A GENERALIZATION OF STRONG CAUSALITY

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ABSTRACT. In this paper we are going to define strong T_n- causality for $n \in N$. These are levels in the causal ladder of spacetime between stable causality and strong causality. A characterization of them is presented. The newly proposed levels are compared with Carter's levels of causal virtuosity. It is also proved that strong T_1- causality implies compact stable causality.

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1. Introduction

Even if a spacetime is causal, it may contain a point p such that there are future directed causal curves leaving arbitrary small neighborhoods of p and then returning. Since an arbitrary small perturbation of the metric in such spacetimes would produce causality violation, these spacetimes seem physically unreasonable. So it is useful to formulate precise conditions which characterize this type of behavior. One such characterization is the strong causality condition. Given $p \in M$, The spacetime M is said to be strongly causal at p if p has arbitrary small causally convex neighborhoods. A spacetime is strongly causal if it is strongly causal at p,

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for every $p \in M$ [1, 3]. We recall that an open set V is called causally convex if any causal curve with endpoints in V is entirely contained in V.

In this paper we are going to generalize the concept of strong causality. To this end it is useful to regard the chronological and causal relations [1, 3] as subsets of $M \times M$:

$$I^+ = \{(p,q) : p \ll q\}\ ,\ J^+ = \{(p,q) : p \leq q\}.$$

The definitions given here are compatible with those used in set theory. The relation $R^+ \subseteq M \times M$ is transitive if for all p, q and $z \in M$, $(p,q) \in R^+$ and $(q,z) \in R^+$ then $(p,z) \in R^+$. The relation R^+ is antisymmetric if for all $p, q \in M$, $(p,q) \in R^+$ and $(q,p) \in R^+$ then $p = q$. R^+ is a causal relation if $I^+ \subseteq R^+$. In addition we take:

- $R^- = \{(q,p) : (p,q) \in R^+\}$
- $R^+(p) = \{q : (p,q) \in R^+\}$
- $R^-(p) = \{q : (q,p) \in R^+\}$.

Given a relation R^+ one can define two operations: (a) closure: $R^+ \to \overline{R^+}$, and (b) transitivization: $R^+ \to R^{+\infty} = \cup_{i=1}^{\infty}(R^+)^i$, that $(R^+)^i = \{(p,q) :\ \exists\, p_1, .., p_{i-1} \in M : (p,p_1) \in R^+, (p_1,p_2) \in R^+, .., (p_{i-1},q) \in R^+\}$, for $i \geq 1$.

Sorkin and Woolgar [8] have defined $K^+ \subseteq M \times M$ as the smallest transitive closed relation which contains I^+. This definition arose from the fact that J^+ is transitive but not necessarily closed and $\overline{J^+}$ is closed but not necessarily transitive. The spacetime (M,g) is $K-$causal if K^+ is antisymmetric. The coincidence of K-causality and stable causality has been proved by Minguzzi [6].

If the operations closure and transitivization are alternatively applied to the relation J^+, a chain of relations is obtained that all of its members are contained in K^+ and its members are J^+, $A^+ = \overline{J^+}$, $A^{+\infty}$, $\overline{A^{+\infty}}$, ... , and so on. For simplicity, we denote J^+, $A^{+\infty}$, $\overline{A^{+\infty}}^{\infty}$,.., by T_0^+, T_1^+, T_2^+,..., respectively. M is T_n- causal if T_n^+ is antisymmetric. If U is an open neighborhood of M, then we denote by J_U^+ the causal relation on the spacetime U with the induced metric. And we have a chain as above on the spacetime U, $T_{0,U}^+$, $T_{1,U}^+$, $T_{2,U}^+$,.., and so on.

In section 2, we are going to define strong T_n- causality for $n \in N$ and give a characterization of these levels. This characterization is similar to that proved by Minguzzi for strong causality [4].

Theorem 1.1. The spacetime (M, g) is strongly causal, if and only if for every $p, q \in M$, $(p, q) \in J^+$ and $(q, p) \in \overline{J^+}$ imply $p = q$.

In this section we also prove that strong T_1- causality implies compact stable causality.

Carter showed that between strong causality and stable causality a sequence of relations could be defined. In section 3, strong T_n- causality is compared with Carter's levels of causal virtuosity.

2. Strong T_n- causality

We can generalize the concept of convexity and as a consequence strong causality in the following manner.

Definition 2.1. An open set U is said to be causally T_n- convex if for every $p,\ q \in U$, $(p, q) \in T_n^+$ implies $T_n^+(p) \cap T_n^-(q) \subseteq U$.

Definition 2.2. A spacetime (M, g) is called strongly T_n- causal if for any $p \in M$, there exist arbitrary small causally T_n- convex neighborhoods, i.e. for every neighborhood U of p, there exists a neighborhood $V \subseteq U$, $p \in V$, such that V is causally T_n- convex in M.

Remark 2.3. Every event p of a spacetime (M, g) admits arbitrary small globally hyperbolic neighborhoods. If U is such a neighborhood the causal relation on the spacetime (U, g), J_U^+, is closed and hence $J_U^+ = T_{i,U}^+ = K_U^+$, for every $i \geq 0$. In the remainder of this paper we suppose that causally T_i- convex neighborhood U is contained in the causally T_i- convex neighborhood V, such that $\overline{U} \subseteq V \subseteq W$ and W is a globally hyperbolic neighborhood.

It is trivial that strong T_n- causality implies T_n- causality. The following theorem shows that $\overline{T_n}$- causality implies strong T_n- causality.

Theorem 2.4. The spacetime (M, g) is strongly T_n- causal if and only if for every $p,\ q \in M$, $(p, q) \in J^+$ and $(q, p) \in \overline{T_n^+}$ imply $q = p$.

We need some preliminary lemmas to prove this theorem.

Lemma 2.5. If U is a causally T_n- convex neighborhood, then $T_{n,U}^+ = T_n^+ |_{U \times U}$. Moreover, $\overline{T_{n,U}^+} = \overline{T_n^+} |_{U \times U}$.

Proof. It is trivial that $T_{n,U}^+ \subseteq T_n^+ |_{U \times U}$. Here the converse will be proved by induction. For $n = 0$, if U is causally convex, then $J^+ |_{U \times U} \subseteq J_U^+$. Suppose that the lemma is true for $n = k$. Let U be a causally T_{k+1}- convex neighborhood and

$(p,q) \in T^+_{k+1}$ for $p,\ q \in U$. There exist $p_1, p_2, .., \ p_r \in M$ such that $(p,p_1) \in \overline{T^+_k}$,.., $(p_r,q) \in \overline{T^+_k}$. Since U is causally T_{k+1}- convex, $p_i \in U$ for $1 \leq i \leq r$. There exist sequences p^n_i, for $1 \leq i \leq r$, p^n and q^n such that $(p^n, p^n_1) \in T^+_k$,.., $(p^n_r, q^n) \in T^+_k$, $p^n_i \to p_i$, $p^n \to p$ and $q^n \to q$. Thus by the hypothesis of induction we have $(p^n, p^n_1) \in T^+_{k,U}$,.., $(p^n_r, q^n) \in T^+_{k,U}$ for sufficiently large n. Hence $(p,p_1) \in \overline{T^+_{k,U}}$, .., $(p_r,q) \in \overline{T^+_{k,U}}$ and as a consequence $(p,q) \in T^+_{k+1,U}$.

Lemma 2.6. Let (M,g) be a strongly T_n- causal spacetime and $(p,q) \in \ T^+_{n+1}$. If U is a causally T_n- convex neighborhood of p with compact boundary, such that $q \notin \ \overline{U}$, then there exists a point $t \in \ \partial\, U$ with $(p,t) \in \ J^+$ and $(t,q) \in \ T^+_{n+1}$.

Proof. This lemma can be proved by induction. For $n=1$, suppose that (M,g) is strongly causal and $(p,q) \in \ {T_1}^+$. There exist $p_1,\ .., p_r \in M,\ r \in N$, such that $(p,p_1) \in A^+, .., \ (p_r,q) \in A^+$. Let $p=p_0$ and $q=p_{r+1}$. We choose the smallest i_0 such that $p_{i_0} \in \ U$ and $p_{i_0+1} \notin \ U$. There are two cases.

Case i. $p_{i_0+1} \in \ \ \partial\, U$. In this case p_{i_0+1} is the required point. Indeed, by assumption of remark 2.3, $\overline{U}$ is contained in $V \subseteq \ W$ that V is causally convex and W is globally hyperbolic. $p_i \in \ V$, for $0 \leq i \leq i_0+1$. By lemma 2.5 and remark 2.3, $(p_i, p_{i+1}) \in \ A^+_V = J^+_V$, for $0 \leq i \leq i_0+1$.

Case ii. $p_{i_0+1} \notin \ \partial\, U$. For $i \leq \ i_0$, $(p_i, p_{i+1}) \in \ A^+$, $p_i,\ p_{i+1} \in \ U$. Again lemma 2.5 and remark 2.3 imply that $(p_i, p_{i+1}) \in \ J^+$.

There are sequences $t_n,\ s_n$ and future directed causal curves γ_n with end points t_n and s_n, such that $t_n \to \ p_{i_0}$ and $s_n \to \ p_{i_0+1}$. Thus γ_n intersects $\partial\, U$ at some point l_n for sufficiently large n. By compactness of $\partial\, U$, there exists a subsequence of l_n that converges to a point t. Since $(t, p_{i_0+1}) \in \ A^+$, $(t,q) \in \ T^+_1$. We also have $(p_{i_0}, t) \in \ A^+_V = J^+_V$. So t is the required point.

Suppose that the lemma holds for $n=k$. We prove that it also holds for $n=k+1$. Let M be a strongly T_k- causal spacetime, $(p,q) \in T^+_{k+1}$ and U be a causally T_k- convex neighborhood of p, with compact boundary. By definition of T^+_{k+1}, there exist $p_1,\ .., p_r \in M,\ r \in N$, such that $(p,p_1) \in \overline{T^+_k}, .., (p_r,q) \in \overline{T^+_k}$. Let $p=p_0$ and $q=p_{r+1}$. Again we choose the smallest i_0 such that $p_{i_0} \in \ U$ and $p_{i_0+1} \notin \ U$. There are two cases.

Case i. $p_{i_0+1} \in \ \partial\, U$. p_{i_0+1} is the required point. The proof is similar to the case $n=1$.

Case ii. $p_{i_0+1} \notin \partial U$. For $i \leq i_0$, $(p_i, p_{i+1}) \in \overline{T_k^+}$ and $p_i,\ p_{i+1} \in U$. Remark 2.3 and lemma 2.5 imply that $(p_i, p_{i+1}) \in \overline{T_{k,U}^+} = J_U^+$, for $i \leq i_0$. There are sequences t_n and s_n such that $t_n \to p_{i_0}$, $s_n \to p_{i_0+1}$ and $(t_n, s_n) \in T_k^+$. By the hypothesis of induction, there exists a point $l_n \in \partial U$ such that $(t_n, l_n) \in J^+$ and $(l_n, s_n) \in T_k^+$. l_n has a subsequence that converges to a point t since ∂U is compact. Therefore $(p_{i_0}, t) \in \overline{T_{k,V}^+} = J_V^+$ and $(t, q) \in T_{k+1}^+$. So t is the required point and the induction is complete.

Lemma 2.7. Let M be a strongly T_n- causal spacetime. If $U$and U', $U' \subseteq U$, are causally T_n- convex neighborhoods with compact boundary and U' is not causally T_{n+1}- convex, then there exist $p,\ q \in U'$ and $t,\ l \in \partial U$, $t \neq l$, such that $(p, t) \in J^+$, $(t, l) \in T_{n+1}^+$ and $(l, q) \in J^+$.

Proof. Since U' is not strongly T_{n+1}- convex, there exist $p,\ q \in U'$ such that $(p, q) \in T_{n+1}^+$ and $T_{n+1}^+(p) \cap T_{n+1}^-(q)$ is not a subset of U'. There exist $p_1, .., p_r \in M$, $r \in N$, and $1 \leq i \leq r$ such that $(p, p_1) \in \overline{T_n^+}$,.., $(p_r, q) \in \overline{T_n^+}$ and $p_i \notin \overline{U}$. (indeed if for every set of points with the above property we have $p_i \in \overline{U}$, for every $1 \leq i \leq r$, then by lemma 2.5 and remark 2.3, $(p, p_1) \in \overline{T_{n,V}^+} = J_V^+$,.., $(p_r, q) \in \overline{T_{n,V}^+} = J_V^+$. Since $U,\ U'$ are causally T_n- convex, $T_{n+1}^+(p) \cap T_{n+1}^-(q) \subseteq U'$).

$(p, p_i) \in T_{n+1}^+$, thus lemma 2.6 implies that there exists $t \in \partial U$ such that $(p, t) \in J^+$ and $(t, p_i) \in T_{n+1}^+$.

We can repeat this process for $(p_i, q) \in T_{n+1}^+$ and obtain a point $l \in \partial U$ such that $(l, q) \in J^+$, $(p_i, l) \in T_{n+1}^+$. Hence we have $(t, l) \in T_{n+1}^+$.

Proof of theorem 2.4. The sufficiency can be proved by induction. For $n = 0$ the induction holds by theorem 1.1. Suppose that the induction holds for $n = k$. We prove that it also holds for $n = k + 1$.

Suppose by contradiction that (M, g) is not strongly T_{k+1}- causal at p. By the hypothesis of induction, we know that (M, g) is strongly T_k- causal. Let U be a causally T_k- convex neighborhood of p with compact boundary and U_ns ($U_{n+1} \subseteq U_n \subseteq U$) , $n = 1, .., \infty$, be open causally T_k- convex neighborhoods that shrink to p. By the hypothesis, there exists a subsequence of U_n that its members are not causally T_{k+1}- convex neighborhoods. Lemma 2.7 implies that there exist $p_n,\ q_n \in U_n$ and $t_n,\ l_n \in \partial U$ such that $(p_n, t_n) \in J^+$, $(t_n, l_n) \in T_{k+1}^+$ and $(l_n, q_n) \in J^+$.

Since ∂U is compact, there exist subsequences t_{n_k} and l_{n_k} that converge to t and l. By remark 2.3, (p, t) and $(l, p) \in \overline{J_V^+} = J_V^+$. Thus $(l, t) \in J^+$, $(t, l) \in \overline{T_{n+1}^+}$ and

$l \neq t$. Indeed, $l = t$ implies that $(p, l) \in J^+$ and $(l, p) \in J^+$, that is a contradiction to the causality of M.

Conversely, suppose by contradiction that (M, g) is strongly T_n-causal but $(p, q) \in J^+$ and $(q, p) \in \overline{T_n^+}$ for $q \neq p$. Let $w \in J^+(p) \cap J^-(q)$, and U be an arbitrary small neighborhood of w whose closure does not contain p and q. There are sequences p_n and q_n such that $(q_n, p_n) \in T_n^+$, $p_n \to p$ and $q_n \to q$. Take $z \in I^+(w) \cap U$ and $r \in I^-(w) \cap U$, then $p \in I^-(z)$ and $q \in I^+(r)$. For sufficiently large n, $p_n \in I^-(z)$ and $q_n \in I^+(r)$, since $I^-(z)$ and $I^+(r)$ are open. Therefore $(r, z) \in T_n^+$. By taking U arbitrary small, $T_n^+(r) \cap T_n^-(z)$ is not entirely contained in U and consequently M is not strongly T_n- causal that is a contradiction.

Remark 2.8. Sorkin and Woolgar [8] have proved that $K-$ causality implies strong $K-$ causality. The converse is trivial.

Corollary 2.9. Let (M, g) be a spacetime and $K^+ = T_n^+$ for some $n \in N$. The spacetime (M, g) is $K-$ causal if and only if for every $p, q \in M$, $(p, q) \in J^+$ and $(q, p) \in K^+$ imply $q = p$.

Proof. The proof is a direct consequence of theorem 2.4 and remark 2.8.

Example 2.10. For an example of a T_1- causal spacetime which is not strongly T_1- causal, see [4] section 5. An example of a strongly T_1- causal spacetime which is not $\overline{T_1}$- causal can be constructed in the similar way(see figure 1). Let (Λ, η) be 2+1 Minkowski spacetime and (t, x, y) be canonical coordinates, $\eta = -dt^2 + dx^2 + dy^2$. Remove planes $t = 0$, $t = 1$ and $t = 2$. Tree holes on the planes (open in plane topology) with the same size but non-aligned centers are not removed. Identify the top hole with that at the bottom cyclically. The causal future of the bottom and middle holes are stopped with two closed disks which are removed. These disks have exactly the same size of light cone at that height. This height is chosen such that the causal past of the middle (resp. the top) hole reaches the edge of the removed disks at d (resp. a). The lightlike geodesic segments cd and ea in the boundary of the light cone, issuing from the bottom hole and the middle hole, are also removed. The metric is induced from Minkowski spacetime. This spacetime is strongly T_1- causal but it is non $\overline{T_1}$- causal. Just take two points x and z in ab and de, respactively. (x, z) and $(z, x) \in \overline{T_1^+}$. Nevertheless we can not find two different points r and s that $(r, s) \in J^+$ and $(s, r) \in \overline{T_1^+}$.

In addition, we can construct an example of a spacetime which is $\overline{T_1}$- causal but it is not T_2- causal by repeating similar possess on Minkowski spacetime by removing the planes $t = 0$, $t = 1$, $t = 2$ and $t = 3$ [5].

Compact stable causality is another level between $\overline{T_1}$- causality and T_1- causality [5]. We prove that strong T_1- causality implies compact stable causality.

Definition 2.11. A spacetime (M, g) is compactly stably causal if for every relatively compact open set B there is a metric $g_B \geq\ g$ such that $g_B > g$ on B, $g_B = g$ on B^C and (M, g_B) is causal.

Theorem 2.12. [5] $\overline{T_1}$- causality implies compact stable causality.

Theorem 2.13. strong T_1- causality implies compact stable causality.

Proof. Suppose by contradiction that (M, g) is strongly T_1- causal but non-compactly stably causal. So, there is a relatively compact open set B such that for every $g' \geq\ g$, $g' > g$ on B, $g' = g$ on B^C, (M, g') is not causal. Using the proof of theorem 2.12 [5], we obtain a sequence of events $p^k \in\ \overline{B}$ such that $(p^k, p^{k+1}) \in\ A^+$ and $p^k \neq\ p^{k+1}$. In addition, for each k there is a sequence of g- causal curves, not all contained in a compact, so that the endpoints of these curves converge to (p^k, p^{k+1}). Note that for every pair of integers $a \leq\ b$, $(p^a, p^b) \in\ T_1^+$.

Since $\overline{B}$ is compact, there is a subsequence of p^k, like p^{k_s}, that converges to a point p. M is not strongly T_1- causal at p. Indeed, by assumption, we know that M is T_1- causal and thus strongly causal. Hence there exists a sequence of causally convex and relatively compact neighborhoods U_n of p, $U_{n+1} \subseteq\ U_n$, that shrink to p. For each n all but a finite number of the points p^{k_s} belong to U_n. $(p^{k_s}, p^{k_s+1}) \in\ A^+$ and $p^{k_s} \neq\ p^{k_s+1}$. If $p^{k_s+1} \in\ U_n$, then there is a sequence of g- causal curves, not all contained in a compact, so that the endpoints of these curves converge to (p^{k_s}, p^{k_s+1}). Since U_n is causally convex, this is a contradiction. Indeed, $k_{s+1} > k_s + 1$ and $(p^{k_s}, p^{k_{s+1}}) \in\ T_1^+$ but $T_1^+(p^{k_s}) \cap\ T_1^-(p^{k_{s+1}})$ is not a subset of U_n. Since this argument is true for every n, M is not strongly T_1- causal.

Remark 2.14. Compact stable causality doesn't imply strong T_1- causality. For an example of non-strongly T_1- causal spacetime which is compactly stably causal see, [4] section 5.

3. Strong T_n- causality and Carter's causal levels

Strong causality condition actually is not strong enough to avoid all simple pathologies. Carter defined a classification by means of new causal relations called n degree causal relations [2]. One can find the following definitions in [2] and [7].

$$<^1 \zeta) \equiv \{x \in M : (J^+(\zeta) \cap \overline{J^-(x)}) \cup (\overline{J^+(\zeta)} \cap J^-(x)) \neq \emptyset\},$$
$$(\zeta >^1 \equiv \{x \in M : (J^-(\zeta) \cap \overline{J^+(x)}) \cup (\overline{J^-(\zeta)} \cap J^+(x)) \neq \emptyset\}.$$

A binary relation between subsets of M are arranged by:

$$\zeta <^1 \tau \Leftrightarrow \tau \subseteq <^1 \zeta), \ \tau >^1 \zeta \Leftrightarrow \tau \subseteq (\zeta >^1 .$$

These relations have been used to define inductive chain of definitions:

$$\begin{aligned} <^n \zeta) &\equiv \{x \in M : (\overline{<^{n-1} \zeta)} \cap (x >) \bigcup (< \zeta) \cap \overline{(x >^{n-1}})\\ &\bigcup (\bigcup_{r=1}^{n-1} <^r \zeta, x >^{n-r}) \neq \emptyset\},\\ (\zeta >^n &\equiv \{x \in M : (< x) \cap \overline{(\zeta >^{n-1})} \bigcup (\overline{<^{n-1} x)} \cap (\zeta >)\\ &\bigcup (\bigcup_{r=1}^{n-1} <^r \zeta, x >^{n-r}) \neq \emptyset\}, \end{aligned}$$

where

$$<^r \zeta, x >^{n-r} \equiv <^r \zeta) \cap (x >^{n-r} .$$

From these relations we can construct binary relations $<^n, >^n$ between subsets of M. Of course all of these definitions are valid for sets consisting of a single point ($x <^r y$ will be used for $\{x\} <^r \{y\}$ and so on).

Definition 3.1. A subset $\zeta \subseteq M$ is said to be virtuous to the nth degree ($n \geq 0$) if no pair of different points $x, y \in \zeta$, satisfies $x <^r y$ and $y <^s x$ with $r + s \leq n$. And ζ is said to be sub-vicious by n degrees ($n \geq 0$) if for any two points $x, y \in \zeta$, we have $x <^r y$ and $y <^s x$ with $r + s \leq n$.

In particular M is said to be infinite causally virtuous if it is nth degree causally virtuous for every n. 0th degree causally virtuous spacetimes are causal spacetimes, the spacetimes virtuous to the 1th degree are distinguishing spacetimes and those virtuous to the 2th degree are strongly causal.

Theorem 3.2. [4] K- causality implies infinite causal virtuosity.

Now the following question may arise. What is the relationship between strong T_n- causality and Carter's levels?

Lemma 3.3. [5] Let $\circ$ denote the composition of relations, then $J^+ \circ A^+ \subseteq A^+$ and $A^+ \circ J^+ \subseteq A^+$.

Lemma 3.4. Let $x <^{n+2} y$, then one of the following statements holds:

(1) There exist a point t and a sequence t_n that converges to t such that (a) $(x, t_n) \in T_n^+$ and $(t, y) \in A^+$ or (b) $(x, t) \in A^+$ and $(t_n, y) \in T_n^+$.

(2) $(x, y) \in T_n^+$.

Moreover, $(x, y) \in T_{n+1}^+$.

Proof. This lemma can be proved by induction. For $n = 1$, suppose that $x <^3 y$. There exists a point t in one of the sets $(\overline{<^2 x)} \cap (y >)$, $(<^2 x) \cap (y >)$, $(< x) \cap \overline{(y >^2)}$ or $(< x) \cap (y >^2)$. Suppose that t belongs to the first set, then we prove that 1 case (a) holds (for the other cases, by a similar argument we can prove that 1 case (b) or 2 holds). There exists a sequence t_n that converges to t, $t_n \in <^2 x)$ and $t \in (y >$. Note that by lemma 3.3, $t \in (y >$ implies $(t, y) \in A^+$.

One of the cases $(\overline{< x)} \cap (t_n >) \neq \emptyset$, $(< x) \cap (t_n >) \neq \emptyset$ or $(< x) \cap \overline{(t_n >)} \neq \emptyset$ holds. Again we suppose that the first case holds. The proof for the other cases is similar to this one. There exists a sequence l_n that $l_n \in (t_n >$ and $l_n \in \overline{< x)}$. Choose the sequence l_n^m that converges to l_n and $l_n^m \in < x)$. Hence $(x, l_n^m) \in A^+$, $(l_n, t_n) \in A^+$ and $(t, y) \in A^+$. By taking a limit when $m \to \infty$, $(x, l_n) \in A^+$. So $(x, t_n) \in T_1^+$ and $(t, y) \in A^+$.

Suppose that the induction holds for $n = k$. We prove that it also holds for $n = k + 1$. If $x <^{k+3} y$, then there exists a point t in one of the sets $\overline{<^{k+2} x}, y >$, $< x, \overline{y >^{k+2}}$ or $<^r x, y >^{k-r+3}$, $1 \leq r \leq k + 2$. If t belongs to the first set, then there exists a sequence t_n that converges to t such that $t_n \in <^{k+2} x)$ and $t \in (y >$. The hypothesis of induction implies that $(x, t_n) \in T_{k+1}^+$ and $(t, y) \in A^+$ (a similar argument can be used for the case that $< x, \overline{y >^{k+2}} \neq \emptyset$). If t belongs to $<^r x$, $y >^{k-r+3}$, $1 \leq r \leq k+2$, then by the hypothesis of induction we have $(x, t) \in T_{r-1}^+$ and $(t, y) \in T_{k-r+2}^+$. Consequently $(x, y) \in T_{k+1}^+$ and 2 holds.

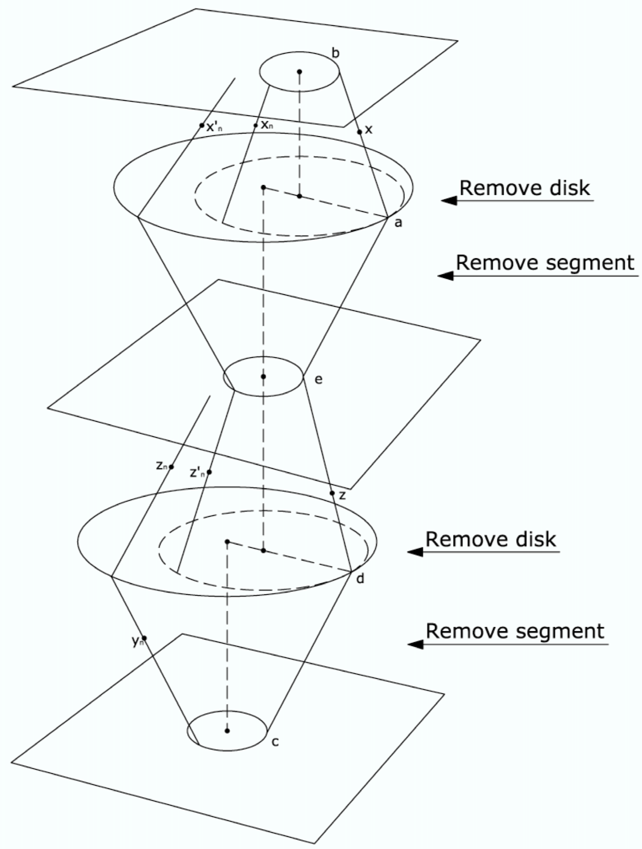

FIGURE 1. The events x_n, y_n and z_n are chosen so that $(x_n, y_n) \in A^+$ and $(y_n, z_n) \in A^+$. Consequently, $(x_n, z_n) \in T_1^+$. In addition, $(z'_n, x'_n) \in T_1^+$. Since (x, z) (resp. (z, x)) can be regarded as a limit of a suitable sequence (x_n, z_n) (resp. (z'_n, x'_n)), this spacetime is not $\overline{T_1^+}$- causal.

Theorem 3.5. Strong T_n- causality implies $n+2$th degree causal virtuosity.

Proof. Suppose by contradiction that x and y are sub-vicious of degree $n+2$ then $x <^r y$ and $y <^s x$ with $r+s \leq n+2$. For example, suppose that $x <^{n+2} y$ and $y <^0 x$. If 1 case (a) (1 case (b)) in lemma 3.4 holds, then there exist a point t and a sequence t_n that converges to t such that $(x, t_n) \in T_n^+$ and $(t, y) \in A^+$ ($(x, t) \in A^+$ and $(t_n, y) \in T_n^+$). Since $(y, x) \in J^+$, lemma 3.3 implies $(t, x) \in A^+$ ($(y, t) \in A^+$). Thus the spacetime is not strongly T_n- causal at t and this is a contradiction. If 2 holds then the spacetime is not T_n- causal. For the cases $x <^r y$ and $y <^s x$, $r, s \leq n+1$, lemma 3.4 implies $(x, y) \in T_{r-1}^+$, and $(y, x) \in T_{s-1}^+$. This is a contradiction to T_n- causality and as a consequence to strong T_n- causality.

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