TOPOLOGICAL CONJUGACY OF RELATIVE TOPOLOGICAL DYNAMICS

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Abstract. In this paper the notion of topological conjugacy from the viewpoint of an observer is considered. Relative minimality, relative transitivity and relative expansiveness are studied. It is proved that: these concepts are invariant up to a relative topological conjugate relation.

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1. INTRODUCTION

When a historian writes a historic book, he registers most of events based on his view and meditation. He has embossed some events and give up to mention some events. In fact we also say that, a historian attributes a number between 0 and 1 to each historic event. The events that are corresponding to zero are neglected by the

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historian and the events that are corresponding to one are more outstanding and highlighted from his view point. We can be described the meditation of the historian as an observer to historic events set. If the set of all history events presented by X then a historian assigns a number between 0 and 1 to every historic event. As the matter of fact this number shows the writer's view point in historic events.

The representation of numbers in computer can be also taken into consideration as an observer. We know that numbers in computer are bounded, although this doesn't important limitation because we can put the numbers in the range of representation of numbers in computer by scaling. Moreover in the representation of integer numbers and finite decimal expansion numbers there is no error. The error in a representation occurs when a number has decimal part with infinite expansion. Now without loss of generality we can consider the numbers of the interval $(0, 1)$. We can take the computer as an observer. So that attributes to each number in $(0, 1)$ a value in $[0, 1]$ by:

$$\mu(x) = 1- \mid x - x^* \mid,$$

where $\mid x - x^* \mid = \min\{\mid x - a \mid : a \in \mathbf{A}\}$ and $\mathbf{A}$ is the set of machine numbers. This is the new approach for studying of numerical methods and numerical analysis of computational algorithms.

Also there are many problems in physics, economics, computer science and other branches of sciences, depend on the view point of an observer.

We assume that X is a nonempty set, an observer μ is a function from X to $[0, 1]$ [2, 3, 4, 5]. Moreover we assume that τ_μ is a relative topology [6] it means that: it is a collection of members of $[0, 1]^X$, which are subsets of μ with the following properties :

i) μ, $\chi_\phi \in \tau_\mu$, where χ is the characteristic function;

ii) If λ_1, $\lambda_2 \in \tau_\mu$, then $\lambda_1 \cap \lambda_2 \in \tau_\mu$;

iii) If $\{\lambda_i : i \in \Gamma\} \subset \tau_\mu$, then $\bigcup_{i\in\Gamma} \lambda_i \in \tau_\mu$.

τ_μ is a mathematical modeling of the notion of topology from the viewpoint of an observer μ.

We say $f : X \to X$ be a relative continuous map or a $\mu-$continuous map if $f^{-1}(\eta) \cap \mu \in \tau_\mu$ for all $\eta \in \tau_\mu$, where $f^{-1}(\eta)(x) = \eta(f(x))$. A relative homeomorphism or a $\mu-$homeomorphism f is a bijection mapping such that f and f^{-1} are $\mu-$continuous mappings.

A triple $(X, \mu, \{\varphi^t : t \in T\})$ is called a relative topological dynamics [1] if :
(i) φ^0 is the identity map on X;
(ii) $\varphi^s \circ \varphi^t = \varphi^{s+t}$ for all $s, t \in T$,
where T is the set of real numbers or the set of integer numbers and $\{\varphi^t : t \in T\}$ is a one parameter family of $\mu-$homeomorphisms from X to X.

For given $\alpha \in [0, 1]$ and $\lambda \in \tau_\mu$ the α-cut of λ is the set $\lambda^{-1}(\alpha, 1]$ and denoted by λ_α.

Theorem 1.1. [1] *Let $(X, \mu, \{\varphi^t : t \in T\})$ be a relative topological dynamics from the viewpoint of the observer μ. Moreover assume that μ_α is an invariant set for φ^t, where $\alpha \in (0, 1]$, and $t \in T$. Then $(\mu_\alpha, \{\varphi^t : t \in T\})$ is a topological dynamics, where the topology of μ_α is $\tau_\mu^\alpha = \{\lambda_\alpha : \lambda \in \tau_\mu\}$.*

We are going to consider the notion of topological conjugacy from the viewpoint of an observer, and we will prove that: relative minimality, relative transitivity, and relative expansiveness will preserve under a relative topological conjugate relation.

2. RELATIVE TOPOLOGICAL CONJUGACY

We begin this section with the definition of relative conjugacy for relative topological dynamics.

Definition 2.1. *Two relative topological dynamics $(X, \mu, \{\varphi^t : t \in T\})$ and $(X, \mu, \{\psi^t : t \in T\})$ are called relative topological conjugate if for given $\alpha \in (0, 1]$ there is a $\mu-$homeomorphism $h : X \longrightarrow X$ such that $h(\mu_\alpha) = \mu_\alpha$ and $(h \circ \varphi^t)(x) = (\psi^t \circ h)(x)$ for each $t \in T$ and $x \in \mu_\alpha$ i.e. the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\varphi^t} & X \\ h \downarrow & & \downarrow h \\ X & \xrightarrow{\psi^t} & X \end{array}$$

Remark 2.2. *Relative topological conjugacy is an equivalence relation on the set of all relative topological dynamics on a space X with observer μ.*

If $(X, \mu, \{\varphi^t : t \in T\})$ is a relative topological dynamics and $x \in X$ then the set $O^\varphi(x) = \{\varphi^t(x) : t \in T\}$ is called the orbit of $x \in X$. If $\alpha \in [0, 1]$ then $(O^\varphi_\mu)^\alpha(x) = O^\varphi(x) \cap \mu^{-1}((\alpha, 1])$ is called the relative orbit of x or $\mu-$orbit of x with

the level α.
The straightforward calculation shows that if h is a conjugacy from $(X, \mu, \{\varphi^t : t \in T\})$ to $(X, \mu, \{\psi^t : t \in T\})$ then $h((O^{\varphi}_{\mu})^{\alpha}(x)) = (O^{\psi}_{\mu})^{\alpha}(h(x))$.

Example 2.3. *Let X be the set of mappings x from $\mathbb{Z}$ to $\{0,1\}$, and let $\mu : X \to [0,1]$ be defined by*

$$\mu(x) = \begin{cases} 0 & if\ x(i) = 0\ for\ all\ i \\ \frac{1}{2} & otherwise. \end{cases}$$

Moreover let T be the set of integer numbers and let τ_μ be the relative topology generated by $\{\mu, \lambda_j, \chi_\emptyset : j \in \mathbb{N} \cup \{0\}\}$, where

$$\lambda_j(x) = \begin{cases} \frac{1}{2^{j+1}} & if\ x(j) = 1\ or\ x(-j) = 1 \\ 0 & otherwise. \end{cases}$$

Then for $t \in T$, the mapping $\varphi^t : X \longrightarrow X$ defined by $\varphi^t(x)(i) = x(i+t)$ is a $\mu-$continuous map, and $(X, \mu, \{\varphi^t : t \in T\})$ is a relative topological dynamics. Also if we define $\psi^t : X \longrightarrow X$ by $\psi^t(x)(i) = x(i-t)$ where $t \in T$, then $h : X \longrightarrow X$ defined by $h(x)(i) = x(-i)$ is a relative conjugacy.

Because: *If $\alpha > 1/2$ then $\mu_\alpha = \emptyset$. If $0 < \alpha < 1/2$ then $\mu_\alpha = X - \{0\}$. We have $x \in \mu_\alpha \Leftrightarrow x \neq 0 \Leftrightarrow h(x) \neq h(0) = 0 \Leftrightarrow h(x) \in \mu_\alpha$. So $h(\mu_\alpha) = \mu_\alpha$. Moreover for each $i \in \mathbb{Z}$ we find*

$$(h \circ \varphi^t)(x)(i) = h(\varphi^t(x)(i)) = x(-i-t)$$

$$(\psi^t \circ h)(x)(i) = \psi^t(h(x)(i)) = x(-i-t).$$

Let $\alpha \in (0,1]$ be given. A relative topological dynamics $(X, \mu, \{\varphi^t : t \in T\})$ is called a quasi-minimal relative topological dynamics on μ_α or briefly "$\mu_\alpha-$quasi-minimal " if for every $x \in \mu_\alpha$, $(O^{\varphi}_{\mu})^{\alpha}(x)$ is a dense subset of μ_α, where the topology of μ_α is τ^{α}_{μ}.

Theorem 2.4. *Let $(X, \mu, \{\varphi^t : t \in T\})$ and $(X, \mu, \{\psi^t : t \in T\})$ be relative topological conjugate. Then $(X, \mu, \{\varphi^t : t \in T\})$ is $\mu_\alpha-$quasi-minimal iff $(X, \mu, \{\psi^t : t \in T\})$ is too.*

Proof. Let $\alpha \in (0,1]$ be given and h be a $\mu-$conjugacy i.e. $h(\mu_\alpha) = \mu_\alpha$ and $h \circ \varphi^t = \psi^t \circ h$ for all $t \in T$. Let $x \in \mu_\alpha$ and λ^α be open in μ_α. So $h^{-1}(\lambda^\alpha)$ is open in μ_α and $h^{-1}(x) \in \mu_\alpha$. Since $(X, \mu, \{\varphi^t : t \in T\})$ is quasi-minimal we can find

$t_0 \in T$ such that $\varphi^{t_0}(h^{-1}(x)) \in h^{-1}(\lambda^\alpha)$. Hence $(h \circ \varphi^{t_0} \circ h^{-1})(x) = \psi^{t_0}(x) \in \lambda^\alpha$ i.e. $\{\psi^t(x) : t \in T\}$ is a dense subset of μ_α. The proof of converse is similar. $\square$

Let $(X, \mu, \{\varphi^t : t \in T\})$ be a relative topological dynamics on a metric space X and let $x \in \mu_\alpha$. For given $\alpha \in (0,1]$ we define
$w_{\mu_\alpha}(x) = \{y \in \mu_\alpha \mid \exists \{t_n\} \subset T \textit{ such that } \varphi^{t_n}(x) \to y \textit{ as } t_n \to +\infty\}$.

Let $(X, \mu, \{\varphi^t : t \in T\})$ be a relative topological dynamics and $\alpha \in (0,1]$ be given. A point $p \in \mu_\alpha$ is called a $\mu_\alpha-$non-wandering if for each neighborhood U of p in τ_μ^α, there exists $0 \neq t \in T$ such that $\varphi^t(U) \cap U \cap \mu_\alpha \neq \emptyset$. We denote the set of $\mu_\alpha-$non-wandering points of $(X, \mu, \{\varphi^t : t \in T\})$ by Ω_{μ_α}.

We say that a relative topological dynamics $(X, \mu, \{\varphi^t : t \in T\})$ is relative transitive if for all $\lambda, \eta \in \tau_\mu - \{\chi_\emptyset\}$ there is $t \in T$ such that $\varphi^t(\lambda) \cap \eta \neq \chi_\emptyset$.

Theorem 2.5. *Let $(X, \mu, \{\varphi^t : t \in T\})$ and $(X, \mu, \{\psi^t : t \in T\})$ be relatively topological conjugate with a $\mu-$conjugacy h. Then $(X, \mu, \{\varphi^t : t \in T\})$ is relative transitive iff $(X, \mu, \{\psi^t : t \in T\})$ is too. Moreover $h\Omega^\varphi_{\mu_\alpha} = \Omega^\psi_{\mu_\alpha}$ and if $\varphi^t(x) = x$ for some $t \in T$, then $\psi^t(h(x)) = h(x)$.*

Proof. Let $(X, \mu, \{\varphi^t : t \in T\})$ be relative transitive. Then for given $\alpha \in (0,1]$ and for all $\lambda^\alpha,\ \eta^\alpha \in \tau_\mu^\alpha$, there exists $t \in T$ such that $\varphi^t(\lambda^\alpha) \cap \eta^\alpha \neq \emptyset$. Let h be a $\mu-$conjugacy. So $h(\mu_\alpha) = \mu_\alpha$ and $h \circ \varphi^t = \psi^t \circ h$ for all $t \in T$. Thus $h\varphi^t(\lambda^\alpha) \cap h(\eta^\alpha) \neq \emptyset$. Hence $\psi^t h(\lambda^\alpha) \cap h(\eta^\alpha) \neq \emptyset$ where $h(\lambda^\alpha),\ h(\eta^\alpha) \in \tau_\mu^\alpha$. One can prove the converse similarly.
Suppose $\Omega^\varphi_{\mu_\alpha}$ and $\Omega^\psi_{\mu_\alpha}$ are the sets of $\mu_\alpha-$non-wandering points of $(X, \mu, \{\varphi^t : t \in T\})$ and $(X, \mu, \{\psi^t : t \in T\})$ respectively. Moreover h be a $\mu_\alpha-$conjugacy, and $p \in \Omega^\varphi_{\mu_\alpha}$. Then for given neighborhood U of p there exists $0 \neq t \in T$ such that $\varphi^t(U) \cap U \cap \mu_\alpha \neq \emptyset$. So $h(U)$ is a neighborhood of $h(p)$ and $h(\varphi^t(U) \cap U \cap \mu_\alpha) \neq \emptyset$. Thus $(h \circ \varphi^t)(U) \cap h(U) \cap \mu_\alpha \neq \emptyset$ and $\psi^t(h(U)) \cap h(U) \cap \mu_\alpha \neq \emptyset$. Therefore $h(p) \in \Omega^\psi_{\mu_\alpha}$ when $p \in \Omega^\varphi_{\mu_\alpha}$ i.e. $h(\Omega^\varphi_{\mu_\alpha}) \subseteq \Omega^\psi_{\mu_\alpha}$. By replacing h with h^{-1} we have $\Omega^\psi_{\mu_\alpha} \subseteq h(\Omega^\varphi_{\mu_\alpha})$. If $\varphi^t(x) = x$ for some $t \in T$ and $x \in \mu_\alpha$, then $h(\varphi^t(x)) = h(x)$. So $\psi^t(h(x)) = h(x)$. $\square$

Definition 2.6. *For given $\alpha \in (0,1]$ we say the function $f : X \longrightarrow \mathbb{C}$ is a $\mu_\alpha-$continuous function if $f^{-1}(O) \cap \mu_\alpha \in \tau_\mu^\alpha$, where O is an open subset of the field of complex numbers $\mathbb{C}$.*

Theorem 2.7. *Let $\alpha \in (0,1]$ be given and let $(X, \mu, \{\varphi^t : t \in T\})$ be relative transitive. Moreover let τ_μ^α has a countable base and there exists a $\mu_\alpha-$continuous function f such that $(f \circ \varphi^t)(x) = f(x)$ where $t \in T$ and $x \in \mu_\alpha$. Then f is constant on μ_α.*

Proof. Since $(X, \mu, \{\varphi^t : t \in T\})$ is relative transitive, then by Theorem 4.1 of [1]. There exists $x_0 \in \mu_\alpha$ such that $(O_\mu^\varphi)^\alpha(x_0)$ is dense in μ_α, where $(O_\mu^\varphi)^\alpha(x_0) = \{\varphi^t(x_0) : t \in T\} \cap \mu^{-1}((\alpha, 1])$. So $f((O_\mu^\varphi)^\alpha(x_0)) = f(x_0)$. $\square$

3. RELATIVE EXPANSIVENESS OF THE RELATIVE HOMEOMORPHISMS

We begin this section with the next three assumptions: X is a metric space, $\alpha \in [0,1]$ is fixed and μ is an observer.

Definition 3.1. *Let $\mathcal{A} = \{\lambda_1^\alpha, \lambda_2^\alpha, \cdots, \lambda_n^\alpha\}$ be an open cover for a compact metric space μ_α. Then for a $\mu_\alpha-$homeomorphism $\varphi : X \longrightarrow X$ we define $\bar{\mathcal{A}} = \{\bar\lambda_1^\alpha, \bar\lambda_2^\alpha, \cdots, \bar\lambda_n^\alpha\}$ and*

$$\bigvee_{i=-N}^{N} \varphi^{-i}\bar{\mathcal{A}} = \bar{\mathcal{A}} \bigvee \varphi^{-1}\bar{\mathcal{A}} \bigvee ... \bigvee \varphi^{-N}\bar{\mathcal{A}} \bigvee \varphi\bar{\mathcal{A}} \bigvee ... \bigvee \varphi^{N}\bar{\mathcal{A}}$$

$$= \{\bar\lambda_{i_0}^\alpha \cap \varphi^{-1}\bar\lambda_{i_1}^\alpha \cap ... \cap \varphi^{-N}\bar\lambda_{i_N}^\alpha \cap ... \cap \varphi\bar\lambda_{i_1}^\alpha \cap ... \cap \varphi^{N}\bar\lambda_{i_N}^\alpha : 1 \le i_0, i_1, ..., i_N \le n\}.$$

Definition 3.2. *A finite open cover $\mathcal{A}$ of μ_α is called an $\alpha-$relative strong generator for a $\mu-$homeomorphism $\varphi : X \longrightarrow X$ if :*
(i) $\varphi(\mu_\alpha) = \mu_\alpha$ and,
(ii) there is a metric $d_\alpha : \mu_\alpha \times \mu_\alpha \longrightarrow \mathbb{R}^+ \cup \{0\}$ such that $(\mu_\alpha, \tau_\mu^\alpha)$ is a compact metric space with the metric d_α, and for every bisequence $\{\bar\lambda_n^\alpha\}_{-\infty}^{\infty}$ of members of $\bar{\mathcal{A}}$ the set $\bigcap_{n=-\infty}^{\infty} \varphi^{-n}\bar\lambda_n^\alpha$ contains at most one point of μ_α.

If this condition is replaced by : " $\bigcap_{n=-\infty}^{\infty} \varphi^{-n}\lambda_n^\alpha$ contains at most one point of μ_α" then $\mathcal{A}$ is called a $\alpha-$relative weak generator.

Theorem 3.3. *If $\varphi : X \longrightarrow X$ is a $\mu_\alpha-$homeomorphism of a compact metric space μ_α with $\varphi(\mu_\alpha) = \mu_\alpha$, then φ has an $\alpha-$relative strong generator iff φ has an $\alpha-$relative weak generator.*

Proof. The definitions imply that an $\alpha-$relative strong generator is an $\alpha-$relative weak generator. Let $\mathcal{B}$ be an $\alpha-$relative weak generator for φ, $\mathcal{B} = \{\eta_1^\alpha, ..., \eta_s^\alpha\}$ and let δ be a Lebesgue number for $\mathcal{B}$. Moreover let $\mathcal{A}$ be a finite open cover by sets λ_i^α having $\mathrm{diam}(\bar{\lambda}_i^\alpha) \leq \delta$. Then if $\lambda_{i_n}^\alpha$ is a bisequence in $\mathcal{A}$ then for all n there exists j_n with $\bar{\lambda}_{i_n}^\alpha \subseteq \eta_{j_n}^\alpha$. Hence $\bigcap\limits_{-\infty}^{\infty} \varphi^{-n} \bar{\lambda}_{i_n}^\alpha \subseteq \bigcap\limits_{-\infty}^{\infty} \varphi^{-n} \eta_{j_n}^\alpha$ which is either empty or a single point. So $\mathcal{A}$ is an $\alpha-$relative strong generator. □

Lemma 3.4. *Let $\varphi : X \longrightarrow X$ be a $\mu_\alpha-$homeomorphism of a compact metric space (μ_α, d_α), and let $\mathcal{A}$ be an $\alpha-$relative strong generator for φ. Then for $\epsilon > 0$, there exists $N > 0$ such that each set in $\bigvee_{-N}^{N} \varphi^{-n}\mathcal{A}$ has diameter less than ϵ.*

Proof. Suppose the first part of the Lemma does not hold. Then there exist $\epsilon > 0$ such that for all $j > 0$, there are x_j, y_j with $d_\alpha(x_j, y_j) > \epsilon$ and there exist $\lambda_{j,i}^\alpha \in \mathcal{A}$, $-j \leq i \leq j$ with $x_j, y_j \in \bigcap\limits_{i=-j}^{j} \varphi^{-i} \lambda_{j,i}^\alpha$. Since μ_α is compact then there are subsequences $x_{j_k} \longrightarrow x$ and $y_{j_k} \longrightarrow y$. So $x \neq y$. Infinitely many of $\lambda_{j_k,0}^\alpha$ coincide. Because $\mathcal{A}$ is finite then $x_{j_k}, y_{j_k} \in \lambda_0^\alpha$ for some λ_0^α. Hence $x, y \in \bar{\lambda}_0^\alpha$. Similarly for each n, infinitely many $\lambda_{j_k,n}^\alpha$ coincide and we can find $\lambda_n^\alpha \in \mathcal{A}$ with $x, y \in \varphi^{-n}\bar{\lambda}_n^\alpha$. Thus $x, y \in \cap_{-\infty}^{\infty} \varphi^{-n} \bar{\lambda}_n^\alpha$, and this contradicts with: $\mathcal{A}$ is an $\alpha-$relative strong generator. □

Now, we introduce the definition of the relative expansiveness of a $\mu_\alpha-$homeomorphism.

Definition 3.5. *A $\mu_\alpha-$homeomorphism φ of a compact metric space (μ_α, d_α), with $\varphi(\mu_\alpha) = \mu_\alpha$, is called to be relative expansive or briefly $\mu_\alpha-$expansive if there exists $\delta > 0$ such that for $x, y \in \mu_\alpha$ with $x \neq y$ there exists $n \in \mathbb{Z}$ such that $d_\alpha(\varphi^n x, \varphi^n y) > \delta$. In this case δ is called $\mu_\alpha-$expansive constant for φ.*

Example 3.6. *Let $X_5 = \prod_{n\in\mathbb{Z}}\{1, 2, 3, 4, 5\}$ denote the space of all sequences taking values $\{1, 2, 3, 4, 5\}$ indexed by $\mathbb{Z}$.*

We define a metric on X_5 *by :*

$$d(x,y)=\begin{cases}(1/2)^N & if\ x\neq y\\ 0 & otherwise,\end{cases}$$

where $x=(x_n)_{n\in\mathbb{Z}}$, $y=(y_n)_{n\in\mathbb{Z}}$, *and*

$$N=min\{i\geq 0\ : x_i\neq y_i\ or\ x_{-i}\neq y_{-i}\}.$$

We define $\mu : X_5 \longrightarrow [0,1]$ *by :*

$$\mu(x)=\begin{cases}1/5 & if\ x_i=5\ for\ some\ i\\ 1/2 & otherwise.\end{cases}$$

For $j\in\{1,2,3,4\}$ *we define* $\lambda_j : X_5 \longrightarrow [0,1]$ *by :*

$$\lambda_j(x)=\begin{cases}1/2 & if\ x_i\neq 5\ for\ all\ i\ and\ x_0=j\\ 0 & otherwise.\end{cases}$$

Moreover we define $\lambda_5 : X_5 \longrightarrow [0,1]$ *by :*

$$\lambda_5(x)=\begin{cases}1/5 & if\ x_0=5\\ 0 & otherwise.\end{cases}$$

Let τ_μ *be the relative topology generated by* $\{\mu,\chi_\emptyset,\lambda_j : j\in\{1,2,3,4,5\}\}$. *Then* $\mathcal{A}=\{\lambda_j^{1/4} : j\in\{1,2,3,4\}\}$ *is a* $1/4-$*relative strong generator for a* $\mu-$*homeomorphism shift map* $\sigma : X_5\longrightarrow X_5$ *defined by* $(\sigma x)_n = x_{n+1},\ \forall n\in\mathbb{Z}$.
Because : For every bisequence $\{\lambda_{j_i}^{1/4}\}_{i=-\infty}^{\infty}$ *of members of* $\mathcal{A}$ *the set* $\bigcap\limits_{-\infty}^{\infty}\sigma^{-i}\lambda_{j_i}^{1/4}=\emptyset$, *where* $\mu_\alpha=\{x=(x_n)_{n\in\mathbb{Z}} : x_n\neq 5\ for\ all\ n\in\mathbb{Z}\}$ *and* μ_α *is compact. Moreover* $\mu-$*homeomorphism* σ *of a compact metric space* $(\mu_{1/4},d_{1/4})$ *is relative expansive. Because if we take* $\delta=1/2$, *then for every* $x,y\in\mu_\alpha$ *and* $x\neq y$ *we find* $N\in\mathbb{Z}^+\cup\{0\}$ *with* $d(x,y)=(1/2)^N$.
If $N=0$, *then* $d(x,y)=d(\sigma^0x,\sigma^0y)=(1/2)^0=1>1/2=\delta$.
If $N=k>0$, *then* $d(\sigma^kx,\sigma^ky)=(1/2)^0=1>1/2=\delta$, *or,* $d(\sigma^{-k}x,\sigma^{-k}y)=(1/2)^0=1>1/2=\delta$.
So $\delta=1/2$ *is a* $\mu_{1/4}-$*expansive constant for* σ.

Theorem 3.7. *Let* φ *be a* $\mu_\alpha-$*homeomorphism of a compact metric space* (μ_α,d_α) *with* $\varphi(\mu_\alpha)=\mu_\alpha$. *Then* φ *is* $\mu_\alpha-$*expansive iff* φ *has an* $\alpha-$*relative strong generator.*

Proof. Let δ be a $\mu_\alpha-$expansive constant for φ and $\mathcal{A}$ be a finite cover by open balls of radius $\delta/2$. Suppose $x, y \in \bigcap_{-\infty}^{\infty} \varphi^{-n}\bar{\lambda}_n^\alpha$ where $\lambda_n^\alpha \in \mathcal{A}$. Then $d_\alpha(\varphi^n x, \varphi^n y) \leq \delta$, for all $n \in \mathbb{Z}$. So $x = y$. Because δ is a $\mu_\alpha-$expansive constant for φ. Therefore $\mathcal{A}$ is an $\alpha-$relative strong generator.
Conversely, suppose $\mathcal{A}$ is an $\alpha-$relative strong generator. Let δ be a Lebesgue number for $\mathcal{A}$. If $d_\alpha(\varphi^n x, \varphi^n y) \leq \delta$, for all $n \in \mathbb{Z}$ then for all n, there exists $\lambda_n^\alpha \in \mathcal{A}$ with $\varphi^n x, \varphi^n y \in \lambda_n^\alpha$ and so $x, y \in \bigcap_{-\infty}^{\infty} \varphi^{-n}\bar{\lambda}_n^\alpha$. Since this intersection contains at most one point we have $x = y$. Hence φ is $\mu_\alpha-$expansive. □

Corollary 3.8. (i) *Expansiveness is independent of the metric.*
(ii) *If $k \neq 0$ then φ is $\mu_\alpha-$expansive iff φ^k is $\mu_\alpha-$expansive.*
(iii) *Expansiveness is a relative topological conjugacy invariant i.e. if $\varphi : X \longrightarrow X$ and $\phi : X \longrightarrow X$ are two $\mu_\alpha-$homeomorphisms with relative topology τ_μ, and $\psi : X \longrightarrow X$ is a $\mu_\alpha-$homeomorphism such that $\psi(\mu_\alpha) = \mu_\alpha$ and $\psi\varphi = \phi\psi$ then φ is $\mu_\alpha-$expansive iff ϕ is too.*

Proof. (i) It is clear.
(ii) If $\mathcal{A}$ is an $\alpha-$relative generator for φ then $\mathcal{A}\bigvee\varphi^{-1}\mathcal{A}\bigvee...\bigvee\varphi^{-(k+1)}\mathcal{A}$ is an $\alpha-$relative generator for φ^k. Also any $\alpha-$relative generator for φ^k is an $\alpha-$relative generator for φ. Because if $\mathcal{B}$ is an $\alpha-$relative generator for φ^k, then $\mid \cap_{-\infty}^{\infty}\varphi^{-ik}\eta_j \mid\leq 1$, where $\eta_j \in \mathcal{B}$. Therefore $\mid \cap_{-\infty}^{\infty}\varphi^{-i}\eta_j \mid\leq 1$. So $\mathcal{B}$ is an $\alpha-$relative generator for φ, where $\mid A \mid$ denote the number of elements of the set A.
(iii) A cover $\mathcal{A}$ is an $\alpha-$relative generator for ϕ iff $\psi^{-1}\mathcal{A}$ is an $\alpha-$relative generator for φ. □

Theorem 3.9. *Let $\varphi : X \longrightarrow X$ be a $\mu_\alpha-$expansive homeomorphism of a compact metric space (μ_α, d_α). Then for each integer $m > 0$ the $\mu_\alpha-$homeomorphism φ^m has only a finite number of fixed points in μ_α.*

Proof. Let δ be a $\mu_\alpha-$expansive constant for φ^m. Suppose $\varphi^m(x) = x$ and $\varphi^m(y) = y$ for $x, y \in \mu_\alpha$. Then either $x = y$ or $d_\alpha(\varphi^m x, \varphi^m y) = d_\alpha(x, y) > \delta$. □

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