the Caputo fractional derivative is used to construct the Lagrangian and the Hamiltonian formulations for nonconservative mechanical systems. The Euler-Lagrange equations are derived using a variational principle. Hamilton's equations of motion are then obtained by introducing a set of momenta and a Hamiltonian in a manner that is more consistent with that used in conventional classical mechanics. The proposed formalism is applied to the damped harmonic oscillator, as an example.

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1. INTRODUCTION

It is generally known that modern classical mechanics deals only with conservative systems; while almost all classical systems are nonconservative. Accordingly, a number of methods have been proposed to handle this paradox [26, 5, 4, 8, 9, 22]. The most recent method is based on fractional derivatives [16, 14, 10, 17, 25], which have played a significant role in various fields of science and engineering [13, 15, 18, 21, 6, 27]. Riewe used fractional derivatives [24, 23] to obtain a version of the Euler-Lagrange equation and Hamilton's equations that are valid for both conservative and nonconservative systems. Riewe's work was generalized by Rabei *et al.* [19] who used the Laplace transforms of fractional derivatives to obtain a general formula for a potential of any arbitrary force, conservative or nonconservative. However, when Riewe applied his theory to the damped harmonic oscillator, he assumed the left Riemann-Liouville fractional derivative to be equal to the right Riemann-Liouville fractional derivative, which is not true.

Agrawal [1] attempted to solve the problem by regarding the left and right Riemann-Liouville fractional derivatives as not equal. He applied the fractional variational problem to obtain the corresponding Euler-Lagrange equation. The passage from the Lagrangian to the Hamiltonian formulation was achieved by Rabei *et al.* [20]. These authors used the methods of ordinary classical mechanics to obtain the Hamiltonian and Hamilton's equations of motion.

The Euler-Lagrange equation derived by Agrawal was not applied for the damped harmonic oscillator. Klimek [11, 12] used the symmetric fractional derivative to derive the Euler-Lagrange equation and Hamilton's equations for nonconservative systems. The method was applied to the fractional harmonic oscillator and to the classical frictional force that is proportional to velocity.

Recently, Agrawal [2] used the Caputo fractional derivative to derive the Euler-Lagrange equation. He demonstrated that in this case the fractional initial condition may be necessary. The fractional Hamilton formulation for systems described by using the Caputo fractional derivative was given by Baleanu and Agrawal [3]. It led to the same result obtained by the fractional Euler-Lagrange equation of motion derived earlier by Agrawal.

In this paper, we use the Caputo fractional derivative to construct a fractional Lagrangian and Hamiltonian formulation for nonconservative systems. The set of

momenta is introduced in a manner that is more consistent with that used in conventional classical mechanics [7]. The proposed formalism is then applied to the damped harmonic oscillator.

The plan of the paper is as follows: in section 2 the fractional Euler-Lagrange equation is formulated by using a variational principle. Section 3 deals with fractional Hamilton's equations. In section 4 the damped harmonic oscillator is investigated, as an example. Section 5 represents the conclusion.

2. FRACTIONAL EULER-LAGRANGE EQUATION

In this section we present briefly the definitions of the left and right Caputo fractional derivatives, together with the corresponding formulation of Lagrangian dynamics by following the steps of the conventional calculus of variations.

The Left Caputo Fractional Derivative is given by

$$(1) \quad {}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} \left(\frac{d}{d\tau}\right)^m f(\tau) d\tau;$$

Likewise, the Right Caputo Fractional Derivative is

$$(2) \quad {}_x^C D_b^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_x^b (\tau-x)^{m-\alpha-1} \left(-\frac{d}{d\tau}\right)^m f(\tau) d\tau.$$

where α satisfies the relation $m-1 \leq \alpha < m$ and Γ denotes Euler's gamma function.

To derive the Euler-Lagrange equation for the nonconservative systems, we will consider the Lagrangian.

$$(3) \quad L = L(q, q_n, \dot{q}_n, t),$$

Where q , q_n and $\dot{q}_n$ represent the generalized coordinate and their time derivatives:

$$(4) \quad q_n = ({}_a^C D_t^\alpha)^n q(t)$$

and

$$(5) \quad q_n' = ({}_t^C D_b^\beta)^n q(t).$$

Here $0 < \alpha, \beta \leq 1$, and n denotes the order of derivative defining the generalized coordinate q ($n=1, \dots, N$).

Following the same pattern of the conventional calculus of variation used in classical mechanics. One can derive the Euler-Lagrange equation by starting with the integral.

$$(6) \quad J = \int_a^b L(q, q_n, q_n', t) dt.$$

To apply the principle of variational calculus, we will define a family of curves:

$$(7) \quad q(t) = q^*(t) + \varepsilon \eta(t),$$

which satisfy the following boundary conditions:

$$(8) \quad \eta(a) = \eta(b) = 0,$$

indicating a fixed ends at a and b .

Since the Caputo fractional derivative is a linear operator, it follows that

$$(9) \quad q_n(t) = q_n^*(t) + \varepsilon \eta_n(t),$$

and

$$(10) \quad q_n'(t) = q_n^{*'}(t) + \varepsilon \eta_n'(t),$$

$\eta_n(t)$ and $\eta_n'(t)$ being defined as

$$(11) \quad \eta_n(t) = ({}_a^C D_t^\alpha)^n \eta(t);$$

$$(12) \quad \eta_n'(t) = ({}_t^C D_b^\beta)^n \eta(t).$$

Differentiating equation (6) with respect to ε , we obtain.

$$(13) \quad \begin{aligned} \frac{dJ}{d\varepsilon} &= \int_a^b \left(\frac{\partial L}{\partial q} \frac{\partial q}{\partial \varepsilon} + \sum_{n=1}^N \frac{\partial L}{\partial q_n} \frac{\partial q_n}{\partial \varepsilon} + \frac{\partial L}{\partial q_n'} \frac{\partial q_n'}{\partial \varepsilon} \right) dt \\ &= \int_a^b \left(\frac{\partial L}{\partial q} \frac{\partial q}{\partial \varepsilon} + \sum_{n=1}^N \frac{\partial L}{\partial q_n} \eta_n(t) + \frac{\partial L}{\partial q_n'} \eta_n'(t) \right) dt. \end{aligned}$$

Using the relation between the Riemann-liouville and the Caputo fractional derivative, which can be written as [2]:

$$(14) \quad {}^C D_t^\alpha \eta(t) = {}_a D_t^\alpha \eta(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \eta(a),$$

and

$$(15) \quad {}^C D_b^\alpha \eta(t) = {}_t D_b^\alpha \eta(t) - \frac{(1-t)^{-\alpha}}{\Gamma(1-\alpha)} \eta(b).$$

the second term on the right in Eq. (13) can be written as:

$$(16) \quad \begin{aligned} \int_a^b \sum_{n=1}^N \frac{\partial L}{\partial q_n} \eta_n(t) dt &= \int_a^b \sum_{n=1}^N \left(\frac{\partial L}{\partial q_n} \right) ({}_a D_t^\alpha)^n \eta(t) dt \\ &= \int_a^b \sum_{n=1}^N \left(\frac{\partial L}{\partial q_n} \right) ({}_a D_t^\alpha)^n \eta(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \eta(a) dt \end{aligned}$$

substituting Eq.(8) in Eq.(16) we get.

$$(17) \quad \int_a^b \sum_{n=1}^N \frac{\partial L}{\partial q_n} \eta_n(t) dt = \int_a^b \sum_{n=1}^N \left(\frac{\partial L}{\partial q_n} \right) ({}_a D_t^\alpha)^n \eta(t) dt$$

using the formula for integration by parts in fractional calculus [1] with the following boundary conditions are fulfilled

$$(18) \quad \frac{\partial L}{\partial q_n}(a) = \frac{\partial L}{\partial q_n}(b) = 0$$

Eq.(17) becomes

$$(19) \quad \int_a^b \sum_{n=1}^N \left(\frac{\partial L}{\partial q_n} \right) \eta_n(t) dt = \int_a^b \sum_{n=1}^N ({}_t D_b^\alpha)^n \left(\frac{\partial L}{\partial q_n} \right) \eta(t) dt$$

Similarly, the third term in Eq.(13) can be written as:

$$(20) \quad \int_a^b \sum_{n=1}^N \frac{\partial L}{\partial q'_n} \eta'_n(t) dt = \int_a^b \sum_{n=1}^N ({}_a D_t^\beta)^n \left(\frac{\partial L}{\partial q'_n} \right) \eta dt.$$

substitution Eq. (17) and Eq. (20) in Eq. (13), we will get

$$(21) \quad \frac{dJ}{d\varepsilon} = \int_a^b \left(\frac{\partial L}{\partial q} \eta + \sum_{n=1}^N ({}_t D_b^\alpha)^n \left(\frac{\partial L}{\partial q_n} \right) \eta + \sum_{n=1}^N ({}_a D_t^\beta)^n \left(\frac{\partial L}{\partial q'_n} \right) \eta \right) dt.$$

For the integral in Eq. (6) to be extremum, the value of $\left(\frac{dJ}{d\varepsilon}\right)$ should be equal to zero. This gives the Euler-Lagrange equation

$$(22) \quad \frac{\partial L}{\partial q} + \sum_{n=1}^N ({}_t D_b^\alpha)^n \left(\frac{\partial L}{\partial q_n} \right) + \sum_{n=1}^N ({}_a D_t^\beta)^n \left(\frac{\partial L}{\partial q'_n} \right) = 0.$$

In the limit $N \rightarrow 1$ and $\alpha \rightarrow 1$, we obtain the classical Euler-Lagrange equation.

3. HAMILTON'S EQUATIONS

Introducing the set of momenta as:

$$(23) \quad p_n = \frac{\partial L}{\partial q_{(n+1)}}; \quad p'_n = \frac{\partial L}{\partial q'_{(n+1)}};$$

and the Hamiltonian

$$(24) \quad H = \sum_{n=0}^{N-1} p_n q_{(n+1)} + p'_n q'_{(n+1)} - L(q, q_n, q'_n, t),$$

we have,

$$(25) \quad \begin{aligned} dH &= \sum_{n=0}^{N-1} q_{(n+1)} dp_n + p_n dq_{(n+1)} + q'_{(n+1)} dp'_n + p'_n dq'_{(n+1)} \\ &\quad - \frac{\partial L}{\partial q} - \sum_{n=1}^N \frac{\partial L}{\partial q_n} dq_n - \sum_{n=1}^N \frac{\partial L}{\partial q'_n} dq'_n - \frac{\partial L}{\partial t} dt \\ &= \sum_{n=0}^{N-1} q_{(n+1)} dp_n + \left(p_n - \frac{\partial L}{\partial q_{(n+1)}} \right) dq_{(n+1)} \\ &\quad + \sum_{n=0}^{N-1} q'_{(n+1)} dp'_n + \left(p'_n - \frac{\partial L}{\partial q'_{(n+1)}} \right) dq'_{(n+1)} - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt \\ &= \sum_{n=0}^{N-1} q_{(n+1)} dp_n + q'_{(n+1)} dp'_n - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt \end{aligned}$$

This means that the Hamiltonian is a function of the form.

$$H = H(q, p_n, p'_n, t).$$

The total differential of this Hamiltonian reads

$$(26) \quad dH = \frac{\partial H}{\partial q} dq + \sum_{n=0}^{N-1} \left(\frac{\partial H}{\partial p_n} dp_n + \frac{\partial H}{\partial p'_n} dp'_n \right) + \frac{\partial H}{\partial t} dt.$$

Comparing Eq.(25) and Eq.(26), we get the following Hamilton's equations of motion:

$$(27) \quad \frac{\partial H}{\partial p_n} = q_{(n+1)};$$

$$(28) \quad \frac{\partial H}{\partial p'_n} = q'_{(n+1)};$$

$$(29) \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q};$$

$$(30) \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

The Euler-Lagrange equation (22) then gives

$$(31) \quad \begin{aligned} \frac{\partial L}{\partial q} &= -\sum_{n=1}^N ({}_t D_b^\alpha)^n \left(\frac{\partial L}{\partial q_n} \right) - \sum_{n=1}^N ({}_a D_t^\beta)^n \left(\frac{\partial L}{\partial q'_n} \right); \\ &= -\sum_{n=0}^{N-1} ({}_t D_b^\alpha)^{n+1} \left(\frac{\partial L}{\partial q_{n+1}} \right) - \sum_{n=0}^{N-1} ({}_a D_t^\beta)^{n+1} \left(\frac{\partial L}{\partial q'_{n+1}} \right); \\ &= -\sum_{n=0}^{N-1} {}_t D_b^{\alpha(n+1)}(p_n) - \sum_{n=0}^{N-1} ({}_a D_t^\beta)^{(n+1)}(p'_n). \end{aligned}$$

Substituting Eq.(31) in Eq.(29), we get

$$(32) \quad \frac{\partial H}{\partial q} = \sum_{n=0}^{N-1} {}_t D_b^{\alpha(n+1)}(p_n) + \sum_{n=0}^{N-1} ({}_a D_t^\beta)^{(n+1)}(p'_n).$$

In the limit $N \rightarrow 1$ and $\alpha, \beta \rightarrow 1^+$, we obtain the set of classical Hamilton's equations of motion.

4. EXAMPLE

Consider the following one-dimensional Lagrangian with left-hand coordinates given by

$$(33) \quad L = \frac{1}{2}mq_2^2 - \frac{1}{2}kq_0^2 + \frac{i\gamma}{2}(q_1)^2,$$

Where

$$q_1 = {}^C D_t^{\frac{1}{2}} q_0; \quad q_2 = {}^C D_t^1 q_0 = \dot{q}.$$

Applying the Euler equation (22), we get.

$$(34) \quad \frac{\partial L}{\partial q_0} + {}_t D_b^{\frac{1}{2}} \left(\frac{\partial L}{\partial q_1} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial q_2} \right) = 0;$$

$$(35) \quad -kq_0 + {}_t D_b^{\frac{1}{2}} (i\gamma)q_1 - m\dot{q}_2 = 0.$$

Using the relationship between the Reimann-Liouville and the Caputo fractional derivative, Eq.(35) becomes

$$(36) \quad -kq_0 + {}^C D_b^{\frac{1}{2}} (i\gamma)q_1 + \frac{(1-t)^{-\alpha}}{\Gamma 0.5} q_1(b) - m\dot{q}_2 = 0$$

According to Eq.(18)

$$(37) \quad -kq_0 + {}^C D_b^{\frac{1}{2}} (i\gamma)q_1 - m\ddot{q}_0 = 0$$

We can relate the left and the right Caputo fractional derivative of order $(\frac{1}{2})$ as follows.

$$(38) \quad \begin{aligned} \int_a^b (\tau-x)^{-0.5} \left(\frac{d}{d\tau} \right) f(\tau) d\tau &= \int_a^x (\tau-x)^{-0.5} \left(\frac{d}{d\tau} \right) f(\tau) d\tau + \int_x^b (\tau-x)^{-0.5} \left(\frac{d}{d\tau} \right) f(\tau) d\tau \\ &= \int_a^x (-1)^{0.5} (x-\tau)^{-0.5} \left(\frac{d}{d\tau} \right) f(\tau) d\tau - \int_x^b (\tau-x)^{-0.5} \left(-\frac{d}{d\tau} \right) f(\tau) d\tau \\ &= \Gamma\left(\frac{1}{2}\right) \left((i) {}^C D_x^{\frac{1}{2}} f(x) - {}^C D_b^{\frac{1}{2}} f(x) \right) \end{aligned}$$

by taking the limit $b \rightarrow a^+$ of Eq. (38)(Klimek, 2001), one gets.

$$(39) \quad (i) {}^C D_t^{\frac{1}{2}} x = ({}^C D_b^{\frac{1}{2}} x).$$

Substituting this formula in Eq. (35), we obtain.

$$(40) \quad -kq_0 + (i_a^C D_t^{\frac{1}{2}})(i\gamma)q_1 - m\dot{q}_2 = 0.$$

Thus,

$$(41) \quad kq_0 + \gamma q_2 + m\dot{q}_2 = 0,$$

Which can be written as

$$(42) \quad kq_0 + \gamma\dot{q}_0 + m\ddot{q}_0 = 0.$$

Equation (42) represent the canonical equation of motion for the Damped Harmonic Oscillator.

Now, let us apply the Hamilton's equations to the Lagrangian in Eq.(33).

According to Eq. (23), the set of momenta can be calculated as

$$(43) \quad \begin{aligned} p_0 &= \frac{\partial L}{\partial q_1} \\ &= i\gamma q_1; \end{aligned}$$

and,

$$(44) \quad \begin{aligned} p_1 &= \frac{\partial L}{\partial q_2} \\ &= m\dot{q}_2. \end{aligned}$$

Then, the Hamiltonian takes the following form

$$(45) \quad \begin{aligned} H &= p_0 q_1 + p_1 q_2 - L \\ &= p_0 q_1 + p_1 q_2 - \frac{1}{2} m \dot{q}_2^2 + \frac{1}{2} k q_0^2 - \frac{i\gamma}{2} q_1^2 \\ &= p_0 \left(\frac{-ip_0}{\gamma} \right) + p_1 \left(\frac{p_1}{m} \right) - \frac{1}{2} m \left(\frac{p_1^2}{m^2} + \frac{1}{2} k q_0^2 - \frac{i\gamma}{2} \left(-\frac{p_0^2}{\gamma^2} \right) \right) \\ &= \frac{-ip_0^2}{\gamma} + \frac{p_1^2}{2m} + \frac{1}{2} k q_0^2 + \frac{ip_0^2}{2\gamma} \\ &= \frac{-ip_0^2}{2\gamma} + \frac{p_1^2}{2m} + \frac{1}{2} k q_0^2. \end{aligned}$$

Applying the set of Hamilton's equations

$$(46) \quad \begin{aligned} \frac{\partial H}{\partial p_0} &= \frac{-ip_0}{\gamma} \\ &= q_1; \end{aligned}$$

$$\begin{aligned}
(47) \quad \frac{\partial H}{\partial p_1} &= \frac{p_1}{m} \\
&= q_2,
\end{aligned}$$

we have

$$\begin{aligned}
(48) \quad \frac{\partial H}{\partial q_0} &= \sum_{n=0}^{N-1} {}^C D_b^{\frac{1}{2}(n+1)}(p_n); \\
kq_0 &= ({}^C D_b^{\frac{1}{2}})(p_0) + ({}^C D_b^1)(p_1) \\
&= ({}^C D_b^{\frac{1}{2}})(i\gamma q_1) - \frac{d}{dt}(mq_2)
\end{aligned}$$

Making use of Eq.(39), we have

$$\begin{aligned}
(49) \quad kq_0 &= (i)({}^C D_t^{\frac{1}{2}})(i\gamma q_1) - m\dot{q}_2 \\
&= -\gamma q_2 - m\dot{q}_2 \\
&= -\gamma \dot{q}_0 - m\ddot{q}_0;
\end{aligned}$$

or

$$(50) \quad kq_0 + \gamma \dot{q}_0 + m\ddot{q}_0 = 0,$$

Which is the canonical equation of motion for the damped harmonic oscillator.

5. CONCLUSION

In this paper, we have obtained the Euler-Lagrange equation and Hamilton's equations for nonconservative systems by using the Caputo fractional derivatives. These equations are then applied to the damped harmonic oscillator, as an example. We note that both the fractional Euler-Lagrange equation and Hamilton's equations give the same equation of motion; this is the same as that obtained in Newton's mechanics. It will be interesting to apply the same procedures to other examples in physics, such as the Brownian-motion problem.

When we consider a Lagrangian containing a derivative of integral order only, the equations reduce to those in conventional classical mechanics. Also the set of momenta introduced is considered more consistent with that used in ordinary classical mechanics.

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