

A STUDY ON RULED SURFACE IN EUCLIDEAN 3-SPACE

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ABSTRACT. In this study, we examined the ruled surfaces in Euclidean 3-space with different way. We obtained some characterizations on the ruled surfaces by using its rulings, curvatures of the base curve, shape operator and Gauss curvature.

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1. INTRODUCTION

In this section, we review some basic concepts on classical differential geometry of space curves and ruled surfaces in Euclidean space. Let $\alpha : I \longrightarrow IR^3$ be a curve with $\alpha'(s) \neq 0$, where $\alpha'(s) = d\alpha(s)/ds$. The arc-length s of a curve $\alpha(s)$ is determined such that $\|\alpha'(s)\| = 1$. Let us denote $T(s) = \alpha'(s)$ and we call $T(s)$ a tangent vector of α at s . We define the curvature of α by $k_1(s) = \sqrt{\|\alpha''(s)\|^2}$. If $k_1(s) \neq 0$, then unit principal normal vector $N(s)$ of the curve at s is given by $\alpha''(s) = k_1(s)N(s)$. The unit vector $B(s) = T(s) \wedge N(s)$ is called unit binormal vector of α at s . Then we have the Frenet-Serret formulae

$$\nabla_T T = k_1 N, \quad \nabla_T N = -k_1 T + k_2 B, \quad \nabla_T B = -k_2 N$$

where $k_2(s)$ is the torsion of α at s . [2]

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A straight line X in IR^3 , such that it is strictly connected to Frenet frame the curve $\alpha(s)$, is represented uniquely with respect to this frame, in the form

$$X(s) = f_1(s)T(s) + f_2(s)N(s) + f_3(s)B(s)$$

where $f_1(s)$, $f_2(s)$ and $f_3(s)$ are smooth functions. As X move along $\alpha(s)$, it generates a ruled surface given by the regular parametrization

$$(1) \quad \varphi(s, v) = \alpha(s) + vX(s)$$

where the components $f_i(s)$, $i = 1, 2, 3$ are scalar functions of the arc-length parameter of the curve $\alpha(s)$. This surface will be denoted by M . the curve $\alpha(s)$ is called a base curve and the various positions of the generating line X are called the rulings of the surface M .

If consecutive rulings of ruled surface in IR^3 intersect, then the surface is to be developable. All the other ruled surfaces are called skew surfaces. If there exists a common perpendicular to two constructive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a striction point. The set of striction points on the ruled surface defines the striction curve [3].

The striction curve of M can be written in terms of the base curve $\alpha(s)$ as

$$(2) \quad \bar{\alpha}(s) = \alpha(s) - \frac{\langle T, \nabla_T X \rangle}{\|\nabla_T X\|^2} X(s)$$

If $\|\nabla_T X\| = 0$ then the ruled surface doesn't have any striction curve. This case characterizes the ruled surface is cylindrical. Thus the base curve can take as a striction curve.

Let P_x be distribution parameter of M , then

$$(3) \quad P_X = \frac{\det(T, X, \nabla_T X)}{\|\nabla_T X\|^2}$$

where ∇ is Levi-Civita connection on E^3 [1]. If the base curve is periodic then M is closed ruled surface. Let M be a closed ruled surface and W be Darboux vector then the Steiner rotation and Steiner translation vectors are

$$(4) \quad D = \oint_{(\alpha)} W \quad \text{and} \quad V = \oint_{(\alpha)} d\alpha$$

respectively. Furthermore, the pitch and angle of pitch of M are

$$(5) \quad L_X = \langle V, X \rangle \quad \text{and} \quad \lambda_X = \langle D, X \rangle$$

respectively [3,4,5].

2. ONE-PARAMETER SPATIAL MOTION IN E^3

Let $\alpha : I \rightarrow E^3$ be a regular curve and $\{T, N, B\}$ be Frenet vectors. The two coordinate systems $\{O, T, N, B\}$ and $\{O'; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are orthogonal coordinate systems in E^3 which represent the moving space and the fixed space, respectively. In generally, during the one parameter spatial motion, each line X in moving space, generates a ruled surface in the fixed space. Since ξ normal to T then $\xi \in Sp\{N, B\}$ and ξ can be choose as $\xi = T \wedge X$ along the line X depending orientation of M . Thus ξ and X can be written as

$$(6) \quad \xi = \cos \psi N + \sin \psi B \quad \text{and} \quad X = \sin \psi N - \cos \psi B$$

where $\psi = \psi(s)$ is the angle between ξ and N along α [6]. We obtained the distribution parameter of the ruled surface M by using (3) and (6). From (6)

$$(7) \quad \nabla_T X = -k_1 \sin \psi T + (\psi' + k_2) \xi$$

and putting this equation in (3), we obtained by direct computation that the distribution parameter is

$$(8) \quad P_X = \frac{\psi' + k_2}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2}$$

which we need to determine the conditions of the ruled surface is developable. It is well known that the ruled surface developable if and only if P_X is zero from [1], so we can state the following theorem.

Theorem 2.1. *A ruled surface is a developable if and only if*

$$\psi = - \int k_2 ds + c$$

is satisfies where c is constant.

In the case $\psi = \pi/2$ and $\psi = \pi$ then we get $P_X = P_N$ and $P_X = P_B$ respectively. Thus distribution parameters are

$$P_N = \frac{k_2}{k_1^2 + k_2^2}, \quad P_B = \frac{1}{k_2}$$

and we obtain

$$\left(\frac{k_1}{k_2}\right)^2 = \frac{P_B}{P_N} - 1$$

Thus we have the following corollary.

Corollary 2.2. *The base curve of the ruled surface is a helix if and only if $\frac{P_B}{P_N}$ is constant.*

On the other hand, from (2) the striction curve of M is

$$\bar{\alpha}(s) = \alpha(s) + \frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2} X(s)$$

In the case of M is a cylindrical ruled surface, we can give the following theorem.

Theorem 2.3. *If M is a cylindrical ruled surface then $k_1 \sin \psi = 0$. In the case $k_1 = 0$ then ruled surface is a plane. In the case $\psi = 2k\pi$, $k \in \mathbb{Z}$ then unit normal vector of M and principal normal vector are on the same direction and both striction curve and base curve are geodesic of M .*

Corollary 2.4. *A cylindrical ruled surface is developable if and only if*

$$k_1 \sin \left(\int k_2 ds \right) = 0$$

is satisfies. In this case the base curve is the planar.

Theorem 2.5. *Let ψ be $\psi' \neq -k_2$ and M be closed ruled surface with in the form (1). Distance between generating lines of M is minimum along the striction curve.*

Proof. Let $X_{\alpha(s_1)}$ and $X_{\alpha(s_2)}$ be generating lines which are passing the points $\alpha(s_1)$ and $\alpha(s_2)$ of the base curve, respectively ($s_1, s_2 \in IR$, $s_1 < s_2$). Distance between these generating lines, along the orthogonal orbits is

$$J(v) = \int_{s_1}^{s_2} \|A\| ds$$

where $A = T + v \nabla_T X$. Then we obtain,

$$J(v) = \int_{s_1}^{s_2} \left(1 - 2vk_1 \sin \psi + (k_1^2 \sin^2 \psi + (\psi' + k_2)^2) v^2 \right)^{\frac{1}{2}} ds$$

If $J(v)$ is minimum for v_0 then $J'(v_0) = 0$ and we get

$$v_0 = \frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2}$$

Thus, the orthogonal orbit is the striction curve of M for $v = v_o$. $\square$

Corollary 2.6. *Let $\bar{\alpha}(s)$ be striction curve of M then $\frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2}$ is constant.*

Proof. Since the tangent vector field of the striction curve is normal to X then $\langle X, \frac{d\bar{\alpha}}{ds} \rangle = 0$.

$$\begin{aligned} \left\langle X, \frac{d\bar{\alpha}}{ds} \right\rangle &= \left\langle X, \left\{ T + \frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2} \nabla_T X \right. \right. \\ &\quad \left. \left. + \frac{d}{ds} \left(\frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2} \right) X \right\} \right\rangle \\ &= \langle X, T \rangle + \frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2} \langle X, \nabla_T X \rangle \\ &\quad + \frac{d}{ds} \left(\frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2} \right) \langle X, X \rangle \end{aligned}$$

then

$$\left\langle X, \frac{d\bar{\alpha}}{ds} \right\rangle = \frac{d}{ds} \left(\frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2} \right)$$

thus we get $\frac{d}{ds} \left(\frac{\lambda}{\lambda^2 + (\psi')^2} \right) = 0$ and so $\frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2}$ is constant $\square$

Theorem 2.7. *Let M be a ruled surface given in the form (1) then $\varphi(t, v_o)$ is a striction point if and only if $\nabla_T X$ is normal to tangent plane at that point on M , where v_o is*

$$v_o = \frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2}$$

Proof. Let suppose that $\varphi(t, v_o)$ be a striction point on the base curve $\alpha(t)$ of M then we must show that $\langle \nabla_T X, X \rangle = 0$ and $\langle \nabla_T X, A \rangle = 0$. Since the vector field X is unit vector then $T[\langle X, X \rangle] = 0$ and we get $\langle \nabla_T X, X \rangle = 0$.

On the other side, from (7) we obtain,

$$\langle \nabla_T X, A \rangle = \{k_1^2 \sin^2 \psi + (\psi' + k_2)^2\} v_o - k_1 \sin \psi$$

Now, by making the substitution $v_o = \frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2}$ then we get $\langle \nabla_T X, A \rangle = 0$. This means that $\nabla_T X$ is normal to tangent plane at the striction point $\varphi(t, v_o)$ on M .

Conversely, since $\nabla_T X$ is normal to tangent plane at the point $\varphi(t, v_o)$ on M then $\langle \nabla_T X, A \rangle = 0$ and from (7) we obtain

$$\{k_1^2 \sin^2 \psi + (\psi' + k_2)^2\} v_o - k_1 \sin \psi = 0$$

Thus we get $v_o = \frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2}$ and so $\varphi(t, v_o)$ is a striction point on M . $\square$

Theorem 2.8. *Absolute value of the Gauss curvature of M is maximum at the striction points on the generating line X and it is*

$$|K|_{\max} = \frac{\{k_1^2 \sin^2 \psi + (\psi' + k_2)^2\}^2}{(\psi' + k_2)^2}$$

Proof. Let M be a ruled surface that given in the form (1) and let Φ be base of the tangent space which is spanning by the unit vectors A_o and X where A_o is the tangent vector of the curve $\varphi(s, v = \text{const.})$ with the arc-length parameter s^* . Hence we write $A_o = \frac{d\varphi}{ds^*} = \frac{d\varphi}{ds} \frac{ds}{ds^*}$ where $\frac{d\varphi}{ds} = A$, $A_o = \frac{1}{\|A\|} A$ and $\frac{ds}{ds^*} = \frac{1}{\|A\|}$. Thus we obtain the following equations after the routine calculations.

$$\begin{aligned} \nabla_{A_o} T &= \frac{k_1}{\|A\|} \{\sin \psi X + \cos \psi \xi\} \\ (9) \quad \nabla_{A_o} \xi &= \frac{1}{\|A\|} \{-k_1 \cos \psi T - (\psi' + k_2) X\} \\ \nabla_{A_o} A &= \frac{1}{\|A\|} \left\{ \begin{aligned} &\{(1 - vk_1 \sin \psi)' - v(\psi' + k_2)k_1 \cos \psi\} T \\ &+ \{(1 - vk_1 \sin \psi)k_1 \sin \psi - v(\psi' + k_2)^2\} X \\ &+ \{(1 - vk_1 \sin \psi)k_1 \cos \psi + (v(\psi' + k_2))'\} \xi \end{aligned} \right\} \end{aligned}$$

On the other hand, we denote $\xi_{\varphi(s,v)}$ as the unit normal vector at the points $\varphi(s, v)$ then from (6), (7) and (9) we get

$$(10) \quad \xi_{\varphi(s,v)} = \frac{1}{\|A\|} \{-v(\psi' + k_2)T + (1 - vk_1 \sin \psi)\xi\}$$

By differentiating both side of (10) with respect to the parameter s then we get

$$\begin{aligned} \frac{d\xi_{\varphi(s,v)}}{ds} &= - \left\{ \begin{aligned} &((v(\psi' + k_2))' + k_1 \cos \psi (1 - vk_1 \sin \psi)) \frac{1}{\|A\|} \\ &+ v(\psi' + k_2) \left(\frac{1}{\|A\|}\right)' \end{aligned} \right\} T \\ (11) \quad &+ \left\{ (1 - vk_1 \sin \psi) \left(\frac{1}{\|A\|}\right)' + \begin{pmatrix} (1 - vk_1 \sin \psi)' \\ -vk_1 \cos \psi (\psi' + k_2) \end{pmatrix} \frac{1}{\|A\|} \right\} \xi \\ &- (\psi' + k_2) \frac{1}{\|A\|} X \end{aligned}$$

Let S be shape operator of M at the points $\varphi(s, v)$ then we can easily obtained that the matrix S_Φ is following with respect to base Φ .

$$S_\Phi = \begin{bmatrix} \langle S(A_o), A_o \rangle & \langle S(A_o), X \rangle \\ \langle S(X), A_o \rangle & \langle S(X), X \rangle \end{bmatrix}$$

Since $\langle S(X), X \rangle = 0$ and $\langle S(A_o), X \rangle = \langle S(X), A_o \rangle$ then the Gauss curvature is

$$K(s, v) = \det S_\Phi = -(\langle S(A_o), X \rangle)^2$$

Suppose that s^* is arc-length parameter of A_o then we get

$$S(A_o) = \nabla_{A_o} \xi_{\varphi(s, v)} = \frac{d\xi_{\varphi(s, v)}}{ds^*} = \frac{d\xi_{\varphi(s, v)}}{ds} \frac{ds}{ds^*} = \frac{1}{\|A\|} \frac{d\xi_{\varphi(s, v)}}{ds}$$

From (10) and (11) we obtained

$$\begin{aligned} S(A_o) = & -\frac{1}{\|A\|} \left\{ \begin{aligned} & ((v(\psi' + k_2))' + k_1 \cos \psi (1 - vk_1 \sin \psi)) \frac{1}{\|A\|} \\ & + v(\psi' + k_2) \left(\frac{1}{\|A\|} \right)' \end{aligned} \right\} T \\ (12) \quad & + \frac{1}{\|A\|} \left\{ (1 - vk_1 \sin \psi) \left(\frac{1}{\|A\|} \right)' + \begin{pmatrix} (1 - vk_1 \sin \psi)' \\ -vk_1 \cos \psi (\psi' + k_2) \end{pmatrix} \frac{1}{\|A\|} \right\} \xi \\ & - (\psi' + k_2) \frac{1}{\|A\|^2} X \end{aligned}$$

Hence, Gauss curvature is

$$(13) \quad K(s, v) = -\frac{(\psi' + k_2)^2}{\|A\|^4}$$

We differentiated both side of (13) with respect to v for find the maximum value of the Gauss curvature along the X on M . Thus we obtain

$$\frac{\partial K(s, v)}{\partial v} = \frac{2(\psi' + k_2)^2 \left\{ v \left(\lambda^2 + (\psi')^2 \right) - \lambda \right\}}{\left\{ v^2 \left\{ k_1^2 \sin^2 \psi + (\psi' + k_2)^2 \right\} - 2vk_1 \sin \psi + 1 \right\}^3} = 0$$

and we get the value of v is $v = \frac{k_1 \sin \psi}{k_1^2 \sin^2 \psi + (\psi' + k_2)^2}$. It is easy to see that, $\varphi(s, v)$ is the striction point and we can say that absolute value of the Gauss curvature of M is maximum at the striction points on the X . Finally, by using (13) then we get

$$(14) \quad |K|_{\max} = \frac{\left\{ k_1^2 \sin^2 \psi + (\psi' + k_2)^2 \right\}^2}{(\psi' + k_2)^2}$$

□

We can write the relation between the Gauss curvature and the distribution parameter by using (8) and (14) is following.

$$|K|_{\max} = \frac{1}{(P_X)^2}$$

Thus we proved the following corollary too.

Corollary 2.9. *Distribution parameter of the ruled surface is depend only generating lines.*

Moreover, the Darboux frame of the surface along base curve is

$$\begin{bmatrix} \nabla_T T \\ \nabla_T X \\ \nabla_T \xi \end{bmatrix} = \begin{bmatrix} 0 & k_1 \sin \psi & k_1 \cos \psi \\ -k_1 \sin \psi & 0 & (\psi' + k_2) \\ -k_1 \cos \psi & -(\psi' + k_2) & 0 \end{bmatrix} \begin{bmatrix} T \\ X \\ \xi \end{bmatrix}$$

and the Darboux vector is

$$W = (\psi' + k_2) T - k_1 \sin \psi X + k_1 \cos \psi \xi$$

Thus we obtain geodesic curvature, geodesic torsion and normal curvature of the ruled surface along its generating lines as follows

$$(15) \quad \kappa_g = k_1 \sin \psi, \quad \tau_g = (\psi' + k_2), \quad \kappa_\xi = k_1 \cos \psi$$

respectively. Note also that if the ruled surface is constant curvature surface with nonzero geodesic curvature then P_X is constant and from (8) and (15) we obtain $\frac{\tau_g^2}{\kappa_g^2 + \tau_g^2} = \text{constant}$. Hence we can give the following theorem.

Theorem 2.10. *A ruled surface is constant curvature surface with nonzero geodesic curvature if and only if $\frac{\tau_g^2}{\kappa_g^2 + \tau_g^2}$ is constant. In the case that the base curve is one of the geodesics of the ruled surface then ruled surface is developable.*

On the other side, Steiner rotation and translation vectors are

$$(16) \quad \begin{aligned} D &= \left(\oint_{(\alpha)} (\psi' + k_2) ds \right) T - \left(\oint_{(\alpha)} k_1 \sin \psi ds \right) X + \left(\oint_{(\alpha)} k_1 \cos \psi ds \right) \xi \\ V &= \oint_{(\alpha)} T ds \end{aligned}$$

respectively, furthermore, the pitch and angle of pitch of M are

$$(17) \quad L_X = 0 \quad \text{and} \quad \lambda_X = - \oint_{(\alpha)} k_1 \sin \psi ds$$

respectively. From (4), (5), (16) and (17), we obtained that $L_T = \oint_{(\alpha)} T ds$, $L_N = L_B = 0$ and

$$\lambda_T = \oint_{(\alpha)} (\psi' + k_2) ds$$

$$\lambda_N = - \left(\oint_{(\alpha)} k_1 \sin \psi ds \right) \sin \psi + \left(\oint_{(\alpha)} k_1 \cos \psi ds \right) \cos \psi$$

$$\lambda_B = \left(\oint_{(\alpha)} k_1 \sin \psi ds \right) \cos \psi + \left(\oint_{(\alpha)} k_1 \cos \psi ds \right) \sin \psi$$

for the special cases $X = T$, $X = N$ and $X = B$.

Corollary 2.11. *If the ruled surface is developable then*

$$(\lambda_N)^2 + (\lambda_B)^2 = \left(\oint_{(\alpha)} k_1 \sin \psi ds \right)^2 + \left(\oint_{(\alpha)} k_2 \cos \psi ds \right)^2$$

is holds.

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