

THE NATURAL LOGARITHM ON TIME SCALES

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ABSTRACT. We define a logarithm function on time scales and present its main properties.

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1. INTRODUCTION

A time scale, denoted by $\mathbb{T}$, is a nonempty closed subset of the real numbers. The calculus on time scales is a relatively new area that unifies the difference and differential calculus, which are obtained by choosing $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$, respectively. The subject was initiated by Bernd Aulbach and Stefan Hilger in the eighties of the XX century [2], and is now under strong current research in many different fields in which dynamic processes can be described with discrete, continuous, or hybrid models [5].

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We define $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$, where $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M , and $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$ with $\rho(m) = m$ if $\mathbb{T}$ has a minimum m . If $\mathbb{T}$ has a left-scattered maximum b , then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{b\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^\kappa$ if there exists a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of t such that $|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$ for all $s \in U$. We call $f^\Delta(t)$ the delta derivative of f at t and say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. In this case we define the delta integral of f from a to b ($a, b \in \mathbb{T}$) by $\int_a^b f(t) \Delta t := F(b) - F(a)$. For concepts and preliminary results on time scales, we refer the reader to [4, 6].

While in many aspects the theory of time scales is already well established and developed [5], in some other is still in the very beginning. An example of the latter is the lack of a logarithm function [3].

In this short note we introduce the logarithm function on time scales, i.e., a function $L : \mathbb{T} \cap (0, +\infty) \rightarrow \mathbb{R}$ characterized by the equation $L(xy) = L(x) + L(y)$ and such that $L(t) = \ln(t)$ when $\mathbb{T} = \mathbb{R}$.

2. THE DEFINITION

Throughout the text we assume $\mathbb{T}$ to be a time scale with at least two positive points, one of them being always one: $1 \in \mathbb{T}$, there exists at least one $t \in \mathbb{T}$ such that $0 < t \neq 1$. We define the *natural logarithm* function on the time scale $\mathbb{T}$ by

$$(1) \quad L_{\mathbb{T}}(t) := \int_1^t \frac{1}{\tau} \Delta \tau, \quad t \in \mathbb{T} \cap (0, +\infty).$$

We remark that since $f(t) = 1/t$ is a rd-continuous function on $\mathbb{T} \cap (0, +\infty)$ it has a delta antiderivative [4, Theorem 1.74]. Therefore, $L_{\mathbb{T}}(t)$ is well defined. The motivation for our definition (1) is clear: $L_{\mathbb{T}}(\cdot)$ is a delta differentiable function satisfying

$$(2) \quad L_{\mathbb{T}}^\Delta(t) = \frac{1}{t}, \quad L_{\mathbb{T}}(1) = 0, \quad \text{and } L_{\mathbb{R}}(t) = \ln(t).$$

Since $L_{\mathbb{T}}^\Delta(t) > 0$ for every $t \in \mathbb{T}^\kappa \cap (0, +\infty)$, $L_{\mathbb{T}}(\cdot)$ is an increasing and continuous function. Moreover, for $t \in \mathbb{T} \cap (0, +\infty)$, $L_{\mathbb{T}}(t) \leq t - 1$; if $t < 1$, then $L_{\mathbb{T}}(t) < 0$; if $t > 1$, then $L_{\mathbb{T}}(t) > 0$. Directly from the definition we have also that $L_{\mathbb{T}}(\sigma(t)) = L_{\mathbb{T}}(t) + \mu(t)L_{\mathbb{T}}^\Delta(t) = L_{\mathbb{T}}(t) + \frac{\mu(t)}{t}$, where $\mu(t) = \sigma(t) - t$ is the graininess function.

We remark that definition (1) is different from both approaches investigated in [3] (see also Remark 2 below). Although simple and intuitive, our definition (1) was not chosen as the natural one in time scales [3]. According to [3], the most natural definition of logarithm (the first approach of [3]) is

$$L(t) = \int_1^t \frac{1}{\tau + 2\mu(\tau)} \Delta\tau.$$

Differently from both approaches followed in [3], we show here that (1) gives a proper logarithm function on time scales.

3. PROPERTIES

We begin by proving an analogous relation in time scales to the equality

$$\frac{d}{dt} \ln(p(t)) = \frac{p'(t)}{p(t)}.$$

Proposition 1. *Assume $p : \mathbb{T} \rightarrow \mathbb{R}^+$ is strictly increasing and $\tilde{\mathbb{T}} := p(\mathbb{T})$ is a time scale. If $p^\Delta(t)$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(3) \quad \frac{\Delta}{\Delta t} L_{\tilde{\mathbb{T}}}(p(t)) = \frac{p^\Delta(t)}{p(t)}.$$

Proof. For each $t \in \mathbb{T}^\kappa$ the chain rule [4, Theorem 1.93] asserts that

$$(L_{\tilde{\mathbb{T}}} \circ p)^\Delta(t) = \left[(L_{\tilde{\mathbb{T}}} \circ p) p^\Delta \right](t) = L_{\tilde{\mathbb{T}}}^\Delta(p(t)) p^\Delta(t).$$

From (2) we know that

$$L_{\tilde{\mathbb{T}}}^\Delta(p(t)) = \frac{1}{p(t)},$$

and the desired result follows. $\square$

Remark 2. Proposition 1 illustrates well the difference between our definition (1) and the second approach of [3] that defines the logarithm in such a way that the delta derivative of the logarithm coincides with the right-hand side of (3) for any rd-continuous and regressive $p(\cdot)$. Differently from [3], condition (3) holds only for a strictly increasing $p(\cdot)$. This makes a difference with the classical calculus. While the chain rule of classical calculus has no monotonicity assumptions, on time scales we need to require $p(\cdot)$ to be strictly increasing. A chain rule on time scales that does not require $p(\cdot)$ to be strictly increasing is possible, but involves a different notion of derivative on time scales — the so called alpha-derivative [1]. We confine ourselves here to the well established delta calculus on time scales and to the question posed

in [3]. The results of the paper are, however, with the necessary changes, easily formulated for the nabla [5, Chapter 3] or alpha [1] calculus on time scales.

Example 3. Let $\mathbb{T} = \mathbb{N}$ and $p(t) = t^2$. It follows from Proposition 1 that

$$(L_{p(\mathbb{N})} \circ p)^\Delta(t) = \frac{t + \sigma(t)}{t^2} = \frac{2t + 1}{t^2} = L_{\mathbb{N}}^\Delta(t) (2 + L_{\mathbb{N}}^\Delta(t)) .$$

We now state the main property of our function $L_{\mathbb{T}}(\cdot)$. It is Theorem 4 that justify the name *logarithm* to the function defined by (1).

Theorem 4. Let $t = ab \in \mathbb{T}$, $a \in \mathbb{T}$ with $a > 0$, and $b \in \mathbb{R}^+$. Then,

$$(4) \quad L_{\mathbb{T}}(ab) = L_{\mathbb{T}}(a) + L_{\mathbb{T}/a}(b) .$$

Proof. By definition,

$$L_{\mathbb{T}}(ab) = \int_1^{ab} \frac{1}{\tau} \Delta\tau$$

and from basic properties of the delta integral one has

$$L_{\mathbb{T}}(ab) = \int_1^a \frac{1}{\tau} \Delta\tau + \int_a^{ab} \frac{1}{\tau} \Delta\tau = L_{\mathbb{T}}(a) + \int_a^{ab} \frac{a}{\tau} \frac{1}{a} \Delta\tau .$$

Using the substitution rule for delta integrals [4, Theorem 1.98] with $f(\tau) = a/\tau$ and $\nu(\tau) = \tau/a$ (so that $\nu^\Delta(\tau) = 1/a$, $\tilde{\mathbb{T}} = \mathbb{T}/a$, and $\nu^{-1}(s) = as$), we arrive to the intended conclusion:

$$\begin{aligned} L_{\mathbb{T}}(ab) &= L_{\mathbb{T}}(a) + \int_{\nu(a)}^{\nu(ab)} (f \circ \nu^{-1})(s) \tilde{\Delta}s \\ &= L_{\mathbb{T}}(a) + \int_1^b \frac{a}{as} \tilde{\Delta}s \\ &= L_{\mathbb{T}}(a) + L_{\tilde{\mathbb{T}}}(b) . \end{aligned}$$

□

Remark 5. The $b > 0$ in Theorem 4 does not belong necessarily to the time scale $\mathbb{T}$. However, b always belong to $\mathbb{T}/a$ because $ab \in \mathbb{T}$ ($s \in \mathbb{T}/a$ if and only if there exists a $t \in \mathbb{T}$ such that $t = as$).

Remark 6. When $\mathbb{T} = \mathbb{R}$ one has $\mathbb{T}/a = \mathbb{R}$ for any $a > 0$. Then, for any a and b positive, the equality (4) reduces to the classical relation $\ln(ab) = \ln(a) + \ln(b)$.

Corollary 7. *Let $a > 0$, $n \in \mathbb{N}$, and $a^k \in \mathbb{T}$ for $k = 0, 1, \dots, n$. Then,*

$$(5) \quad L_{\mathbb{T}}(a^n) = \sum_{k=0}^{n-1} L_{\mathbb{T}/a^k}(a).$$

Proof. For $n = 1$ we have $L_{\mathbb{T}}(a) = L_{\mathbb{T}/1}(a)$. Let $n \in \mathbb{N}$. Assume that (5) is true for any time scale $\mathbb{T}$ and for n . Then, from Theorem 4, $L_{\mathbb{T}}(a^{n+1}) = L_{\mathbb{T}}(a \cdot a^n) = L_{\mathbb{T}}(a) + L_{\mathbb{T}/a}(a^n)$. Notice that for $k = 0, 1, \dots, n-1$ now $a^k \in \mathbb{T}/a$. Hence, from the inductive assumption for the time scale $\mathbb{T}/a$, we can state that $L_{\mathbb{T}}(a^{n+1}) = L_{\mathbb{T}}(a) + \sum_{k=0}^{n-1} L_{(\mathbb{T}/a)/a^k}(a) = \sum_{k=0}^n L_{\mathbb{T}/a^k}(a)$. $\square$

Remark 8. Let $a > 0$, $n \in \mathbb{N}$. If $\mathbb{T} = \mathbb{R}$, then $a^n \in \mathbb{R}$ and $\mathbb{R}/a^k = \mathbb{R}$. Then, $\ln(a^n) = L_{\mathbb{R}}(a^n) = \sum_{k=0}^{n-1} L_{\mathbb{R}}(a) = n \ln a$.

Corollary 9. *Let $0 < x \in \mathbb{T}$, and $0 < x/y \in \mathbb{T}$. Then,*

$$(6) \quad L_{\mathbb{T}}\left(\frac{x}{y}\right) = L_{\mathbb{T}}(x) - L_{(y\mathbb{T})/x}(y).$$

Proof. Let $z = x/y$. Then, $x = yz$ and by Theorem 4

$$L_{\mathbb{T}}(x) = L_{\mathbb{T}}(yz) = L_{\mathbb{T}}(z) + L_{\mathbb{T}/z}(y),$$

that is, $L_{\mathbb{T}}(z) = L_{\mathbb{T}}(x) - L_{\mathbb{T}/z}(y)$. $\square$

Remark 10. The positive real y in Corollary 9 does not necessarily belong to $\mathbb{T}$. However, $y \in \mathbb{T}/z$ with $z = x/y$.

Remark 11. If $y = x$, then (6) gives $L_{\mathbb{T}}(1) = 0$.

Remark 12. In the particular case $x = 1$ Corollary 9 gives

$$L_{\mathbb{T}}\left(\frac{1}{y}\right) = -L_{y\mathbb{T}}(y).$$

Example 13.

i) Let $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n, n \in \mathbb{N}_0\}$, $q > 1$. Then,

$$L_{\mathbb{T}}(t) = (q-1) \frac{\log t}{\log q}, \quad L_{\mathbb{T}}(\mathbb{T}) = (q-1)\mathbb{N}_0.$$

ii) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$. Then, $L_{\mathbb{T}}(t) = \frac{(q-1)\log t}{\log q}$. Moreover, $L_{\overline{q^{\mathbb{Z}}}}(q^{\mathbb{Z}}) = (q-1)\mathbb{Z}$.

For the particular case $q = 2$, $L_{\mathbb{T}}(\cdot)$ gives a map from $2^{\mathbb{Z}}$ to $\mathbb{Z}$.

iii) For $\mathbb{T} = \mathbb{N}$ the logarithm is a harmonic function (see [4, Example 1.45]):

$$L_{\mathbb{N}}(1) = H_0 = 0, \quad L_{\mathbb{N}}(n) = \sum_{k=1}^{n-1} \frac{1}{k} = H_{n-1}, \quad n > 1,$$

$$L_{\mathbb{N}}(\mathbb{N}) = \{H_n : n \in \mathbb{N}_0\}.$$

4. CONCEPT OF CONVEXITY/CONCAVITY

The fact that functions $L_{\mathbb{T}}(\cdot)$ are continuous and increasing is a direct consequence of the definition. In the standard case $\mathbb{T} = \mathbb{R}$, another important property of the natural logarithm is that it is a concave function. To prove that we have now a similar property, one needs first to define what we mean by convexity and concavity of a function on an interval of a generic time scale $\mathbb{T}$. As before, let $\mathbb{T}$ be a time scale consisting of at least two points.

Definition 14. Let I be an interval in $\mathbb{R}$ such that the set $I_{\mathbb{T}} := I \cap \mathbb{T}$ is a nonempty subset of $\mathbb{T}$. A function f defined and continuous on $I_{\mathbb{T}}$ is called *convex on $I_{\mathbb{T}}$* if for any $t_1, t_2 \in I_{\mathbb{T}}$

$$(7) \quad (t_2 - t)f(t_1) + (t_1 - t_2)f(t) + (t - t_1)f(t_2) \geq 0 \quad \forall t \in I_{\mathbb{T}} \cap [t_1, t_2].$$

Similarly, f is said to be *concave on $I_{\mathbb{T}}$* if (7) holds with ≥ 0 substituted by ≤ 0 .

Remark 15. Note that $I_{\mathbb{T}} \subset \mathbb{T}$ is closed or open, finite or infinite.

Remark 16. For $\mathbb{T} = \mathbb{R}$ Definition 14 agrees with the standard definition of convexity and concavity of a function. For an arbitrary $\mathbb{T}$ it may happen that $I_{\mathbb{T}}$ consists only of one or two points. Then, f is convex and concave on such $I_{\mathbb{T}}$. Let $t_1, t_2 \in I_{\mathbb{T}}$ and $t_1 < t_2$. If $t \in I_{\mathbb{T}}$ and $t_1 \leq t \leq t_2$, then $t = \alpha t_1 + (1 - \alpha)t_2$ with $\alpha = \frac{t_2 - t}{t_2 - t_1}$, $1 - \alpha = \frac{t - t_1}{t_2 - t_1}$. Thus, we define convexity using convex combinations of points from $I_{\mathbb{T}}$. Indeed, then the condition (7) can be rewritten as $f(t) = f(\alpha t_1 + (1 - \alpha)t_2) \leq \alpha f(t_1) + (1 - \alpha)f(t_2)$.

Theorem 17. Let function f be defined on $I_{\mathbb{T}} := I \cap \mathbb{T}$ and Δ -differentiable on $I_{\mathbb{T}}^{\kappa}$. If f^{Δ} is nondecreasing (nonincreasing) on $I_{\mathbb{T}}^{\kappa}$, then f is convex (concave) on $I_{\mathbb{T}}$.

Proof. If $I_{\mathbb{T}}$ consists of less than three points, then f is simultaneously convex and concave. Let us assume that $t \in I_{\mathbb{T}}$ and $t_1 < t < t_2$. Then, the condition (7) can be

rewritten in the following equivalent form:

$$(8) \quad \frac{f(t) - f(t_1)}{t - t_1} \leq \frac{f(t_2) - f(t)}{t_2 - t}.$$

Let us prove this relation. From the mean value theorem [5, Theorem 1.14] we have the existence of points $\tau_1, \xi_1 \in [t_1, t)$ and $\tau_2, \xi_2 \in [t, t_2)$ such that

$$(9) \quad f^\Delta(\tau_1) \leq \frac{f(t) - f(t_1)}{t - t_1} \leq f^\Delta(\xi_1), \text{ and } f^\Delta(\tau_2) \leq \frac{f(t_2) - f(t)}{t_2 - t} \leq f^\Delta(\xi_2).$$

As $t_1 \leq \xi_1 < \tau_2$, inequality (8) holds from the assumption $f^\Delta(\xi_1) \leq f^\Delta(\tau_2)$: $\frac{f(t) - f(t_1)}{t - t_1} \leq f^\Delta(\xi_1) \leq f^\Delta(\tau_2) \leq \frac{f(t_2) - f(t)}{t_2 - t}$. $\square$

Proposition 18. *Let $\mathbb{T}$ be a time scale with at least two positive points, one of them being one: $1 \in \mathbb{T}$, there exists at least one $t \in \mathbb{T}$ such that $0 < t \neq 1$. The natural logarithm $L_{\mathbb{T}}(\cdot)$ is concave on $\mathbb{T} \cap (0, +\infty)$.*

Proof. Since $L_{\mathbb{T}}^\Delta(t) = \frac{1}{t}$ is a decreasing function on $\mathbb{T} \cap (0, +\infty)$, the result follows as a corollary of Theorem 17. $\square$

The following result is a consequence of Theorem 17 and [5, Corollary 1.16]:

Corollary 19. *Let function f be defined and continuous on an interval $I_{\mathbb{T}}$ and let f^{Δ^2} exist finite on $I_{\mathbb{T}}^{\kappa^2}$. Then, f is convex (concave) on $I_{\mathbb{T}}$ if $f^{\Delta^2}(t) \geq 0$ ($f^{\Delta^2}(t) \leq 0$) for all $t \in I_{\mathbb{T}}^{\kappa^2}$.*

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