

ANALYSIS OF LEMAITRE-TOLMAN-BONDI COSMOLOGICAL MODELS IN A SHORT RANGE COLLAPSE/GENERATION INTERVAL BY A PUISEUX SERIES

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ABSTRACT. The spherically symmetric dust dominated models, usually called Lemaitre-Tolman-Bondi models, are investigated. The final/initial behavior of such a Universe is studied in proximity of collapse/generation times by an expansion in fractional Puiseux series. Our analysis shows that in a short range of collapse/generation interval the spatial initial curvature doesn't influence the evolution: in proximity of these times, a generic Lemaitre-Tolman-Bondi Universe evolves analogously to the initially spatially Euclidean model.

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1. INTRODUCTION

The Lemaitre-Tolman-Bondi (LTB) models are spatially spherically symmetric models where the matter source is a perfect fluid with zero pressure, i.e. dust, usually inhomogeneous in the radial direction [14, 18, 6].¹

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¹ In the homogeneous case LTB-models reduce to the usual Robertson–Walker models.

These simple models, describing dust Universes, such as stellar systems and spherical gravitational collapse (see e.g. [16]), has the advantage of allowing a number of predictions on evolution.

Precisely, in widely accepted cosmological models describing the present scenario of the expanding Universe, space-time curvatures are nonvanishing through the Universe [9, 10, 11, 8]; so, in some recent works we have analyzed the influence of the initial spatial curvature on the evolution of a spatially spherical Universe [1, 2] and we have suggested to analyze its evolution through a suitable formal series, the Puiseux series [3].

So here, generalizing our previous results [3], we will study the evolution of LTB Universes in proximity of the collapse/generation times by a Puiseux series.

More precisely we have shown that although in general the evolution of a spatially spherical dust Universe depends on the initial spatial curvature [6, 1, 2, 4], if we consider an expansion of the exact solutions of evolution equations in fractional Puiseux series ², in proximity of the collapse/generation times, a generic spatially spherical dust Universe evolves analogously to the spatially Euclidean case. This result allows a generalization, in a suitable interval of time, of some of the previous results relative to the spatially Euclidean case [12, 13], to the two spatially non-Euclidean cases.

Hence we stress that the Puiseux series is a very useful computational method that can be largely applied also in a cosmological scenario: providing a way to smoothly represent the solution at singularities, it makes simpler the analysis of the gravitational collapse since through an expansion in this formal series of the solution of the evolution equations we show that it is possible to compare Euclidean and non Euclidean Universes.

The layout of the paper is the following: in the next section we recall the form equations in the case of spherical symmetry and in section 3 we study the behavior of the Universe in proximity of the collapse/generation times.

The first step is to write the exact solutions of the main evolution equation in implicit form. Then, for different signs of principal curvature, to expand in fractional

² A formal series of the form $\sum_{n=m}^{\infty} a_n z^{n/k}$ where m and k are integers such that $k \geq 1$ is called fractional power series or fractional Puiseux series (see e. g. [17, 7]).

Puiseux series the exact implicit solutions and hence analyze them in a short range collapse/generation interval.

2. EVOLUTION EQUATIONS

Since in the following we are going to consider dust Universes with spatial spherical symmetry, we want to briefly summarize the previous main results in a form inspired by [6, 13].

We will consider a dust system C which generates, during its evolution, a Riemannian manifold, which has spatial spherical symmetry around a physical point O (see [15, Chap. XII, §11 p.408]); the metric can then be given the form [15, 6]: ³

$$(1) \quad ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = A^2(t, r) dr^2 + B^2(t, r)(d\theta^2 + \sin^2 \theta d\varphi^2) - c^2 dt^2 ,$$

where t is the proper time of each particle, r, θ, φ are co-moving spherical coordinates and we can interpret $B(t, r)$ as the intrinsic radius, at time t , of the geodesic sphere with its centre at the centre of symmetry O and co-moving radius r . More precisely, at any point $\frac{1}{B^2}$ represents the gaussian curvature of the geodesic sphere with its centre at the centre of symmetry O and passing through the point [15, Chap. XII, §11 p.410].

Hereafter, we will call r -spheres $S(r)$ the geodesic spheres with centre at O and co-moving radius r , and r -shells $\Sigma(r)$ their surfaces (i.e. the dust initially distributed on the surface of the geodesic sphere with center at O and radius r (r -sphere)).

So, in accordance with [13] we assign each particle of an r -shell the initial intrinsic radius $B(0, r)$ as radial co-moving coordinate r :

$$(2) \quad B(0, r) = r .$$

Moreover, the coordinates (r, θ, φ, t) are adapted to the physical frame identified by the field of unit vectors tangent to the geodesic congruence formed by the world lines of the particles [13, 1, 2].

If we put $a(r) = A(0, r)$, the metric of the initial O -sphere V_3 takes the form:

$$(3) \quad d\sigma^2 = \gamma_{ij} dx^i dx^j = a^2(r) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) ,$$

³ In accordance with [15] (but differently from [13]) the latin indices will vary from 1 to 3, whereas the Greek indices will vary from 1 to 4.

where, since the material system $\mathcal{C}$ constitutes an irrotational and geodesics system of reference, $\gamma_{ij} \equiv g_{ij}$ and we can introduce the standard time t as an absolute time [4, 5].

The system of field and conservation equations

$$(4) \quad \begin{aligned} G_{\alpha\beta} &= -\chi T_{\alpha\beta} \\ \nabla_\alpha T_\beta^\alpha &= 0 \end{aligned}$$

is equivalent to the following system of evolution equations [6, 13]⁴

$$(5) \quad \begin{cases} A(t, r) &= a(r) B'(t, r) \\ \dot{B}^2(t, r) &= c^2 \left(\frac{1}{a^2(r)} - 1 \right) + \frac{2 G_N m_e(r)}{B(t, r)} \\ \mu(t, r) &= \frac{\mu_0(r) r^2}{B'(t, r) B^2(t, r)} \end{cases}$$

where $\mu(t, r)$ is the mass density, $\mu_0 = \mu(0, r)$ is the initial mass density, and $m_e(r)$ is the so-called “*Euclidean mass*” [13]

$$(6) \quad m_e(r) = 4\pi \int_0^r \mu_0(s) s^2 ds .$$

Note that if the initial space-like manifold V_3 is Euclidean, $m_e(r)$ is the mass of dust contained within the O -sphere $S(r)$ [13].

Actually, equations (5)₁ and (5)₂ come from Bondi’s original equations, while equation (5)₃ is the equation of conservation of material energy (see [13]).

Remark 1. In [13] it was demonstrated that, as a consequence of evolution equation (5)₂, given a spherical dust Universe, for each material r -shell there exists a corresponding time $T(r)$ at which the dust distributed on the r -shell is collapsed into the symmetry center, so that we have a function $t = T(r)$ which satisfies $B(T(r), r) = 0$.

⁴ Hereafter, a dot will denote differentiation with respect to t and a prime differentiation with respect to r .

3. STUDY OF THE BEHAVIOR OF THE UNIVERSE IN PROXIMITY OF THE COLLAPSE/GENERATION TIMES

Now we want to study the behavior of the Universe in proximity of the collapse/expansion times through an expansion in fractional Puiseux series.

By introducing the two functions [1, 2]

$$(7) \quad \varepsilon(r) = \frac{1}{a^2(r)} - 1 ,$$

that represents the percentage deviation of the metric coefficient $a^2(r)$ from the Euclidean value $a^2 = 1$, and

$$R_s(r) = \frac{2 G_N m_e(r)}{c^2} ,$$

that is formally the Schwarzschild radius associated with the Euclidean mass $m_e(r)$ of the generic r -sphere, we can give equation (5)₂ the simple form

$$(8) \quad \dot{B}^2 = c^2 \left(\varepsilon + \frac{R_s(r)}{B} \right) ,$$

formally analogous to the integral of energy for a body of unit mass, freely falling toward or from an attracting mass m_e fixed in O , in classical mechanics (see [2]).

Now we will focus our attention on a given single r -shell (that is, we will consider r as a temporarily-fixed parameter), so we can regard the intrinsic radius $B = B(t; r)$ as a function of time only, and r , $\varepsilon(r)$, $m_e(r)$, ... as constants.

There are three possible kinds of evolution, depending on the sign of ε .

3.1. Euclidean spatial initial manifold V_3 : $\varepsilon = 0$.

The case $\varepsilon = 0$ corresponds to the spatially Euclidean case $a^2(r) = 1$, already studied in [13]; it is the only case where it is possible to solve (8) explicitly for B . Now, following also [9, 10, 11, 8], we are going to summarize some of the previous results [13], which will be useful in the following.

Since $\varepsilon(r) = 0$, evolution equation (8) takes the simple form

$$(9) \quad \dot{B}^2 = c^2 \frac{R_s(r)}{B} ,$$

whose solution is

$$(10) \quad t - T_e(r) = \pm \frac{2 B^{\frac{3}{2}}}{3 c \sqrt{R_s(r)}} , = \pm \frac{1}{3} \sqrt{\frac{2 B^3}{G_N m_e(r)}} ,$$

where, from remark 1, for each given r -shell, $T_e(r)$ is the time $T(r)$ at which the dust distributed on the r -shell is collapsed into the symmetry center (see [13, 1]) and we have to choose the plus sign if Universe is initially expanding ($\dot{B}(0; r) > 0$), or the minus sign if Universe is initially contracting ($\dot{B}(0; r) < 0$).

We can calculate $T(r)$ through the initial condition $B(0; r) = r$ (see [13]):

$$(11) \quad T_e(r) \equiv T(r) = \pm \frac{2 B^{\frac{3}{2}}}{3 c \sqrt{R_s(r)}} = \mp \frac{1}{3} \sqrt{\frac{2 r^3}{G_N m_e(r)}} .$$

Then we can write the solutions of equations (9) in the form [13]:

$$(12) \quad B(t, r) = r \left(1 - \frac{t}{T_e(r)} \right)^{\frac{2}{3}} .$$

3.2. Non-Euclidean spatial initial manifold V_3 .

Now we are going to show that in proximity of the times of generation or collapse the evolution has the same behavior as the spatially Euclidean case ($\varepsilon = 0$) apart from its initial geometry (which is determined by the sign of ε).

In particular, also in the two non Euclidean cases, the function $T(r)$ will approximately have the same form (11) as in the Euclidean case.

3.2.1. Non-Euclidean spatial initial manifold V_3 with $\varepsilon < 0$.

Since $\varepsilon(r) < 0$, equation (8) becomes

$$(13) \quad \dot{B}^2 = c^2 \left(-|\varepsilon| + \frac{R_s(r)}{B} \right) .$$

By introducing the new function [2]

$$(14) \quad B_M(r) = \frac{R_s(r)}{|\varepsilon(r)|}$$

that represents the maximum value that the intrinsic radius of the given r -shell can reach and that corresponds to the turning point in the analogy with a falling body (see [2]), we can write equation (13) as

$$(15) \quad \dot{B}^2 = c^2 |\varepsilon| \left(\frac{B_M}{B} - 1 \right) .$$

Apart from the factor $c^2 |\varepsilon|$, Eq. (13) is the well-known differential equation of the brachistocrone, hence its general integral is (see [9, 10, 11, 8, 1])

$$(16) \quad t - T_1(r) = \pm \frac{1}{\sqrt{|\varepsilon|} c^2} \left[B_M \arctan \left(\frac{\sqrt{B}}{\sqrt{B_M - B}} \right) - \sqrt{B(B_M - B)} \right] ,$$

where, for each given r -shell, $T_1(r)$ is the time $T(r)$ at which the dust distributed on the r -shell is collapsed into the symmetry center (see remark 1 and [13, 1]) and we have to choose the plus sign if Universe is initially expanding ($\dot{B}(0; r) > 0$) or the minus sign if Universe is initially contracting ($\dot{B}(0; r) < 0$).

We can again calculate $T_1(r)$ through the initial condition $B(0; r) = r$ (see [1])

$$(17) \quad T_1(r) = \mp \frac{1}{\sqrt{|\varepsilon|} c^2} \left[B_M \arctan \left(\sqrt{\frac{r}{B_M - r}} \right) - \sqrt{r(B_M - r)} \right] .$$

Expansion of the solution in fractional Puiseux series.

It is not possible to invert the implicit solution (16) with respect to B , but we can approximate it by an opportune fractional power series (or Puiseux series). Note that it is not possible to expand the second member of (16) in a simple power series with respect to B , but we can expand it in the Mac Lauren series with respect to $\sqrt{B}$ thus obtaining a fractional power series. As it is known, the fractional power series are particular cases of Puiseux series. Precisely we have:

$$(18) \quad \begin{aligned} t - T_1(r) &= \pm \frac{1}{\sqrt{|\varepsilon|} c^2} \left[B_M \arctan \left(\frac{\sqrt{B}}{\sqrt{B_M - B}} \right) - \sqrt{B(B_M - B)} \right] = \\ &= \pm \frac{1}{\sqrt{|\varepsilon|} c^2} \left(\frac{2 B^{\frac{3}{2}}}{3 \sqrt{B_M}} + \frac{B^{\frac{5}{2}}}{5 B_M^{\frac{3}{2}}} + \frac{3 B^{\frac{7}{2}}}{28 B_M^{\frac{5}{2}}} + \dots \right) \end{aligned}$$

Obviously, if $T_1(r)$ is also expanded in fractional Puiseux series, from (17) we obtain

$$(19) \quad \begin{aligned} T_1(r) &= \mp \frac{1}{\sqrt{|\varepsilon|} c^2} \left[B_M \arctan \left(\frac{\sqrt{r}}{\sqrt{B_M - r}} \right) - \sqrt{r(B_M - r)} \right] = \\ &= \mp \frac{1}{\sqrt{|\varepsilon|} c^2} \left(\frac{2 r^{\frac{3}{2}}}{3 \sqrt{B_M}} + \frac{r^{\frac{5}{2}}}{5 B_M^{\frac{3}{2}}} + \frac{3 r^{\frac{7}{2}}}{28 B_M^{\frac{5}{2}}} + \dots \right) \end{aligned}$$

By truncating the fractional series (18) to the first term (with accuracy $3/2$), we find

$$t - T_1(r) = \pm \frac{1}{3} \sqrt{\frac{2 B^3}{G_N m_e(r)}} ,$$

where

$$T_1(r) = \mp \frac{1}{3} \sqrt{\frac{2 r^3}{G_N m_e(r)}} ,$$

which corresponds to the first term of the Puiseux expansion (19).

So in our approximation we found the same expressions (10) and (11) that characterize the case $\varepsilon = 0$.

Remark 2. *In the spatially non-Euclidean case $\varepsilon < 0$, in proximity of the generation or collapse times, each r -shell expands or collapses with the same behavior as in the spatially Euclidean case $\varepsilon = 0$; moreover, the function $T_1(r)$ has, with good accuracy, the form (11).*

3.3. Non-Euclidean spatial initial manifold V_3 with $\varepsilon > 0$.

Since $\varepsilon(r) > 0$, we can write the function $B_M(r)$ (14) in the form

$$B_M(r) = \frac{R_s(r)}{\varepsilon(r)}.$$

Note that in this case $B_M(r)$ does not represent a maximum of the function $B(t; r)$. Moreover, Eq. (8) becomes

$$(20) \quad \dot{B}^2 = \varepsilon c^2 \left(1 + \frac{B_M(r)}{B} \right).$$

The general integral of the non-linear quadratic equation (20) is (see [9, 10, 11, 8, 1])

$$(21) \quad t - T_2(r) = \pm \frac{1}{\sqrt{\varepsilon c^2}} \left[\sqrt{B(B + B_M)} - B_M \operatorname{arsinh} \left(\frac{\sqrt{B}}{\sqrt{B_M}} \right) \right],$$

where, for each given r -shell, $T_2(r)$ is again the time $T(r)$ at which the dust distributed on the r -shell is collapsed into the symmetry center (see remark 1 and [13, 1]) and we have to choose the plus sign if Universe is initially expanding ($\dot{B}(0; r) > 0$) or the minus sign if Universe is initially contracting ($\dot{B}(0; r) < 0$).

We can calculate $T_2(r)$ through the initial condition $B(0; r) = r$ from Eq. (21) (see [1]):

$$(22) \quad T_2(r) = \mp \frac{1}{\sqrt{\varepsilon c^2}} \left[\sqrt{r(r + B_M)} - B_M \operatorname{arsinh} \left(\frac{\sqrt{r}}{\sqrt{B_M}} \right) \right].$$

3.3.1. Expansion of the solutions in fractional Puiseux series.

Also in this case, since it's not possible to solve (21) explicitly with respect to B , we can approximate the exact solution by a Puiseux series:

$$(23) \quad \begin{aligned} t - T_2(r) &= \pm \frac{1}{\sqrt{\varepsilon c^2}} \left[\sqrt{B(B + B_M)} - B_M \operatorname{arsinh} \left(\frac{\sqrt{B}}{\sqrt{B_M}} \right) \right] = \\ &= \pm \frac{1}{\sqrt{\varepsilon c^2}} \left[\frac{2 B^{\frac{3}{2}}}{3 \sqrt{B_M}} - \frac{B^{\frac{5}{2}}}{5 B_M^{\frac{3}{2}}} + \frac{3 B^{\frac{7}{2}}}{28 B_M^{\frac{5}{2}}} + \dots \right] \end{aligned}$$

Obviously, if $T_2(r)$ is also expanded in fractional Puiseux series, from (22) we obtain

$$(24) \quad \begin{aligned} T_2(r) &= \mp \frac{1}{\sqrt{\varepsilon c^2}} \left[\sqrt{r(r+B_M)} - B_M \operatorname{arsinh} \left(\frac{\sqrt{r}}{\sqrt{B_M}} \right) \right] = \\ &= \mp \frac{1}{\sqrt{\varepsilon c^2}} \left[\frac{2r^{\frac{3}{2}}}{3\sqrt{B_M}} - \frac{r^{\frac{5}{2}}}{5B_M^{\frac{3}{2}}} + \frac{3r^{\frac{7}{2}}}{28B_M^{\frac{5}{2}}} + \dots \right] \end{aligned}$$

By truncating the fractional series (23) to the first term (with accuracy 3/2), we find

$$t - T_2(r) = \pm \frac{1}{3} \sqrt{\frac{2B^3}{G_N m_e(r)}},$$

where

$$T_2(r) = \mp \frac{1}{3} \sqrt{\frac{2r^3}{G_N m_e(r)}},$$

which corresponds to the first term of the Puiseux expansion (24).

So in our approximation, we again found the same equations that characterize the case $\varepsilon = 0$.

Remark 3. *Also in the spatially non-Euclidean case $\varepsilon > 0$, in proximity of the generation or collapse times, each r -shell expands or collapses with the same behavior as in the Euclidean case $\varepsilon = 0$; moreover, the function $T_2(r)$ has, with good accuracy, the form (11).*

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