

ANGLE CONTRACTION BETWEEN GEODESICS

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ABSTRACT. We consider here a generalization of a well known discrete dynamical system produced by the bisection of reflection angles that are constructed recursively between two lines in the Euclidean plane. It is shown that similar properties of such systems are observed when the plane is replaced by a regular surface in $\mathbb{R}^3$ and lines are replaced by geodesics. An application of our results to the classification of points on the surface as elliptic, hyperbolic or parabolic is also presented.

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1. INTRODUCTION

Let us consider the following simple problem. In the Euclidean plane take two rays starting at point V and forming an acute angle of measure μ . Denote the rays by L_A and L_B and take an arbitrary transversal segment A_1B_1 with $A_1 \in L_A$, and $B_1 \in L_B$. Keep constructing further segments with respect to the following rule: To create a new transversal, take the angle between the most recently constructed transversal and the corresponding ray, and let the new transversal be the bisector of this angle. In particular, if we have as our last transversal $B_{k-1}A_k$, we take the bisector of $\angle B_{k-1}A_kV$, denote its intersection with line L_B by B_k and use the

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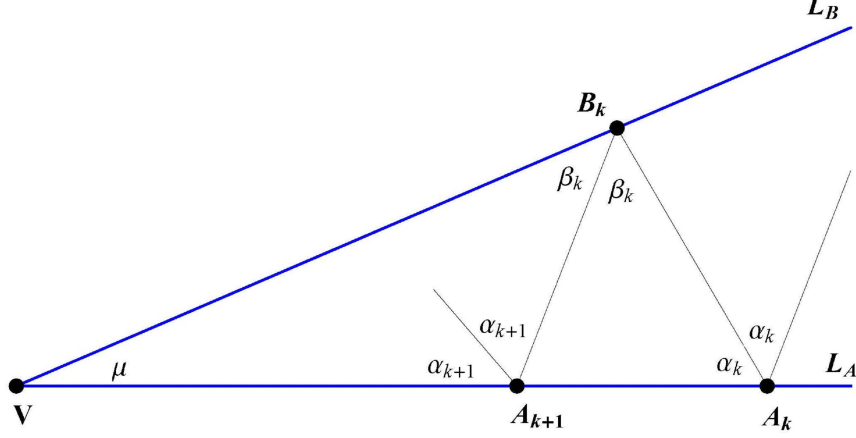


FIGURE 1

transversal $A_k B_k$ for the next step (see Figure 1). Clearly, this process creates not only two sequences of points $\{A_k\} \in L_A$, $\{B_k\} \in L_B$ but also two sequences of angles $\{\alpha_k\}$, and $\{\beta_k\}$, where $\alpha_k = \angle B_k A_k V$, and $\beta_k = \angle A_{k+1} B_k V$.

By expressing each sequence of angles as a first order difference equation, it is easy to show that

$$\lim_{k \rightarrow \infty} \alpha_k = \lim_{k \rightarrow \infty} \beta_k = \frac{\pi - \mu}{3}$$

irrespective of the choice of the initial transversal segment. We show below that this property holds under more general circumstances. Specifically, we replace the plane with a smooth surface in $\mathbb{R}^3$, where our triangles will be geodesic triangles, and drop the condition that the angles are created by bisection.

2. A GENERALIZATION OF THE PROBLEM

Let $S \subset \mathbb{R}^3$ be a regular surface locally parametrized by a differentiable vector function $\mathbf{r}(u, v) : U \rightarrow \mathbb{R}^3$, with U being open in $\mathbb{R}^2$. All curves we consider below will be *regular parametrized curves*, that is, differentiable vector functions $\mathbf{c} : [a, b] \rightarrow \mathbb{R}^3$ with $\dot{\mathbf{c}}(t) = d\mathbf{c}(t)/dt \neq 0$ for all values of the parameter $t \in (a, b)$.

By a *geodesic* on S we will mean a unit speed regular curve $\mathbf{c}(s)$ (parametrized by the arc length) on S such that the second derivative $\mathbf{c}''(s)$ is the zero vector or perpendicular to the surface.

To avoid the cases when two distinct points on S can be joined by different geodesics, we will consider only “small” triangles formed by two geodesics intersecting at a point V and a third geodesic intersecting the first two transversally at points A and B . By a “small” triangle we will mean one contained in an open subset of S , which is a normal neighborhood of all its points. Proof of the existence of such a neighborhood for each point $p \in S$ can be found in §4-7 of [1].

If $\gamma_v : [0, b] \rightarrow S, \gamma_v(0) = P, \gamma'_v(0) = v$ and $\gamma_w : [0, c] \rightarrow S, \gamma_w(0) = P, \gamma'_w(0) = w$ are two geodesics at P , the angle between these two geodesics is defined to be the angle between v and w . If X is on γ_v and Y is on γ_w , we will denote this angle either by $\angle XPY$ or by $\angle YPX$. Assume now that we are given a “small” triangle $\triangle VAB$ on a regular surface $S \subset \mathbb{R}^3$ and denote $\angle BVA$ by μ , and assume that $\mu < \pi$.

As was the case in the Euclidean plane, we can construct first order difference equations for the angles created by bisection, where now our transversals will be geodesic segments. Let $A_1 = A$, $B_1 = B$ and $\alpha_1 := \angle VA_1B_1$. Assume also that we are given two continuous functions $p : S \rightarrow \mathbb{R}^+$ and $q : S \rightarrow \mathbb{R}^+$ taking only positive values. To construct the point A_2 on the geodesic segment VA_1 , we consider the geodesic $\gamma_{\beta_1} : [0, 1] \rightarrow S$ such that $\gamma_{\beta_1}(0) = B_1$ and the angle between $\gamma'_{\beta_1}(0) \in T_{B_1}(S)$ and the geodesic segment B_1V is

$$\beta_1 = \frac{\angle VB_1A_1}{1 + q(B_1)}.$$

Geodesic $\gamma_{\beta_1}([0, a])$ (with a large enough) will intersect transversally the segment VA_1 at a point, which we denote by A_2 . Then we use function $p : S \rightarrow \mathbb{R}^+$ and divide angle $\angle B_1A_2V$ by the geodesic $\gamma_{\alpha_2} : [0, 1] \rightarrow S$ to create the point B_2 on segment VB_1 , where $\gamma_{\alpha_2}(0) = A_2$ and the angle between $\gamma'_{\alpha_2}(0) \in T_{A_2}(S)$ and segment A_2V is defined by

$$\alpha_2 = \frac{\angle VA_2B_1}{1 + p(A_2)}.$$

Point B_2 is defined as the intersection of VB_1 with geodesic $\gamma_{\alpha_2}([0, a])$ (see Figure 2, where we abbreviated $p(A_m)$ and $q(B_n)$ as p_m and q_n respectively).

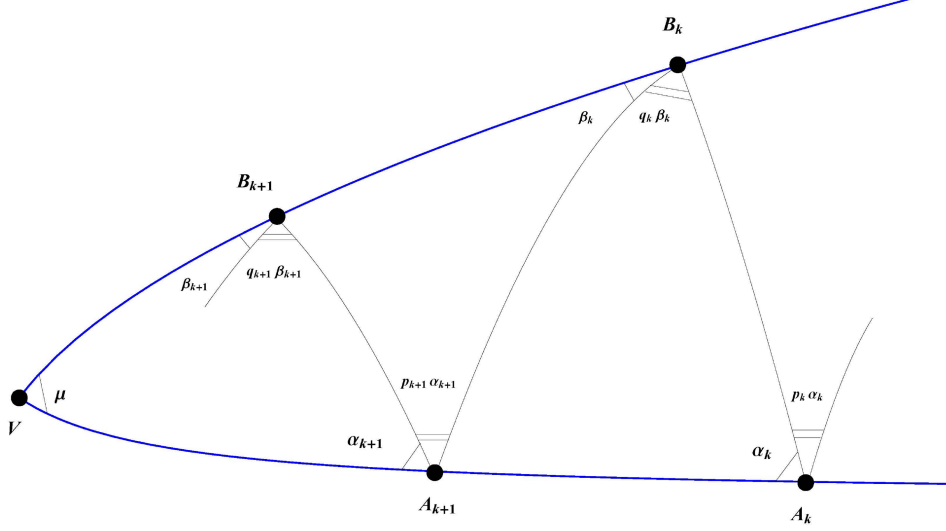


FIGURE 2

Then we iterate this procedure to construct two sequences of points $\{A_k\}_{k=1}$, $\{B_k\}_{k=1}$, and angles

$$\alpha_{k+1} = \frac{\angle V A_{k+1} B_k}{1 + p(A_{k+1})}, \text{ and } \beta_k = \frac{\angle V B_k A_k}{1 + q(B_k)}$$

Here arises a natural question. Do the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge, and if so, to what values? In the next section we will prove the following

Theorem. *For an arbitrary small triangle $\triangle VAB$ on a regular surface S in $\mathbb{R}^3$ with the angle $\angle AVB = \mu < \pi$ and two continuous positive functions $p : S \rightarrow \mathbb{R}^+$ and $q : S \rightarrow \mathbb{R}^+$, we have*

$$\lim_{n \rightarrow \infty} \alpha_n = \frac{q(V) \cdot (\pi - \mu)}{p(V) + q(V) + p(V) \cdot q(V)}$$

and

$$\lim_{n \rightarrow \infty} \beta_n = \frac{p(V) \cdot (\pi - \mu)}{p(V) + q(V) + p(V) \cdot q(V)}$$

Corollary 1. *If, in addition to the conditions of the theorem we assume that functions p and q are equal, then*

$$\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = \frac{\pi - \mu}{2 + p(V)}$$

Recall that a point $x \in S$ is called elliptic when the Gauss curvature $K(x) > 0$, hyperbolic when $K(x) < 0$ and parabolic when $K(x) = 0$. Our next corollary shows that there exist functions $p(x)$ and $q(x)$ so that the limits of our sequences of angles will distinguish the type of the point V .

Corollary 2. *Let $k_1(x)$ and $k_2(x)$ denote two principal curvatures at $x \in S$. If*

$$p(x) = 1 + |K(x) \cdot (k_1(x) + k_2(x))| \text{ and } q(x) = 1 + |K(x)| \cdot (|k_1(x)| + |k_2(x)|)$$

then the pairs $(\lim \alpha_n, \lim \beta_n)$ are different for different type of points.

Proof. Straightforward computations show that in the case of a parabolic, elliptic, and hyperbolic points correspondingly, we have

$$\lim \alpha_n = \lim \beta_n = \frac{\pi - \mu}{3}, \quad \lim \alpha_n = \lim \beta_n < \frac{\pi - \mu}{3}, \quad \lim \alpha_n \neq \lim \beta_n.$$

□

3. PROOF OF THE THEOREM

The proof of our Theorem relies on a generalization of the Banach contraction mapping principle and a special case of the local version of the Gauss-Bonnet Theorem. Let us first recall the contraction principle.

The Contraction Theorem Consider a complete metric space (M, dist) . Let $T : M \rightarrow M$ be a contraction mapping with Lipschitz constant $k \in (0, 1)$, and suppose $\alpha \in M$ is the fixed point of T . Let $\{\varepsilon_n\}$ be a sequence of positive numbers converging to zero, and suppose $\{\alpha_n\} \subseteq M$ satisfies:

$$\text{dist}(\alpha_{n+1}, T(\alpha_n)) \leq \varepsilon_n.$$

Then the sequence $\{\alpha_n\}$ converges to α . We refer the reader to chapter 3 of [2] for a proof of this theorem.

Proof of Theorem. Let's apply the Gauss-Bonnet Theorem to geodesic triangles $A_k B_k A_{k+1}$ and $A_k B_k V$. Using notations from figure 2, one will get the following:

$$\iint_{A_k B_k A_{k+1}} K \, dS = \alpha_k + q(B_k) \cdot \beta_k + (\pi - (1 + p(A_{k+1})) \cdot \alpha_{k+1}) - \pi,$$

which is equivalent to the formula

$$(1) \quad \alpha_{k+1} = \frac{\alpha_k}{1 + p(A_{k+1})} + \frac{q(B_k) \cdot \beta_k}{1 + p(A_{k+1})} - \frac{1}{1 + p(A_{k+1})} \cdot \iint_{A_k B_k A_{k+1}} K \, dS,$$

and similarly

$$\iint_{A_k B_k V} K \, dS = \alpha_k + (1 + q(B_k)) \cdot \beta_k + \mu - \pi,$$

which gives β in terms of α and μ :

$$(2) \quad \beta_k = \frac{\pi - \mu}{1 + q(B_k)} - \frac{\alpha_k}{1 + q(B_k)} + \frac{1}{1 + q(B_k)} \cdot \iint_{A_k B_k V} K \, dS.$$

Substituting (2) into (1) one easily obtains

$$\alpha_{k+1} =$$

$$\frac{\alpha_k + q(B_k) \cdot (\pi - \mu)}{(1 + p(A_{k+1})) \cdot (1 + q(B_k))} + \frac{q(B_k)}{(1 + p(A_{k+1})) \cdot (1 + q(B_k))} \cdot \iint_{A_k B_k V} K \, dS - \frac{1}{1 + p(A_{k+1})} \cdot \iint_{A_k B_k A_{k+1}} K \, dS$$

Since $p(X) > 0$ and $q(X) > 0$ for an arbitrary point $X \in S$, the map

$$T(\varphi) := \frac{1}{(1 + p(V)) \cdot (1 + q(V))} \cdot \varphi + \frac{q(V) \cdot (\pi - \mu)}{(1 + p(V)) \cdot (1 + q(V))}$$

will be a contraction with Lipschitz constant $1/[(1 + p(V)) \cdot (1 + q(V))] \in (0, 1)$. Its fixed point α can be found by a straight forward computation

$$\alpha = \frac{q(V) \cdot (\pi - \mu)}{p(V) + q(V) + p(V) \cdot q(V)}.$$

To apply the contraction theorem for this map T , we set $M = \mathbb{R}$ with the standard distance function $\text{dist}(x, y) := |x - y|$. Then it follows that

$$\varepsilon_k := |\alpha_{k+1} - T(\alpha_k)| = \left| \frac{\alpha_k + q(B_k) \cdot (\pi - \mu)}{(1 + p(A_{k+1})) \cdot (1 + q(B_k))} - T(\alpha_k) \right| +$$

$$\left| \frac{q(B_k)}{(1+p(A_{k+1})) \cdot (1+q(B_k))} \cdot \iint_{A_k B_k V} K dS - \frac{1}{1+p(A_{k+1})} \cdot \iint_{A_k B_k A_{k+1}} K dS \right|$$

Thus, to prove our theorem for the sequence $\{\alpha_n\}$, it is enough to show that $\{\varepsilon_k\} \rightarrow 0$ when $k \rightarrow \infty$. For this we will show that the difference of two fractions and each of the double integrals in the formula for ε_k converge to zero. The Gaussian curvature of a regular surface is a smooth function, and hence it is universally bounded on a small triangle. Since the union of all triangles $\triangle A_k B_k A_{k+1}$ is contained in the small triangle $\triangle VAB$, we must have

$$\sum_{k=1}^{\infty} \iint_{A_k B_k A_{k+1}} |K| dS < \iint_{ABV} |K| dS < \infty,$$

and hence the general term of the series converges to zero. As for the integrals over the triangles $A_k B_k V$, we first notice that both sequences $\{A_k\}$ and $\{B_k\}$ converge to the vertex V (see Lemma below). Therefore length of each of three geodesics $L(A_k, V)$, $L(B_k, V)$ and $L(A_k, B_k)$ approaches zero, which implies that the regions in $\mathbb{R}^2$ corresponding to the triangles $\triangle A_k B_k V$ get shrunk to a point. Hence

$$\iint_{A_k B_k V} K dS \xrightarrow{k \rightarrow \infty} 0.$$

Convergence to zero of

$$\frac{\alpha_k + q(B_k) \cdot (\pi - \mu)}{(1+p(A_{k+1})) \cdot (1+q(B_k))} - T(\alpha_k)$$

follows from the lemma since both functions $p, q : S \rightarrow \mathbb{R}^+$ are continuous. Thus, $\varepsilon_k \rightarrow 0$, when $k \rightarrow \infty$, and hence our result for $\{\alpha_k\}$. To finish the proof take the limit at $k \rightarrow \infty$ of each side in formula (2) to obtain the desired answer for the sequence $\{\beta_k\}$. $\square$

Lemma. $\lim_{k \rightarrow \infty} B_k = \lim_{k \rightarrow \infty} A_k = V$

Proof. Suppose we are given a geodesic segment $PQ \subset S$. As above, we will designate here the length of such a segment by $L(P, Q)$. Since our triangle $\triangle B_1 V A_1$ is a small one, it is easy to see that the sequence of points $\{B_k\}$ converges to a point, P , on the geodesic segment $B_1 V$. Assume that $P \neq V$. Then $\forall \varepsilon > 0$, sufficiently small, $\exists N$, a large enough natural number, such that $\forall k \geq N$, B_k will be in a

neighborhood of radius ε centered at P and contained in $\exp_p(D_\varepsilon)$. Further, we assume that ε is so small that $\exp_p(D_\varepsilon)$ has no points in common with the segment A_1V . This implies that $\exists c > 0$ such that for all large enough n, m we also have $L(A_n, B_m) > c$. Consider a regular parametrization $\gamma(t) : [0, \theta] \rightarrow B_1V$ of the geodesic B_1V such that $\gamma(0) = B_1$, $\gamma(\theta) = V$ and $d\gamma(t)/dt \neq 0$, $\forall t \in (0, \theta)$. Let us denote by $t_k \in [0, \theta]$, $t_1 = 0$, the parameter value corresponding to the point $B_k \in B_1V$, i.e. $\gamma(t_k) = B_k$. Since the sequence $\{B_k\}$ converges and $\gamma(t)$ is smooth, the corresponding sequence of parameters $\{t_k\}$ will converge as well.

As the next step, we use the exponential map to show that existence of the limit point P with $L(P, V) > 0$ implies the convergence to zero of both sequences $\{\alpha_k\}$ and $\{\beta_k\}$. Assume that n is large enough and recall that $\exp_{A_n} : T_{A_n}S \rightarrow S$ is a diffeomorphism locally and its image contains our small triangle $\triangle ABV$. In particular, we can consider the preimage of the two geodesics segments B_nA_n , and A_nB_{n+1} , which will be line segments in $T_{A_n}S$. We denote those lines by $(\tilde{B}_nA_n)$ and $(A_n\tilde{B}_{n+1})$ respectively (i.e. $\exp_{A_n}^{-1}(B_n) = \tilde{B}_n$). Since $\exp_{A_k}$ preserves the angles and lengths of rays through the point A_k , we have that

$$\angle \tilde{B}_nA_n\tilde{B}_{n+1} = \angle B_nA_nB_{n+1} = p(A_n) \cdot \alpha_n \text{ and } |A_n\tilde{B}_{n+i}| = L(A_n, B_{n+i}) > c,$$

where $i \in \{0, 1\}$. We also have the following inequalities:

$$L(\tilde{B}_{n+1}, \tilde{B}_n) \geq \text{dist}[\tilde{B}_{n+1}, (A_n\tilde{B}_n)] > c \cdot \sin(p(A_n) \cdot \alpha_n),$$

where $L(\tilde{B}_{n+1}, \tilde{B}_n)$ denotes the length of the curve $\exp_{A_n}^{-1}[(B_{n+1}B_n)] \subset T_{A_n}S$ and can be computed by the integral

$$L(\tilde{B}_{n+1}, \tilde{B}_n) = \int_{t_n}^{t_{n+1}} |(\exp_{A_n}^{-1} \circ \gamma)'(u)| du.$$

Since the composition $\exp_{A_n}^{-1} \circ \gamma(t)$ is smooth, there exists a positive constant K such that for all large enough n , we have

$$K \cdot |t_{n+1} - t_n| \geq L(\tilde{B}_{n+1}, \tilde{B}_n) > c \cdot \sin(p(A_n) \cdot \alpha_n),$$

which implies that the sequence of angles $\{\alpha_n\}$ converges to zero. Recalling (2), we conclude that the sequence $\{\beta_n\}$ converges to zero as well.

Now let's look at the triangle $\triangle A_kB_kV$ and consider the geodesic through A_k and a point Z on the geodesic segment VB_k , including both end points. Since

Z belongs to a closed and bounded curve, there is such geodesic $A_k Z$ of maximal length. Let's denote this length by L_k , that is

$$L_k := \max_{Z \in B_k V} \{L(A_k, Z)\}.$$

Now we take the circular sector of radius L_k , denoted by $\Delta S(A_k, L_k)$, centered at A_k and bounded between two geodesics $A_k V$ and $A_k B_k$ with angle α_k at A_k . Clearly, $\Delta A_k B_k V \subset \Delta S(A_k, L_k)$, and therefore

$$\iint_{A_k B_k V} |K| \, dS \leq \iint_{\Delta S(A_k, L_k)} |K| \, dS.$$

Since the sequence $\{L_k\}$ is certainly bounded and $\{\alpha_k\} \rightarrow 0$, when $k \rightarrow \infty$, the areas of $\Delta S(A_k, L_k)$ will approach zero, which implies that

$$\iint_{A_k B_k V} K \, dS \xrightarrow{k \rightarrow \infty} 0.$$

Now one can use formula (2) to deduce that $\mu = \pi$, which will contradict the assumption that $\mu < \pi$ in the triangle $\triangle ABV$. A similar argument for $\{A_k\}$ also results in a contradiction. $\square$

REFERENCES

- [1] M.P. de Carmo, Differential geometry of curves and surfaces: Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976.
- [2] M.A. Khamsi, W.A. Kirk, An introduction to metric spaces and fixed point theory: Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 2001.