SUBORDINATION AND SUPERORDINATION RESULTS FOR KOMATU INTEGRAL TRANSFORMS

T.N. SHANMUGAM[1], S. SIVASUBRAMANIAN[2], C. RAMACHANDRAN[3]

[1]DEPARTMENT OF MATHEMATICS
COLLEGE OF ENGINEERING, ANNA UNIVERSITY, CHENNAI-25, INDIA
E-MAIL: DRTNS2001@YAHOO.COM
[2]DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE OF ENGINEERING, ANNA UNIVERSITY-CHENNAI
SARAM-604 307, INDIA
E-MAIL: SIVASAISASTHA@REDIFFMAIL.COM
[3]DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE OF ENGINEERING, ANNA UNIVERSITY-CHENNAI
VILLUPURAM, INDIA
E-MAIL: CRJSP2004@YAHOO.COM

Abstract. Let $\mathcal{A}$ be the class of all normalized analytic functions $f(z)$ in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let q_1 and q_2 be univalent in Δ with $q_1(0) = q_2(0) = 1$. We give some applications of first order differential subordination and superordination to obtain sufficient conditions for a normalized analytic functions f with $f(0) = f'(0) - 1 = 0$ to satisfy

$$q_1(z) \prec \left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \prec q_2(z),$$

where $I_a^{\sigma} f$ is the Komatu integral operator on $\mathcal{A}$.

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1. INTRODUCTION

Let $\mathcal{H}(\Delta)$ be the class of all analytic functions f defined in an open disk of the form $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}(\Delta)$ consisting of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

There are various operators in the literature, related to the study of function theory. O ne such operator, called Komatu operator which generalizes various operators extensively (for example see, Balasubramanian et al [2]) is given by

$$I_a^\sigma f(z) \quad = \quad \frac{(1+a)^\sigma}{z^a \Gamma(\sigma)} \int_0^z \left[log \frac{z}{t}\right]^{\sigma-1} t^{a-1} f(t)dt; \tag{2}$$

$a > 0$, $\sigma > 0$ and $f \in \mathcal{A}$. It can be observed easily that,

$$I_a^\sigma f(z) \quad = \quad z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right)^\sigma a_n z^n.$$

In particular for $a = 1$, this operator have been discussed by Jung, Kim and Srivastava [7] and various results were obtained which are generalized in this paper. More precisely, the properties of

$$I_1^\sigma f(z) = \frac{(1+a)^\sigma}{z\Gamma(\sigma)} \int_0^z \left[log \frac{z}{t}\right]^{\sigma-1} f(t)dt, \sigma > 0, f \in \mathcal{A}, \tag{3}$$

subordinate to certain convex univalent functions were discussed. This operator $I_1^\sigma f$ is closely related to multiplier transformations studied earlier by Flett [6]. It follows from (3) that one can define the operator$I_1^\sigma f$ for any real number σ. The operator $I_a^1 f$ is the Bernardi operator of the convex univalent function $\dfrac{z}{1-z}$ and $I_1^1 f$ was investigated by Libra[8], where $a \in \mathcal{N} = 1, 2, ...$ was studied by Bernardi [3].

Let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{4}$$

then p is a solution of the differential superordination (1.3). (If f is subordinate to F, then F is called a superordinate of f.) An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.3). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.3) is said to be the best subordinant. Recently Miller and Mocanu [9] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{5}$$

Using the results of Miller and Mocanu [9], Bulboaca [4] considered certain classes of first order differential superordinations as well as superordination-preserving operators [3]. Using the
results of [4]. Shanmugam et al. [12] obtained sufficient conditions for a normalized analytic function $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

respectively where, q_1 and q_2 are given univalent functions in Δ. Also, Tuneski [13] obtained a sufficient condition for starlikeness of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. In the present investigation, we obtain sufficient conditions for a normalized analytic function f to satisfy

$$q_1(z) \prec \left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \prec q_2(z).$$

2. Preliminaries

We shall need the following definition and results to prove our main results. In this paper unless otherwise mentioned α, β, γ and δ are complex numbers.

Definition 1. *[[9], Definition 2,p.817] Denote by Q, the set of all functions f that are analytic and injective on $\bar{\Delta} - E(f)$, where*

$$E(f) = \left\{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Theorem 1. *[[9], Theorem 3.4 h, p.132] Let q be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(\omega) \neq 0$ when $\omega \in q(\Delta)$.*

Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that,

(1) $Q(z)$ is starlike univalent in Δ, and
(2) $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \Delta$.

If p is analytic in Δ with $p(\Delta) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{6}$$

then $p \prec q$ and q is the best dominant.

By taking $\theta(\omega) := \alpha + \beta(\omega) + \gamma(\omega)^2$ and $\phi(\omega) := \delta$ in Theorem 1, we get the following.

Lemma 1. *Let q be univalent in Δ with $q(0) = 1$. Further assuming that*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} + \frac{\beta}{\delta} + \frac{2\gamma}{\delta}q(z)\right] > 0.$$

If p is analytic in Δ, with $p(\Delta) \subseteq D$ and

$$\alpha + \beta p(z) + \gamma[p(z)]^2 + \delta zp'(z) \prec \alpha + \beta q(z) + \gamma[q(z)]^2 + \delta zq'(z),$$

then $p \prec q$ and q is the best dominant.

Theorem 2. [5] *Let q be univalent in Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) $\Re\left[\frac{\vartheta'(q(z))}{\varphi(q(z))}\right] > 0$ *for $z \in \Delta$, and*
(2) $Q(z) = zq'(z)\varphi(q(z))$ *is starlike univalent function in Δ.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subset D$, and $\vartheta\{p(z)\} + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta\{q(z)\} + zq'(z)\varphi(q(z)) \prec \vartheta\{p(z)\} + zp'(z)\varphi(p(z)), \tag{7}$$

then $q \prec p$ and q is the best subordinant.

By taking $\vartheta(\omega) := \alpha + \beta\omega + \gamma(\omega)^2$ and $\varphi(\omega) := \delta$ in Theorem 2, we get the following .

Lemma 2. *Let q be univalent in $\Delta, q(0) = 1$. Further assuming that*
$\Re\left[\left(\frac{\beta+2\gamma q(z)}{\delta}\right) q'(z)\right] > 0.$
If $p \in \mathcal{H}[q(0), 1] \cap Q$, and $\alpha + \beta p(z) + \gamma[p(z)]^2 + \delta zp'(z)$ is univalent, and

$$\alpha + \beta q(z) + \gamma[q(z)]^2 + \delta zq'(z) \prec \alpha + \beta p(z) + \gamma[p(z)]^2 + \delta zp'(z),$$

then $q \prec p$ and q is the best subordinant.

3. Subordination results for analytic functions

By using Lemma 1) we first prove the following

Theorem 3. *Let q be univalent in Δ with $q(0)=1$, and satisfying*

$$\Re\left[1+\frac{zq''(z)}{q'(z)}+\frac{\beta}{\delta}+\frac{2\gamma}{\delta}q(z)\right]>0. \tag{8}$$

Let

$$\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z):=\Psi_1(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)+\Psi_2(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z) \tag{9}$$

where $\Psi_1(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)$ and $\Psi_2(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)$ are given by

$$\Psi_1(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z):=\alpha+\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda}\left[\beta+\gamma\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda}\right]$$

$$\Psi_2(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z):=$$

$$\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda}\lambda\delta\left\{(a+1)\frac{I_a^{\sigma}(f(z))}{I_a^{\sigma+1}(f(z))}-a\frac{I_a^{\sigma-1}(f(z))}{I_a^{\sigma}(f(z))}-1\right\}.$$

If $f\in\mathcal{A}$ satisfies

$$\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)\prec\alpha+\beta q(z)+\gamma[q(z)]^2+\delta zq'(z) \tag{10}$$

then,

$$\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda}\prec q(z).$$

and q is the best dominant.

Proof. Define the function $p(z):=\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda}$.
Then a simple computation shows that

$$\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)=\alpha+\beta p(z)+\gamma[p(z)]^2+\delta zp'(z).$$

Now using (10) and Lemma 2. □

Corollary 1. *Let $q(z)=\dfrac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 3.1. Further assuming that (8) holds. If $f\in\mathcal{A}$ satisfies*

$$\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)\prec\alpha+\beta\left(\frac{1+Az}{1+Bz}\right)+\gamma\left(\frac{1+Az}{1+Bz}\right)^2+\frac{\delta(A-B)z}{(1+Bz)^2},$$

then,

$$\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \prec \frac{1+Az}{1+Bz}.$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Also if $q(z) = \frac{1+z}{1-z}$ *, then for* $f \in \mathcal{A}$*, we have*

$$\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z) \prec \alpha + \beta\left(\frac{1+z}{1-z}\right) + \gamma\left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2},$$

$$\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

4. Superordination results for analytic function

Theorem 4. *Let* q *be univalent in* Δ *with* $q(0) = 1$*. If* $f \in \mathcal{A}$ *,* $\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$*, with*

$$\Re\left[\left(\frac{\beta + 2\gamma q(z)}{\delta}\right) q'(z)\right] > 0. \tag{11}$$

and $\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)$ *is univalent in* Δ*, with*

$$\alpha + \beta q(z) + \gamma[q(z)]^2 + \delta z q'(z) \prec \Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z), \tag{12}$$

then, a

$$q(z) \prec \left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda}$$

and q *is the best subordinant.*

Proof. Theorem 4 follows by an application of Lemma 2. □

By taking $q(z) = \frac{1+Az}{1+Bz}$ and $\frac{1+z}{1-z}$ we get the following.

Corollary 2. *Let* q *be univalent in* Δ*. Also let* $f \in \mathcal{A}$ *,* $\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$*. Further assuming that (11) holds. If* $\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)$ *is univalent in* Δ*, and*

$$\alpha + \beta\left(\frac{1+Az}{1+Bz}\right) + \gamma\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\delta(A-B)z}{(1+Bz)^2} \prec \Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z),$$

then,

$$\frac{1+Az}{1+Bz} \prec \left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Let $q(z) = \frac{1+z}{1-z}$, *then for* $f \in \mathcal{A}$, $\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$. *Further assuming that (11) holds. If* $\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)$ *is univalent in* Δ, *and*

$$\alpha + \beta\left(\frac{1+z}{1-z}\right) + \gamma\left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2} \prec \Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z),$$

then,

$$\frac{1+z}{1-z} \prec \left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda}$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

Combining the results of the subordination and superordination we get, the following "Sandwich Theorems".

5. Sandwich Theorems

Theorem 5. *Let* q_1 *and* q_2 *be univalent in* Δ *and satisfies (11) and (8) respectively. If* $f \in \mathcal{A}$, $\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$, *and* $\Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)$ *is univalent in* Δ. *Further if*

$$\alpha+\beta q_1(z)+\gamma[q_1(z)]^2+\delta z q_1'(z) \prec \Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z) \prec \alpha+\beta q_2(z)+\gamma[q_2(z)]^2+\delta z q_2'(z)$$

then

$$q_1(z) \prec \left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \prec q_2(z),$$

and q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}, q_2(z) = \frac{1+A_2 z}{1+B_2 z} (-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1)$, we have the following corollary.

Corollary 3. *If* $f \in \mathcal{A}$, $\left(\frac{I_a^{\sigma+1}(f(z))}{I_a^{\sigma}(f(z))}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$, *and*

$$\Psi_1(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a,A_1,B_1;z) \prec \Psi(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a;z)$$

$$\prec \Psi_2(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a,A_2,B_2;z)$$

then

$$\frac{1+A_1z}{1+B_1z} \prec \left(\frac{I_a^{\sigma+1}(f(z))}{I^\sigma(f(z))}\right)^\lambda \prec \frac{1+A_2z}{1+B_2z}$$

where

$$\Psi_1(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a,A_1,B_1;z) := \alpha+\beta\left(\frac{1+A_1z}{1+B_1z}\right)+\gamma\left(\frac{1+A_1z}{1+B_1z}\right)^2+\frac{\delta(A_1-B_1)z}{(1+B_1z)^2},$$

and

$$\Psi_2(\alpha,\beta,\gamma,\delta,\lambda,\sigma,a,A_2,B_2;z) := \alpha+\beta\left(\frac{1+A_2z}{1+B_2z}\right)+\gamma\left(\frac{1+A_2z}{1+B_2z}\right)^2+\frac{\delta(A_2-B_2)z}{(1+B_2z)^2}.$$

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