JACOBI STABILITY OF LINEARIZED GEOMETRIC DYNAMICS

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Abstract. This paper is dedicated to the study of geometric dynamics (an Euler-Lagrange prolongation of a flow on a Riemannian manifold) from the point of view of KCC theory, Jacobi stability and Lyapounov stability. Section 1 recalls the geometrical roots of Jacobi stability and announces the subject of the paper. Section 2 introduces the variational ODEs (Jacobi fields ODEs), the KCC differential invariants for a second order ODE system, and defines the Jacobi stability. Section 3 studies the KCC differential invariants associated to geometric dynamics. Section 4 describes various linearizations of geometric dynamics. Section 5 studies the Jacobi stability for linearized geometric dynamics around a stationary point of the field. Section 6 shows that the linearized geometric dynamics around a critical point of the energy can be Jacobi stable or unstable. Section 7 proves the Lyapounov instability of Jacobi fields ODEs along geometric dynamics trajectories.

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1. INTRODUCTION

Geometric representation and Jacobi stability are useful tools for second order ordinary differential equations ([2], [3], [9]). The concept of Jacobi stability initially appeared in the context of differential geometry for geodesics of Riemann or Finsler manifold and then was extended to trajectories of second order ODE ([1]- [11]). This is the differential geometric theory of the variational equations for deviation of whole trajectories to nearby ones. The paper [11] interprets the Jacobi stability as the *robustness* of the dynamical system.

It is well known that linear stability analysis of a second order of ODEs involves changing this system into a first order system of ODEs and *linearization* (via Taylor formula) or *extended linearization* (via integral formula along a ray). If a steady state is a stable one, then small perturbation away from this zero speed-point will not change the dynamics of the system. Around a stable state it appears a *basin of attraction* of the system, which is not specified by the linear analysis. On the other hand, the KCC analyzes the stability of a *whole trajectories in a region*, i.e., the perturbations represent trajectories close-by to the reference trajectory. This geometrical method, even when derived at a stationary point, yields information about the behavior of trajectories in an open region surrounding this point ([10]).

The topics of this paper include: geometric representation and Jacobi stability for second order ODE (Section 2), the Euler-Lagrange prolongation of a flow on a Riemannian manifold (geometric dynamics) (Section 3), various linearizations of geometric dynamics (Section 4), the Jacobi instability of linearized geometric dynamics around a stationary point (Sections 5), respectively around a critical point of the energy (Section 6), the Lyapounov instability of Jacobi fields ODEs along geometric dynamics trajectories (Section 7).

2. GEOMETRIC REPRESENTATION AND JACOBI STABILITY FOR SECOND ORDER ODE

The concepts of geometric representation, Jacobi stability and the KCC theory (D. Kosambi [8], E. Cartan [6] and S. S. Chern [7]) are interrelated. We start with some basics of KCC theory.

Let $x = (x^1, ..., x^n)$, $\dot{x} = \dfrac{dx}{dt}$ and t be $2n+1$ coordinates in an open connected subset $D \subset \mathbf{R}^{2n+1}$. We consider a second order differential system

$$\frac{d^2x^i}{dt^2} + h^i(x, \dot{x}, t) = 0, \quad i = \overline{1, n}, \tag{2.1}$$

where each function $h^i(x, \dot{x}, t)$ is $\mathbf{C}^\infty$ ([1], [2],[10], [11]). Though the left hand member of the system (2.1) is a vector field, the "acceleration" term $\dfrac{d^2x^i}{dt^2}$, respectively the "force" term $-h^i(x, \dot{x}, t)$ are not contravariant vector fields with respect to general diffeomorphisms

$$\overline{x}^i = f^i(x^1, \ldots, x^n), \; i = \overline{1, n}, \quad \bar{t} = t.$$

The basic idea of KCC theory was to change the system (2.1) into an equivalent system (same solutions), where each term has a geometrical meaning. In order to do that, were defined five tensor fields, called *KCC differential invariants*. The theory starts by replacing the usual derivative $\dfrac{d}{dt}$ by the *KCC-covariant derivative* $\dfrac{D}{dt}$. This acts on a contravariant vector field $\xi^i(t)$ as

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + \frac{1}{2} h^i_{;r} \xi^r, \tag{2.2}$$

where ";" indicates partial differentiation with respect to the velocity $\dot{x}$. Using the KCC-covariant derivative (2.2), the system (2.1) changes into

$$\frac{D\dot{x}^i}{dt} = \varepsilon^i, \; \varepsilon^i = \frac{1}{2} h^i_{;r} \frac{dx^r}{dt} - h^i \tag{2.3}$$

where ε^i is a contravariant vector field on D called *the first KCC differential invariant.* From the physical point of view, the contravariant vector field ε^i represents an *external force* ([1]).

Remark 2.1. The functions $h^i = h^i(x, \dot{x}, t)$ are 2-homogeneous in $\dot{x}$ if and only if $\varepsilon^i = 0$. In differential geometry, $\varepsilon^i = 0$ is a necessary and sufficient condition for a semispray to be a spray ([10]). Particularly, for the geodesic spray of a Riemannian or Finsler manifold, the first KCC differential invariant vanishes.

The KCC theory starts with the study of trajectories $\overline{x}^i(t)$ which are slightly deviated upon a certain trajectory $x^i(t)$ of (2.1), i.e., $\bar{x}^i(t) = x^i(t) + \eta\xi^i(t)$, $0 < \eta \ll 1$. Precisely, one studies the dynamics of the system in variations. In this way, one gets the *variational ODEs (Jacobi fields ODEs)*

$$\frac{d^2\xi^i}{dt^2} + h^i_{;r}\frac{d\xi^r}{dt} + h^i_{,r}\xi^r = 0, \tag{2.4}$$

where "," indicates partial differentiation with respect to x. Using now the KCC-covariant derivative (2.2), one obtains

$$\frac{D^2\xi^i}{dt^2} = \frac{d^2\xi^i}{dt^2} + \frac{1}{2}h^i_{;r}\frac{d\xi^r}{dt}$$

$$+\frac{1}{2}\left[(h^i_{,j;r}\frac{dx^j}{dt} + h^i_{;j;r}\frac{d^2x^j}{dt^2} + \frac{\partial h^i_{;r}}{\partial t})\xi^r + h_{i;r}\frac{d\xi^r}{dt} + \frac{1}{2}h_{i;j}h^j_{;r}\xi^r\right].$$

In this way, the differential system (2.4) rewrites in the covariant form

$$\frac{D^2\xi^i}{dt^2} = -h^i_r\xi^r + \frac{1}{2}\left[-h^i_{;j;r}h^j\xi^r + h^i_{,j;r}\frac{dx^j}{dt}\xi^r + \frac{\partial h^i_{;r}}{\partial t}\xi^r + \frac{1}{2}h^i_{;j}h^j_{;r}\xi^r\right]$$

or equivalently,

$$\frac{D^2\xi^i}{dt^2} = P^i_r\xi^r, \tag{2.5}$$

where

$$P^i_j = -h^i_{,j} - \frac{1}{2}h^i_{;r;j}h^r + \frac{1}{2}h^i_{,r;j}\frac{dx^r}{dt} + \frac{1}{4}h^i_{;r}h^r_{;j} + \frac{1}{2}\frac{\partial h^i_{;j}}{\partial t}$$

is a tensor field called the *deviation curvature tensor* or *the second KCC differential invariant* of the system (2.1).

The next definition extends the notion of Jacobi stability, passing from geodesics to the trajectories of a second order system of ODE. For more details, see [17].

Definition 2.1.([2],[1]) *The trajectories of the system (2.1) are called Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor* P^j_i *are strictly negative everywhere; otherwise, they are called Jacobi unstable.*

The *third, fourth and fifth KCC differential invariants* of the system (2.1) are respectively (see also [10])

$$R^i_{jk} = \frac{1}{3}(P^i_{j;k} - P^i_{k;j}), \quad B^i_{jkl} = R^i_{jk;l}, \quad D^i_{jkl} = h^i_{;j;k;l}.$$

3. THE KCC DIFFERENTIAL INVARIANTS ASSOCIATED TO GEOMETRIC DYNAMICS

We remind first some basic elements of *geometric dynamics which describes a geodesic motion in a gyroscopic field of forces* ([12]-[18]). The theory of geometric dynamics can be equally applied to any kind of flow on a manifold endowed with a geometric structure capable to produce *square of length* (density of energy) and *derivatives*. The standard example includes a vector field $X \in \chi(M)$ on a Riemannian manifold $(M, g = (g_{ij}))$. To change the flow

$$\frac{dx^i}{dt}(t) = X^i(x(t)),\ i = \overline{1,n} \tag{3.1}$$

into a geodesic motion under a gyroscopic field of forces, we have to prolongate the system by derivation, and after modifying this prolongation to obtain an Euler-Lagrange system

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0$$

associated to the *least squares Lagrangian*

$$L = \frac{1}{2}\left\| \frac{dx^i}{dt} - X^i(x) \right\|_g^2, i = \overline{1,n}. \tag{3.2}$$

The Euler-Lagrange extension (geometric dynamics) of the flow (3.1), on the Riemannian manifold $(M, g = (g_{ij}))$, is

$$\frac{d^2x^i}{dt^2} + G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = F^i_j\frac{dx^j}{dt} + \nabla^i f, \tag{3.3}$$

where G^i_{jk} are the components of the Levi-Civita connection ∇ of (M, g), and

$$F^i_j = \nabla_j X^i - g^{ih}g_{kj}\nabla_h X^k,\ \nabla^i f = g^{ih}g_{kj}(\nabla_h X^k)X^j,\ i,j,k = \overline{1,n}.$$

The trajectories of the Euler-Lagrange extension (3.3), include the field lines of the vector field X and curves which are transversal to the field lines. All these trajectories are geodesics in a proper geometric structure.

The components $\dfrac{d^2x^i}{dt^2} + G^i_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt}$ represent the acceleration on the Riemannian manifold and the components $F^i_j\dfrac{dx^j}{dt} + \nabla^i f$ represent the gyroscopic force. We destroy the geometrical character of members of ODEs (3.3) splitting the terms in the following form

$$\frac{d^2x^i}{dt^2} + h^i(x,\dot{x},t) = 0,\ h^i(x,\dot{x},t) = G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} - F^i_j\frac{dx^j}{dt} - \nabla^i f. \tag{3.4}$$

To attend the KCC geometrical characterization of geometric dynamics, we need

Proposition 3.1. *The Euler-Lagrange system (3.4) defines the following five KCC differential invariants*

$$\varepsilon^i = \frac{1}{2}F^i_j\frac{dx^j}{dt} + \nabla^i f,$$

$$P^i_j = \left(\nabla_j F^i_k - \frac{1}{2}\nabla_k F^i_j\right)\frac{dx^k}{dt} + \nabla_j\nabla^i f + \frac{1}{4}F^i_r F^r_j,$$

$$R^i_{jk} = \frac{1}{2}\left(\nabla_j F^i_k - \nabla_k F^i_j\right), \ B^i_{jkl} = 0, \ D^i_{jkl} = 0.$$

Proof. Using (3.4) we have

$$\begin{aligned} h^i_{,r} &= \frac{\partial G^i_{jk}}{\partial x^r}\frac{dx^j}{dt}\frac{dx^k}{dt} - \frac{\partial F^i_j}{\partial x^r}\frac{dx^j}{dt} - \frac{\partial \nabla^i f}{\partial x^r}, \quad h^i_{;r} = 2G^i_{rk}\frac{dx^k}{dt} - F^i_r, \\ h^i_{,r;p} &= 2\frac{\partial G^i_{kp}}{\partial x^r}\frac{dx^k}{dt} - \frac{\partial F^i_p}{\partial x^r}, \quad h^i_{;r;p} = 2G^i_{rp}. \end{aligned}$$

Also, we take into account the formulas for the five KCC differential invariants and the Riemannian curvature tensor

$$G^i_{jkl} = \frac{\partial G^i_{lj}}{\partial x^k} - \frac{\partial G^i_{kj}}{\partial x^l} + G^i_{kr}G^r_{lj} - G^i_{lr}G^r_{kj},$$

which satisfies $G^i_{jkl} = -G^i_{jlk}$ (skew-symmetry).

The deviation tensor field P^i_j determines the variational ODEs (Jacobi fields ODEs)

$$\frac{D^2\xi^i}{dt^2} = P^i_j\xi^j,$$

associated to the geometric dynamics (3.4).

In the following, we will state the form of the deviation curvature tensor around a stationary point of geometric dynamics generated by a stationary point of the initial flow:

Corollary 3.1. *Let* $\dfrac{dx^i}{dt} = X^i(x),\ i = \overline{1,n}$ *be a flow and* $x^* = \left(x^{*1}, ..., x^{*n}\right)$ *be a stationary point. Consider the Euler-Lagrange extension (3.4) with the stationary point* $(x^*, \dot{x}^* = 0)$. *Then, the tensor* P *associated to (3.4) has the components*

$$P^i_j(x^*, 0) = g^{ih}(x^*)g_{kl}(x^*)\frac{\partial X^k}{\partial x^h}(x^*)\frac{\partial X^l}{\partial x^j}(x^*)$$

$$+\frac{1}{4}\left(\frac{\partial X^i}{\partial x^r}(x^*) - g^{ih}(x^*)g_{kr}(x^*)\frac{\partial X^k}{\partial x^h}(x^*)\right)\left(\frac{\partial X^r}{\partial x^j}(x^*) - g^{rh}(x^*)g_{kj}(x^*)\frac{\partial X^k}{\partial x^r}(x^*)\right).$$

Proof. We use $X(x^*) = 0$ and $\dfrac{dx}{dt}^* = 0$ and

$$\begin{aligned} F^i_j &= \nabla_j X^i - g^{ih}g_{kj}\nabla_h X^k = \frac{\partial X^i}{\partial x^j} - g^{ih}g_{kj}\frac{\partial X^k}{\partial x^h}, \\ \nabla^i f &= g^{ih}g_{kl}(\nabla_h X^k)X^l = g^{ih}g_{kl}\frac{\partial X^k}{\partial x^h}X^l. \end{aligned}$$

4. GEOMETRIC DYNAMICS LINEARIZATION

4.1. **Linearization around an equilibrium point of the vector field** [16]**.** Suppose $x = 0$ is an equilibrium point of the vector field X on a Riemannian manifold (M, g). By the linearization of the dynamical system $\frac{dx}{dt}(t) = X(x(t))$ around the point $x = 0$ we understand the linear dynamical system $\frac{dx}{dt}(t) = A(0)x$, where $A(x) = \frac{\partial X}{\partial x}(x)$ is the Jacobian matrix. By the linearization of (M, g) around the point $x = 0$ we understand the Euclidean space (T_0M, δ_{ij}). Of course, this involves the constants $g_{ij}(0)$ and a suitable selection of coordinates (normal coordinates) around the point $x = 0$. The diagram of interchanging the linearization process with those of passing to geometric dynamics is commutative. Consequently the linearization of geometric dynamics is

$$\frac{d^2x}{dt^2}(t) = (A - A^T)\frac{dx}{dt}(t) + A^TAx(t),$$

where $A = A(0)$.

4.2. **Linearization around a critical point of the energy** [16]**.** Now let us consider $x = 0$ as a critical point of the energy f which is not a zero of the vector field X. Then the linearization of the Riemannian structure and of the geometric dynamics around this point are

$$g_{ij}(0) = \delta_{ij},\ \frac{d^2x}{dt^2}(t) = (A - A^T)\frac{dx}{dt}(t) + Bx(t),$$

where $B = A^TA + < X(0), \mathrm{Hess}\,(X)(0) >$.

4.3. **Linearization around a trajectory** [16]**.** Let $x(t)$ be a solution of geometric dynamics (3.3). It can be a field line or a trajectory transversal to the field lines. If $x(t)$ is a field line, then the linearization is based on $g_{ij}(t) = g_{ij}(x(t))$, $A(t) = A(x(t))$. If $x(t)$ is not a field line, then the linearization is based on $g_{ij}(t) = g_{ij}(x(t))$, $B(t) = A^T(x(t))A(x(t)) + < X(x(t)), \mathrm{Hess}\,(X)(x(t)) >$.

4.4. **Extended linearization around an equilibrium point of the vector field.** By the linearization of (M, g) around the point $x = 0$ we understand the Euclidean space (R^n, δ_{ij}). We fix the point $x_0 \in R^n$. Each point $x \in R^n \setminus \{x_0\}$ is joined to x_0 by a unique ray $\gamma_{x_0x}(t) = x_0 + t(x - x_0)$ and so R^n is star-shaped.

Let $h : R^n \to R$ be a function of class C^∞. The *extended linearization* of the function h around a point x_0 is

$$h(x) = h(x_0) + \int_0^1 \frac{d}{dt} h(\gamma_{x_0 x}(t))dt$$

$$= h(x_0) + (x^i - x_0^i) \int_0^1 \frac{\partial h}{\partial x^i}(\gamma_{x_0 x}(t))dt = h(x_0) + g_i(x)(x^i - x_0^i),$$

where

$$g_i(x) = \int_0^1 \frac{\partial h}{\partial x^i}(\gamma_{x_0 x}(t))dt,\ g_i(x_0) = \frac{\partial h}{\partial x^i}(x_0).$$

Suppose $x = 0$ is an equilibrium point of the vector field X on the Euclidean space (R^n, δ_{ij}). By the extended linearization of the dynamical system $\frac{dx}{dt}(t) = X(x(t))$ around the point $x = 0$ we understand the extended linear dynamical system

$$\frac{dx}{dt}(t) = A(x(t))x(t),\ \ A(x) = \int_0^1 \frac{\partial X}{\partial x}(\gamma_{0x}(t))dt,\ A(0) = \frac{\partial X}{\partial x}(0).$$

The geometric dynamics associated to the extended linear dynamical system is

$$\frac{d^2x}{dt^2}(t) = \left(A(x(t)) - A^T(x(t)) + \frac{\partial A}{\partial x}(x(t))x(t) - \left(\frac{\partial A}{\partial x}\right)^T (x(t))x(t) \right) \frac{dx}{dt}(t)$$

$$+ \left(\left(\frac{\partial A}{\partial x}\right)^T (x(t))x(t) + A^T(x(t)) \right) A(x(t))x(t).$$

It produces the extended linearization

$$\frac{d^2x}{dt^2}(t) = \left(A(x(t)) - A^T(x(t)) + \frac{\partial A}{\partial x}(x(t))x(t) - \left(\frac{\partial A}{\partial x}\right)^T (x(t))x(t) \right) \frac{dx}{dt}(t)$$

$$+ A^T(x(t))A(x(t))x(t).$$

The diagram of interchanging the extended linearization process with those of passing to geometric dynamics is not commutative.

4.5. **Extended linearization around a critical point of the energy.** Now let us consider $x = 0$ as a critical point of the energy f which is not a zero of the vector field X on (R^n, δ_{ij}). Then the extended linearization of the geometric dynamics (3.3) around this point is

$$\frac{d^2x}{dt^2}(t) = \left(A(x(t)) - A^T(x(t)) + \frac{\partial A}{\partial x}(x(t))x(t) - \left(\frac{\partial A}{\partial x}\right)^T (x(t))x(t) \right) \frac{dx}{dt}(t)$$

$$+ B(x(t))x(t).$$

where $B(x(t)) = A^T(x(t))A(x(t)) + < X(x(t)), \mathrm{Hess}\,(X)(x(t)) >$.

5. JACOBI STABILITY FOR LINEARIZED GEOMETRIC DYNAMICS AROUND A STATIONARY POINT OF THE FIELD

Let $A = \left(\dfrac{\partial X^i}{\partial x^j}(x^*)\right)$ be the Jacobi matrix of the initial flow, let $H = \left(\dfrac{\partial^2 f}{\partial x^i \partial x^j}(x^*)\right)$ be the Hessian of the energy f, let $\Omega = \left(\Omega^i_j(x^*)\right)$ be the matrix of gyroscopic force and $P = \left(P^i_j(x^*, 0)\right)$ be the matrix associated to the deviation curvature tensor. Using the Euclidean metric $g_{ij} = \delta_{ij}$, we have seen in [17] that the matrix of the tensor P associated to (3.4) is

$$P = H + \frac{1}{4}\Omega^2,\ H = A^T A,\ \Omega = A - A^T$$

or

$$P = \frac{1}{4}\left[A^2 + \left(A^T\right)^2 - AA^T + 3A^T A\right].$$

Theorem 5.1. *1) The deviation curvature tensor P is symmetric with respect to the Riemannian metric g. 2) The eigenvalues of deviation curvature tensor P are real numbers. 3) The matrix A^2 is symmetric if and only if $P = (A^T - \frac{1}{2}\Omega^T)(A + \frac{1}{2}\Omega)$.*

Theorem 5.2. *1) If the matrix A is symmetric, then $P = A^T A$ and its eigenvalues are positive numbers. 2) Let $S = A + A^T, \Omega = A - A^T$. If $S\Omega = \Omega S$, then $P = S^2$ and its eigenvalues are positive numbers. 3) If $\mathrm{Tr}\, A^2 = 0$, then at least one of the eigenvalues of P is positive. 4) If the matrix A is skew-symmetric, then $P = 0$. 5) Let λ be an eigenvalue of P and x the associated eigenvector. If $x^T A^T A x < \frac{1}{4} x^T \Omega^T \Omega x$, then $\lambda < 0$.*

Proof. 1) The matrices A and A^T have the same proper values. On the other hand

$$(A - \lambda I)(A^T - \lambda I) = A^T A - \lambda^2 I = P - \lambda^2 I.$$

2) The positivity of eigenvalues is a consequence of $P = S^2 = S^T S$.

3) We can verify easily that $\mathrm{Tr}\, P = \mathrm{Tr}\, A^2$.

5) Our statement is a consequence of the formula

$$\lambda = \frac{x^T P x}{x^T x}.$$

The previous theorem shows that the linearized geometric dynamics around a stationary point of the field can be Jacobi stable or unstable, depending on the matrix A.

6. JACOBI STABILITY FOR LINEARIZED GEOMETRIC DYNAMICS AROUND A CRITICAL POINT OF THE ENERGY

The paper [16] points out that geometric dynamics can be also linearized around a nonzero critical point $x = 0$ of the density of energy f (which is not necessarily a zero of the vector field). In this case, the linearized system has the form

$$\frac{d^2x}{dt^2} = \left(A - A^T\right)\frac{dx}{dt} + \left(A^T A + \langle X(0), Hess(X)(0)\rangle\right)x. \tag{6.1}$$

Following the same steps as in the previous section, we find the following result

Theorem 6.1. *If (5.1) is the linearized geometric dynamics around a critical point of the energy, then the matrix of the tensor P is*

$$P = A^T A + \frac{1}{4}\Omega^2 + H_X,\ H_X = \sum_k X^k(0) Hess(X^k)(0).$$

Theorem 6.2. *1) The deviation curvature tensor P is symmetric with respect to the Riemannian metric g. 2) The eigenvalues of deviation curvature tensor P are real numbers.*

We can state a theorem similar to Theorem 5.2 and consequently, the linearized geometric dynamics around a critical point of the energy can be Jacobi stable or unstable, depending on the matrix A and the matrix H_X.

7. LYAPOUNOV INSTABILITY OF JACOBI FIELDS ODEs ALONG GEOMETRIC DYNAMICS TRAJECTORIES

7.1. **Case of linearized geometric dynamics around a stationary point of the field.** Let

$$P = H + \frac{1}{4}\Omega^2,\ H = A^T A,\ \Omega = A - A^T$$

or

$$P = \frac{1}{4}\left[A^2 + \left(A^T\right)^2 - AA^T + 3A^T A\right].$$

and the second order system $\frac{D^2\xi}{dt^2} = P\xi$, or equivalently

$$\frac{d^2\xi}{dt^2} + 2\left(-A + A^T\right)\frac{d\xi}{dt} + \left[\frac{3}{4}\left(A - A^T\right)^2 - A^T A\right]\xi = 0. \tag{7.1}$$

Being a system with constant coefficients, the solutions of this second order system have the form $\xi = ue^{\lambda t}$, $t \in R$, where u is a nonzero solution of the linear system

$$\left(\lambda^2 I + 2\left(-A + A^T\right)\lambda + \frac{3}{4}\left(A - A^T\right)^2 - A^T A\right)u = 0,$$

and λ is a solution of the equation

$$\mathcal{P}(\lambda) = \det\left(\lambda^2 I + 2(-A + A^T)\lambda + \frac{3}{4}(A - A^T)^2 - A^T A\right) = 0. \qquad (7.2)$$

For $B = 2\left(-A + A^T\right)$ and $C = \frac{3}{4}\left(A - A^T\right)^2 - A^T A$, the previous equation rewrites

$$\det\left(\lambda^2 I + B\lambda + C\right) = 0,$$

where $B = (b_{ij})_{i,j=\overline{1,n}}$, $C = (c_{ij})_{i,j=\overline{1,n}}$ with

$$\begin{cases} b_{ij} + b_{ji} = 0,\ i \neq j;\ b_{ij} = 0,\ i = j \\ c_{ij} - c_{ji} = 0,\ \forall i, j = \overline{1, n}. \end{cases}$$

With the notations above, the polynomial (7.2) is

$$\mathcal{P}(\lambda) = \begin{vmatrix} \lambda^2 + c_{11} & b_{12}\lambda + c_{12} & b_{13}\lambda + c_{13} & ... \\ -b_{12}\lambda + c_{12} & \lambda^2 + c_{22} & b_{23}\lambda + c_{23} & ... \\ -b_{13}\lambda + c_{13} & -b_{23}\lambda + c_{23} & \lambda^2 + c_{33} & ... \\ ... & ... & ... & ... \end{vmatrix}.$$

Because $\mathcal{P}(\lambda) = \mathcal{P}(-\lambda)$, it follows that the polynomial $\mathcal{P}(\lambda)$ has real roots of the form $\pm\lambda$ and/or complex roots of the form $\pm\mu \pm i\gamma$.

Summarizing, we can state the following important result

Theorem 7.1. *In the case of ODEs (7.1), the equation $\mathcal{P}(\lambda) = 0$ has either pairs of real zero-sum eigenvalues, either pairs of complex zero-sum eigenvalues. Consequently, the stationary point $\xi = 0, \frac{d\xi}{dt} = 0$ is a neutral saddle.*

7.2. **Case of linearized geometric dynamics around a critical point of the energy.** Let

$$\mathcal{P} = A^T A + \frac{1}{4}\Omega^2 + H_X,\ H_X = \sum_k X^k(0) Hess(X^k)(0),$$

and the second order system $\frac{D^2\eta}{dt^2} = P\eta$. The solutions of this second order system have the form $\eta = ue^{\lambda t}$, $t \in R$, where u is a nonzero solution of the linear system

$$\left(\lambda^2 I + 2\left(-A + A^T\right)\lambda + \frac{3}{4}\left(A - A^T\right)^2 - A^T A - H_X\right) u = 0, \qquad (7.3)$$

and λ is a solution of the equation

$$\det\left(\lambda^2 I + 2\left(-A + A^T\right)\lambda + \frac{3}{4}\left(A - A^T\right)^2 - A^T A - H_X\right) = 0. \qquad (7.4)$$

Because the matrix $B = 2\left(-A + A^T\right)$ is skew-symmetric and the matrix $C = \frac{3}{4}\left(A - A^T\right)^2 - A^T A - H_X$ is symmetric, the equation (7.4) has real roots of the form $\pm\lambda$ and/or complex roots of the form $\pm\mu \pm i\gamma$.

As a conclusion, we state the following important result

Theorem 7.2. *If (5.1) is the linearized geometric dynamics around a critical point of the energy, then the equation (7.4) has either pairs of real eigenvalues with zero-sum, either pairs of complex eigenvalues with zero-sum. Consequently, the stationary point* $\xi = 0, \dfrac{d\xi}{dt} = 0$ *is a neutral saddle.*

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