

HIGHER DIMENSIONAL COSMOLOGICAL MODEL OF THE UNIVERSE WITH TIME DEPENDENT EQUATION OF STATE

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ABSTRACT. Higher dimensional cosmological model of the universe is obtained by assuming variable cosmological constant term Λ of the form: $\Lambda \sim \frac{\dot{R}^2}{R^2}$ and $\Lambda \sim \rho$ with equation of state $p = f(t)\rho$. Dynamical behaviors of the model for the gamma law equation of state is studied.

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1. INTRODUCTION

Kaluza-Klein (KK) theories have shown how gravity and electromagnetism can be unified from Einstein's field equations generalized to five dimensions. The idea has been later extended to include other type of interaction in $(4+d)$ dimensional modes where the isometries of the spontaneously compactified d spacelike dimensions account for the gauge symmetries of the effective four dimensional (4D) theory (Salam and Strathdee [6]). It is generally known that in KK theories with spontaneous dimensional reduction a very large cosmological constant in four dimensions

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almost always arises, whereas from observational data $\Lambda \sim 10^{-56} cm^{-2}$. However for noncompact internal space one can envision dynamical mechanisms leading to a vanishing 4D cosmological constant (Wetterich [7]). However, it is interesting to point out that geometrically such solutions do not correspond to the topology of direct product of 4D and extra dimensional spaces.

Several workers have recently obtained exact solutions using higher-dimensional space-time for both cosmological and non-cosmological cases with or without matter (Chatterjee et al.[3]; Banerjee et al.[2]; Mayers and Perry, [5]). But exact solutions with a time-dependent equation of state in higher dimensions are not much known in the literature. In a recent communication, (Boutoros [2]) obtained solutions for an LRS Bianchi I model, with a time-dependent equation of state in the form of $P = f(t)$. But their solution have been shown to be untenable for the entire range of the cosmic time because of inappropriate scale transformation adopted by Hajj-Boutros in their formalism.

In this paper we obtained the higher dimensional solution by assuming equation of state of the form $p = f(t)\rho$.

2. MODEL AND FIELD EQUATIONS

For the five dimensional cosmological model considered the following line element

$$(1) \quad ds^2 = dt^2 - R^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + A^2 d\psi^2.$$

The Einstein field equations are given as

$$(2) \quad G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = -8\pi[(p + \rho)\mu_i\mu_j - pg_{ij}],$$

where Λ is cosmological constant, ρ and p are the mass density and isotropic fluid pressure.

By substituting $A = R^n$ and from Eq. (1) and Eq. (2) field equations are given by

$$(3) \quad \frac{8\pi\rho + \Lambda}{3} = (n+1) \left(\frac{\dot{R}}{R} \right)^2,$$

$$(4) \quad 8\pi p - \Lambda = - \left[(n+2) \frac{\ddot{R}}{R} + (n^2 + n + 1) \left(\frac{\dot{R}}{R} \right)^2 \right],$$

$$(5) \quad \frac{8\pi p - \Lambda}{3} = - \left[\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R} \right)^2 \right].$$

Case I : $\Lambda = \alpha \left(\frac{\dot{R}}{R} \right)^2$

Above field equations become

$$(6) \quad \frac{8\pi\rho}{3} = a_0 \left(\frac{\dot{R}}{R} \right)^2,$$

$$(7) \quad \frac{8\pi p}{3} = - \left[\frac{\ddot{R}}{R} + \left(1 - \frac{\alpha}{3} \right) \left(\frac{\dot{R}}{R} \right)^2 \right],$$

$$(8) \quad 8\pi p = - \left[(n+2) \frac{\ddot{R}}{R} + (n^2 + n + 1 - \alpha) \left(\frac{\dot{R}}{R} \right)^2 \right],$$

where $a_0 = (n+1 - \frac{\alpha}{3})$ and the dot "." denotes differentiation with respect to the proper time t , $R(t)$ is the scale factor.

Solve Eq. (6) and Eq. (7) for the following two cases :

$$(i) \quad p = \mu\rho \quad ; \quad \mu \neq \text{constant},$$

$$(ii) \quad p = \mu\rho \quad ; \quad \mu = \text{constant}.$$

Differentiating Eq. (6) with respect to t , we have

$$(9) \quad \frac{\ddot{R}}{R} = \frac{1}{a_0} \left[\frac{8\pi\rho}{3} + \frac{4\pi\dot{\rho}R}{3\dot{R}} \right].$$

Using Eq. (6) and Eq. (9), Eq.(7) can be rewritten as

$$(10) \quad \frac{\dot{R}}{R} = - \frac{\dot{\rho}}{2a_0[p + c_0\rho]},$$

where $c_0 = \frac{2-\frac{\alpha}{3}}{a_0}$.

Using Eq. (10) we can rewrite Eq. (6) as

$$(11) \quad a_0 \left(- \frac{\dot{\rho}}{2a_0[p + c_0\rho]} \right)^2 - \frac{8\pi\rho}{3} = 0.$$

Case (a):

Here we assume a time-dependent equation of state

$$(12) \quad p = f(t)\rho,$$

where $\mu = f(t)$ is an arbitrary known function of t .

Then eq.(11) takes the form

$$(13) \quad \frac{\dot{\rho}}{2a_0\rho[f(t) + c_0]} = -S\sqrt{\frac{8\pi\rho}{3a_0}},$$

where $S = 1$ or -1 .

Using Eq. (12) and Eq. (13) one can rewrite Eq. (10) as

$$(14) \quad \frac{\dot{R}}{R} = +S\sqrt{\frac{8\pi\rho}{3a_0}}.$$

Integrating eq.(13), one obtains

$$(15) \quad \rho = \frac{3}{8\pi a_0[c + \int(f(t) + c_0)dt]^2},$$

where c is the constant of integration.

Using Eq.(15), one can integrate Eq.(14) to get

$$(16) \quad R = de^{\frac{S}{a_0} \int \frac{dt}{c + \int(f(t) + c_0)dt}},$$

where d is constant of integration.

Thus the solution of Eq.(6) and Eq.(7) are given by Eq.(12), Eq.(15) and Eq.(16), where Eq.(15) and Eq.(16) are in integrating form.

We have $\rho > 0$ for $f(t)$ such that

$$\left| \frac{3}{8\pi a_0[c + \int(f(t) + c_0)dt]^2} \right| \geq 1.$$

Again, if we set $f(t) = \mu = \text{constant}$, $S = 1$ and $d = 1$, we obtain

$$(17) \quad R = [c + (\mu + c_0)t]^{\frac{1}{a_0(\mu + c_0)}},$$

$$(18) \quad \rho = \frac{3}{8\pi a_0[c + (\mu + c_0)t]^2}.$$

which is the solution of Eq.(6) and Eq.(7).

Case (b):

Here we study Eq. (6) and Eq. (7) with a gamma law equation of state:

$$(19) \quad p = \gamma\rho, \quad -1 \leq \gamma \leq 1,$$

here $\mu = \gamma$.

One can eliminate ρ from Eq. (6) and Eq. (7) to get

$$(20) \quad (\gamma a_0 + b_0)\dot{R}^2 + \ddot{R}R = 0.$$

where $b_0 = 1 - \frac{\alpha}{3}$.

Integrating Eq.(20) one obtains

$$(21) \quad \dot{R} = S \frac{c_1}{R^{\gamma a_0 + b_0}},$$

where $S = +1$ or -1 and c_1 is integrating constant.

For this case of $c_1 \neq 0$, one can integrate Eq.(21) to get

$$(22) \quad \begin{aligned} R &= [d_0(stc_1 + c_2)]^{\frac{1}{d_0}}, & \text{for } \gamma \neq -1; -\frac{1}{2}, \\ R &= [e_0(stc_1 + c_3)]^{\frac{1}{e_0}}, & \text{for } \gamma = -1, \\ R &= [f_0(stc_1 + c_4)]^{\frac{1}{f_0}}, & \text{for } \gamma = -\frac{1}{2}, \end{aligned}$$

where $d_0 = \gamma a_0 + b_0 + 1$, $e_0 = b_0 - a_0 + 1$, $f_0 = b_0 - \frac{1}{2}a_0 + 1$ and c_2, c_3 and c_4 are integrating constant.

Using Eq.(21) from Eq.(6) one obtains

$$(23) \quad \begin{aligned} \rho &= \frac{3a_0}{8\pi} \frac{c_1^2}{R^{2d_0}}, & \text{for } \gamma \neq -1; -\frac{1}{2}, \\ \rho &= \frac{3a_0}{8\pi} \frac{c_1^2}{R^{2e_0}}, & \text{for } \gamma = -1, \\ \rho &= \frac{3a_0}{8\pi} \frac{c_1^2}{R^{2f_0}}, & \text{for } \gamma = -\frac{1}{2}. \end{aligned}$$

If $c_1 > 0$ for all cases, we have $\rho > 0$, i.e. subject to the all above conditions on the integration constant, one obtains a realistic physical model.

Using Eq.(21), one obtains the model decelerations parameter $q(t) = -\frac{R\ddot{R}}{\dot{R}^2}$ as follows:

$$(24) \quad \begin{aligned} q(t) &= \gamma a_0 + b_0, & \text{for } \gamma \neq -1, -\frac{1}{2}, \\ q(t) &= b_0 - a_0, & \text{for } \gamma = -1, \end{aligned}$$

$$q(t) = b_0 - \frac{1}{2}a_0, \quad \text{for } \gamma = -\frac{1}{2}.$$

To discuss the cosmological evolution of the scale factor $R(t)$, we consider a test particle of mass m and a linear coordinate q^n under the action of potential (Dhara [4])

$$(25) \quad \nu(q) = \lambda q^n - \frac{\Lambda m}{12} q^2.$$

Then the lagrangian of the test particle is given by

$$(26) \quad L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - \lambda q^n + \frac{\Lambda m}{12} q^2,$$

where $\dot{q} = \frac{dq}{dt}$, etc. and the Euler-Lagrange equation is given by

$$(27) \quad m\ddot{q} + \lambda n q^{n-1} - \frac{\Lambda m q}{6} = 0,$$

whereas the conserved total energy of the particle E is given by

$$(28) \quad E = \frac{1}{2}m\dot{q}^2 + \lambda q^n - \frac{\Lambda m}{12} q^2.$$

Using Eq.(28) and Eq.(27) can be rewritten as

$$(29) \quad q\ddot{q} - \frac{n}{2}\dot{q}^2 + \frac{nE}{m} + \frac{\Lambda(n-2)}{12}q^2 = 0.$$

Equations (20) and (29) will describe the same dynamic behaviour only if the following two conditions are satisfied

$$(30) \quad n = -2(\gamma a_0 + b_0), \quad E = 0.$$

Thus n is directly related to the nature of the matter content filling the realistic cosmological models. The relation between the γ , n , the potential energy and the nature of the model are shown in the following table:

TableI

γ	n	nature of the model	$v(q)$
-1	$2(a_0 - b_0)$	vaccum	$\lambda q^{2(a_0 - b_0)} - Z$
$-\frac{1}{3}$	$-\frac{2}{3}(3b_0 - a_0)$	string	$\lambda q^{-\frac{2}{3}(3b_0 - a_0)} - Z$
0	$-2b_0$	dust	$\lambda q^{-2(b_0)} - Z$
$\frac{1}{3}$	$-\frac{2}{3}(a_0 + 3b_0)$	R.D.	$\lambda q^{-\frac{2}{3}(a_0 + 3b_0)} - Z$
1	$-2(a_0 + b_0)$	stiff matter	$\lambda q^{-2(a_0 + b_0)} - Z$

where R.D. are radiation dominated and $Z = \frac{\Lambda m}{12} q^2$.

From the relation Eq.(30) we can conclude that the model behaves relativistic flat.

3. CASE II : $\Lambda = \beta\rho$

The field equations become

$$(31) \quad \frac{(8\pi + \beta)\rho}{3} = (n+1) \left(\frac{\dot{R}}{R} \right)^2,$$

$$(32) \quad \frac{8\pi p - \beta\rho}{3} = - \left[\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R} \right)^2 \right],$$

$$(33) \quad 8\pi p - \beta\rho = - \left[(n+2) \frac{\ddot{R}}{R} + (n^2 + n + 1) \left(\frac{\dot{R}}{R} \right)^2 \right].$$

Differentiate Eq.(31) with respect to t we have

$$(34) \quad \frac{\ddot{R}}{R} = \left[\frac{1}{n+1} \left(\frac{8\pi + \beta}{3} \right) \left(\frac{\dot{\rho} R}{2\dot{R}} + \rho \right) \right].$$

Using Eq.(31) and Eq.(34), Eq.(32) can be rewritten as

$$(35) \quad \frac{\dot{R}}{R} = A_0 \left[- \frac{\dot{\rho}}{(p + B_0\rho)} \right],$$

where $A_0 = \frac{8\pi + \beta}{16\pi(n+1)}$ and $B_0 = \frac{16\pi + (1-n)\beta}{8\pi(n+1)}$.

Using Eq.(35), we can rewrite Eq.(31) as

$$(36) \quad (n+1) \left(\frac{-\dot{\rho} A_0}{p + B_0\rho} \right)^2 - \left(\frac{8\pi + \beta}{3} \right) \rho = 0.$$

Case (c):

Taking Eq.(12) as case (c).

Then Eq.(36) takes the form

$$(37) \quad \frac{\dot{\rho} A_0}{\rho(f(t) + B_0)} = -S \sqrt{\frac{8\pi + \beta}{3(n+1)}} \rho.$$

Using Eq.(12) and Eq.(37), one can rewrite Eq.(35) as

$$(38) \quad \frac{\dot{R}}{R} = +S \sqrt{\frac{8\pi + \beta}{3(n+1)}} \rho.$$

Integrating Eq.(37), one obtains

$$(39) \quad \rho = \frac{12A_0^2(n+1)}{(8\pi + \beta)[c + \int (f(t) + B_0)dt]^2},$$

where c is constant of integration.

Using Eq.(39) one can integrate Eq.(38), to get

$$(40) \quad R = ae^{2SA_0 \int \frac{dt}{c + \int (f(t) + B_0)dt}},$$

where a is the constant of integration.

Thus the solution of Eq.(31) and Eq.(32) are given by eq.(12), Eq.(39) and Eq.(40), where Eq.(39) and Eq.(40) are in integrating form.

We have $\rho > 0$ for $f(t)$ such that

$$\left| \frac{12A_0^2(n+1)}{(8\pi + \beta)[c + \int (f(t) + B_0)dt]^2} \right| \geq 1.$$

Again, if we set $f(t) = \mu$, $S = 1$ and $a = 1$, we obtain

$$(41) \quad R = [c + (\mu + B_0)t]^{\frac{2A_0}{(\mu + B_0)}},$$

$$(42) \quad \rho = \frac{12A_0^2(n+1)}{(8\pi + \beta)[c + (\mu + B_0)t]^2}.$$

Case (d)

Taking Eq.(19) as a case (d).

One can eliminate ρ from Eq.(31) and Eq.(32), to get

$$(43) \quad [\gamma(n+1) + 1 - C_0]\dot{R}^2 + \ddot{R}R = 0$$

where $C_0 = \frac{\beta(n+1)(\gamma+1)}{8\pi + \beta}$

Integrating Eq.(43), one obtains

$$(44) \quad \dot{R} = S \frac{d_1}{R^{\gamma(n+1)+1-C_0}}.$$

For the case of $d_1 \neq 0$, one can integrate Eq.(44) to get

$$R = [E_0(std_1 + d_2)]^{\frac{1}{E_0}}, \quad \text{for } \gamma \neq -1, -\frac{1}{2},$$

$$(45) \quad R = [F_0(std_1 + d_3)]^{\frac{1}{E_0}}, \quad \text{for } \gamma = -1,$$

$$R = [G_0(std_1 + d_4)]^{\frac{1}{G_0}}, \quad \text{for } \gamma = -\frac{1}{2},$$

where $E_0 = \gamma(n+1) + 2 - C_0$, $F_0 = -C_0 - n + 1$, $G_0 = -\frac{n}{2} - C_0 + \frac{3}{2}$ and d_2 , d_3 and d_4 are integrating constant .

Using Eq.(44) from Eq.(31), one obtains

$$\rho = \frac{3(n+1)}{8\pi + \beta} \frac{d_1^2}{R^{2E_0}}, \quad \text{for } \gamma \neq -1, -\frac{1}{2},$$

$$(46) \quad \rho = \frac{3(n+1)}{8\pi + \beta} \frac{d_1^2}{R^{2F_0}}, \quad \text{for } \gamma = -1,$$

$$\rho = \frac{3(n+1)}{8\pi + \beta} \frac{d_1^2}{R^{2G_0}}, \quad \text{for } \gamma = -\frac{1}{2}.$$

If $d_1 > 0$ for all cases we have $\rho > 0$, i.e. subjected to the above conditions on the integration constant, one obtain a realistic physical model.

Using Eq.(44), one obtains the model deceleration parameter $q(t) = -\frac{R\ddot{R}}{\dot{R}^2}$ as follows:

$$q(t) = \gamma(n+1) + 1 - C_0, \quad \text{for } \gamma \neq -1, -\frac{1}{2},$$

$$(47) \quad q(t) = -(n + C_0), \quad \text{for } \gamma = -1,$$

$$q(t) = \frac{1}{2} - \frac{n}{2} - C_0, \quad \text{for } \gamma = -\frac{1}{2}.$$

Equation (43) and (29) will describe the same dynamic behavior only if the following two conditions are satisfied :

$$(48) \quad n = -2(\gamma(n+1) + 1 - C_0), \quad E = 0.$$

Thus n is directly related to the nature of the matter content filling the realistic cosmological models . The relation between the γ , n , the potential energy and the nature of the model are shown in the following table:

Table : II

γ	n	nature of the model	$v(q)$
-1	$2(n + C_0)$	vacuum	$\lambda q^{2(n+C_0)} - Z$
$-\frac{1}{3}$	$-\frac{2}{3}(2 - n - 3C_0)$	string	$\lambda q^{-\frac{2}{3}(2-n-3C_0)} - Z$
0	$-2(1 - C_0)$	dust	$\lambda q^{-2(1-C_0)} - Z$
$\frac{1}{3}$	$-\frac{2}{3}(n + 4 - 3C_0)$	R.D.	$\lambda q^{-\frac{2}{3}(n+4-3C_0)} - Z$
1	$-2(n + 2 - C_0)$	stiff matter	$\lambda q^{-2(n+2-C_0)} - Z$

From the relation Eq.(48) we can conclude that the model behaves relativistic flat.

4. CONCLUSION

This work is generalization of the work obtained earlier by Dhara [4] in the absence of cosmological constant Λ . In this paper we have solved the field equations for Kaluza-Klein type cosmological model of the universe, in the presence of cosmological constant term of the form $\Lambda \sim (\frac{\dot{R}}{R})^2$ and $\Lambda \sim \rho$. The solutions are found for the cases by using equation of state $p = \mu\rho$ where μ is time dependent and constant. The dynamical behaviors of the solutions for the gamma law equation of state are also studied.

For the first case $\Lambda = \alpha(\frac{\dot{R}}{R})^2$, by putting the arbitrary function $f(t)$, we obtained scale factor as in the increasing function of t in Eq.(17). The matter density ρ is calculated in Eq.(18). It is observed that when $t \rightarrow \infty$, the matter density ρ vanishes. In case (b), by assuming equation of state we obtained the matter density ρ in the term of scale factor in Eq.(23). The relation γ , n , potential energy and the nature of the model is given in the table I.

Similarly for case II $\Lambda = \beta\rho$ with equation of state from Eq.(12), we obtained the scale factor and matter density in Eq.(39) and Eq.(40) respectively. For the particular value of $f(t)$ we obtained scale factor and ρ in Eq.(41) and Eq.(42) respectively. Similarly we obtained scale factor as in the increasing function of t and matter density again vanishes as $t \rightarrow \infty$. Similar results are obtained in the case (d) for $p = \gamma\rho$ as the results obtained in case (b). The relation γ , n , potential energy and the nature of the model is given in the table II.

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