

DIFFERENTIAL CONNECTIONS ON TOP SPACES

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ABSTRACT. In this paper we construct differential connections such as horizontal lines and we show that: they have Lie semigroup structure with an identity element. We also show that differential connections structures can be lifted by covering top spaces, and if two connections are parallel, then their lifts are parallel.

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1. INTRODUCTION

Top spaces are an extension of Lie groups [1], and they are particularly a class of manifolds. They are also an extension of topological generalized group [8].

Note that topological generalized groups have different meaning with the Vanger generalized groups ([14]) and semigroups admitting relative inverse ([2, 3]).

In this paper we construct differential connections on top spaces, as horizontal lines in a top space, and lift them by the covering top spaces. We will construct a Lie semigroup structure with an identity element on differential connections. In fact we deduce two structures (vertical [7] and horizontal).

Let us to recall the definition of a top space [5, 11].

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A set $\mathcal{T}$ is called a top space if:

- (i) $\mathcal{T}$ is a differentiable manifold;
- (ii) $\mathcal{T}$ is a topological generalized group [10] ;
- (iii) The maps $m_1 : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ and $m_2 : \mathcal{T} \rightarrow \mathcal{T}$ defined by $m_1(t, s) = ts$ and $m_2(t) = t^{-1}$ are differentiable maps;
- (iv) $e(ts) = e(t)e(s)$ for all $t, s \in T$, where e is the identity function of T .

Let us to recall a main theorem in the lifting theory of Lie groups which we will use of it.

Theorem 1.1. [8] *Let $\mathcal{X}$ be a connected manifold and a covering space of $\mathcal{Y}$ with the covering projection $\mathcal{P}$, such that $\mathcal{P}(x_0) = y_0$; where $x_0 \in \mathcal{X}$, and $y_0 \in \mathcal{Y}$. Let $\mathcal{Z}$ be a connected and simply connected manifold, and $\mathcal{F} : \mathcal{Z} \rightarrow \mathcal{X}$ be a differentiable map such that $\mathcal{F}(z_0) = x_0$, where $z_0 \in \mathcal{Z}$. Then there exists a unique differentiable map $\mathcal{H} : \mathcal{Z} \rightarrow \mathcal{Y}$ such that: (i) $\mathcal{H}(z_0) = y_0$; (ii) $\mathcal{P} \circ \mathcal{H} = \mathcal{F}$.*

In this paper we will prove the following statements:

1. Every differentiable connection with zero point $w \in \mathcal{W}$ in a connected top space can be lifted to a unique differential connection under a covering of top spaces, with the zero point $t \in \mathcal{P}^{-1}(w)$.
2. If ξ_1, ξ_2 are two parallel one-parameter connections, then there lifts under covering top spaces are parallel.
3. Let T be a top space, and $\zeta : [0, 1] \rightarrow T$ be a parallel differential connection. Then $\zeta(0, 1]$ is Lie semigroup with an identity element.
4. Let T be a top space and $e(T)$ be a countable set. Let $a, b \in T$. Then there is no any connection between a and b .
5. Let T be a top space of dimension d . If the Lie group $e^{-1}(e(t_0))$ has dimensional d , then the tangent space of T_t is equal to the tangent space of the Lie group $e^{-1}(e(t))$, where $t \in T$.

2. DIFFERENTIABLE CONNECTION

In this section we are going to define differential connection in top spaces and we will lift them by covering top spaces. We know that every top space is a union of disjoint diffeomorphic Lie groups [13](we can suppose that these Lie groups are vertical lines in a top space).

Definition 2.1. A differentiable connection in a top space $\mathcal{T}$ is a one to one C^∞ -map $\xi : [0, 1] \rightarrow \mathcal{T}$ such that $\text{Card}(\xi([0, 1]) \cap e^{-1}(e(t))) \leq 1$. An element of $\xi(o)$ is called a zero point of ξ

Example 2.2. Euclidian space R with the product $a.b = a$, is top space. The mapping $\xi : [0, 1] \rightarrow R$ defined by $\xi(t) = t^3 + 1$ is a differentiable connection in R .

Example 2.3. The subset:

$$\mathcal{T} = \{(\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha) \mid \alpha \in (0, 2\pi), \beta \in (0, \pi)\}$$

of S^2 with the product:

$$\begin{aligned} ((\sin \alpha_1 \cos \beta_1, \sin \alpha_1 \sin \beta_1, \cos \alpha_1), (\sin \alpha_2 \cos \beta_2, \sin \alpha_2 \sin \beta_2, \cos \alpha_2)) \\ \mapsto (\sin \alpha_1 \cos \beta_2, \sin \alpha_1 \sin \beta_2, \cos \alpha_1) \end{aligned}$$

is a top space. In this top space every one to one C^∞ map $\xi : [0, 1] \rightarrow \mathcal{T}$ is a differentiable connection.

Definition 2.4. Two differentiable connection ξ and γ are called parallel, if $(\xi([0, 1])) \cap (\gamma([0, 1])) = \emptyset$.

Example 2.5. In the top space of example 2.2, the differentiable one-parameter connection $\xi(t) = \text{Arctan}(t)$ and the differentiable one-parameter connection $\gamma(t) = \text{Arctan}(t) + 5$ are parallel.

Example 2.6. Let $T^n = \mathbb{R}^n / \mathbb{Z}^n$ be the n -dimensional torus with the product: $((\alpha_1, \dots, \alpha_n) + \mathbb{Z}^n, (\beta_1, \dots, \beta_n) + \mathbb{Z}^n) \mapsto (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n) + \mathbb{Z}^n$. Then two differentiable one-parameter connections $\xi(t) = (\frac{\text{Arctan}(t)}{5}, \dots, \frac{\text{Arctan}(t)}{5}) + \mathbb{Z}^n$ and $\gamma(t) = (\frac{\text{Arctan}(t)}{5} + 1, \dots, \frac{\text{Arctan}(t)}{5} + 1) + \mathbb{Z}^n$ are parallel.

Definition 2.7. Let $\mathcal{T}, \mathcal{W}$ be two top spaces, a covering projection $\mathcal{P} : \mathcal{T} \rightarrow \mathcal{W}$ is called a covering of top spaces, if $\mathcal{P}(t_1 t_2) = \mathcal{P}(t_1) \mathcal{P}(t_2)$, for all $t_1, t_2 \in \mathcal{T}$ [12].

Note: If $\mathcal{W}$ is a topological group, then $\mathcal{P}(\mathcal{T})$ with the induced topology of $\mathcal{W}$ is a Lie group.

Theorem 2.8. Every differentiable connection with zero point $w \in \mathcal{W}$ in a connected top space can be lifted to a unique differential connection under a covering of top spaces, with the zero point $t \in \mathcal{P}^{-1}(w)$.

Proof: Let ξ be a differentiable connection in the connected space $\mathcal{W}$. Theorem 1.1 implies there is a differentiable map γ in $\mathcal{T}$ such that $\mathcal{P} \circ \gamma = \xi$. We show that γ is

a differentiable connection. We proof this fact in the following two steps:

(I) We show that, if $t_1, t_2 \in \mathcal{T}$ and $t_1, t_2 \in e^{-1}(e(t))$, for some $t \in \mathcal{T}$, then $\mathcal{P}(t_1), \mathcal{P}(t_2) \in e^{-1}(e(\mathcal{P}(t)))$. At first we show that $e(\mathcal{P}(t)) = \mathcal{P}(e(t))$, for all $t \in \mathcal{T}$. We have $\mathcal{P}(e(t))\mathcal{P}(t) = \mathcal{P}(e(t)t) = \mathcal{P}(t)$. Since $e(t)t = te(t)$ then the uniqueness of identity implies $e(\mathcal{P}(t)) = \mathcal{P}(e(t))$. Now $\mathcal{P}(t_1)e(\mathcal{P}(t)) = \mathcal{P}(t_1)\mathcal{P}(e(t)) = \mathcal{P}(t_1e(t)) = \mathcal{P}(t_1)$, with this manner $\mathcal{P}(t_2)e(\mathcal{P}(t)) = \mathcal{P}(t_2)$. So $e(\mathcal{P}(t))$ is the right identity of $\mathcal{P}(t_1), \mathcal{P}(t_2)$. With the same process we can show that $e(\mathcal{P}(t))$ is the left identity.

(II) If $\gamma(a), \gamma(b) \in e^{-1}(e(t))$, for $t \in \mathcal{T}$ and $a, b \in [0, 1]$, then $\mathcal{P} \circ \gamma(a) = \xi(a)$ and $\mathcal{P} \circ \gamma(b) = \xi(b)$. The step (I) implies $\xi(a) = \xi(b)$ and then $a = b$.

Theorem 2.9. If ξ_1, ξ_2 are two parallel one-parameter connections, then there lifts under covering top spaces are parallel.

Proof: If γ_i is the lifting of ξ_i , for $i = 1, 2$ and $\gamma_1(a) = \gamma_2(b)$, for $a, b \in [0, 1]$, then $\xi_1(a) = \xi_2(b)$. So ξ_1 and ξ_2 are not parallel.

Corollary 2.10. Let $\mathcal{T}, \mathcal{W}$ be two top spaces and $\mathcal{P} : \mathcal{T} \rightarrow \mathcal{W}$ be a covering projection. If $w \in \mathcal{W}$ and $\text{Card}(\mathcal{P}^{-1}(w)) = k$, then for every connection in $\mathcal{W}$ we have k lifted parallel connections in $\mathcal{T}$.

Now we are going to construct a horizontal parallel Lie semigroup with an identity element. So every top spaces have two horizontal Lie semigroup with identity element and vertical Lie subgroup structures.

Definition 2.11. Let $\zeta : [0, 1] \rightarrow T$ be differential connection in top space T . We define $\zeta(a) * \zeta(b) = \zeta(ab)$, for all $a, b \in (0, 1]$.

Theorem 2.12. Let T be a top space, $\zeta : [0, 1] \rightarrow T$ be a parallel differential connection. Then $\zeta(0, 1]$ is Lie semigroup with an identity element.

Proof: We define $\zeta(a) * \zeta(b) = \zeta(ab)$, for all $a, b \in (0, 1]$. It is clear that $*$ is closed, associative and $\zeta(1)$ is the identity element of $\zeta(0, 1]$. So clearly $(\zeta(0, 1], *)$ is a Lie group.

Theorem 2.13. Let T be a top space and $e(T)$ be a countable set. Let $a, b \in T$. Then there is no any connection between a and b .

Proof: In [13] proved that T is disjoint union of diffeomorphic Lie groups $e^{-1}(e(t))$, $t \in T$. Since $e(T)$ is a countable set and every connection is one to

one with noncountable domain $[0, 1]$, then there is no any connection between a and b .

Example 2.14. The set $R - \{0\}$ with the product $(a, b) \mapsto a|b|$ is top space, in this top space there is no any connection between each two points $a, b \in R - \{0\}$. Because $e(R - \{0\}) = \{-1, 1\}$, $e^{-1}(e(-1)) = R^-$ and $e^{-1}(e(+1)) = R^+$.

Theorem 2.15. Let T be a top space of dimension d . If the Lie group $e^{-1}(e(t_0))$ has dimensional d , then the tangent space T_t is equal to the tangent space of the Lie group $e^{-1}(e(t))$, where $t \in T$.

Proof: We know that T is a disjoint union of diffeomorphic Lie groups $e^{-1}(e(t))$, for all $t \in T$. So the dimension of $e^{-1}(e(t))$ is d . Therefore the tangent space of T in t is isomorphic to the tangent space of $e^{-1}(e(t))$, for all $t \in T$.

3. CONCLUSION

Let us to conclude this paper by sketching two problems:

Let T be a connected top space and let $\text{Card}(e(T)) \geq c$ for all $t \in T$ and a positive constant c .

1. Is there a differential connection ζ , such that t be a member of the rang of ζ ?
2. Can we write the tangent space of T by horizontal and vertical structures?

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