SEIBERG-WITTEN LIKE EQUATIONS ON 8-MANIFOLDS WITH STRUCTURE GROUP SPIN(7)

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Abstract. Seiberg-Witten monopole equations are defined on 4-manifolds and solution space of this equations yields differential topological invariants. In this work we consider 8-dimensional Riemannian manifolds with structure group Spin(7). Such manifolds have a distinguished 4-form. By using this 4-form we define self-dual 2-form in dimension eight which is consistent with the well known self-duality notion in 8-dimension given by CDFN (see [5]) . We then express Seiberg-Witten like equations in 8-dimension as analogues of Seiberg-witten equations in 4-dimension.

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1. Introduction

To write Seiberg-Witten equations on a 4-manifolds we need some structures such as Riemannian metric, spin$^c-$structure, spinor bundle, Dirac operator and self-duality notion for a 2-form. To express Seiberg-Witten like equations on 8-dimension we need similar structures for 8-dimensional manifolds.

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It is known that Seiberg-Witten equations in 4-dimension consist of two equations. The first one is harmonicity condition of the spinors and this condition is linear and can be stated in any dimension for $spin^c$ manifolds. The second equation couples the self-dual part of a $2-$form with a spinor field and it is non-linear. There is no natural generalization of the second equation to higher dimensions, because self-duality of $2-$forms in Hodge sense is not meaningful in higher dimensions. On the other hand there are various definitions of the self-duality in higher dimensions (see [2, 4, 5, 6]). The one given in [5] is most popular among physical and mathematical community. There are some analogies of Seiberg-Witten equations in higher dimensions (see [1, 7, 12, 13]). In this work we deal with $8-$dimensional compact manifolds M with structure group $Spin(7)$ that is, the transition maps of the tangent bundle $T(M)$ of M take their values in the group $Spin(7) \subset SO(8)$. If the structure group of M is $Spin(7)$ then M is orientable and *spin*, with preferred spin structure and orientation (see [11]). Recall that $Spin(8)$ is the double cover of $SO(8)$ and an 8-dimensional manifold is called a *spin* one if the structure group $SO(8)$ of M can be lifted to $Spin(8)$ (see [8, 11]). Also if M is *spin* manifold it is automatically a $spin^c$ manifold that is, an 8-dimensional manifold is called a $spin^c$ one if the structure group $SO(8)$ of M can be lifted to the group $Spin^c(8) = (Spin(8) \times S^1)/\mathbb{Z}_2$ (see [8, 11]). Hence there is a vector bundle S on M which is called spinor bundle and we can express Dirac operator on it and we write first Seiberg-Witten like equation on such manifolds. For the second part of the equation we need a kind of self-duality notion of 2-forms. At this point it is important that if the structure group $SO(8)$ of M reduces to the group $Spin(7)$ then there exist a distinguished non-vanishing $4-$form Φ on M which is called fundamental form (see [10]). By using the fundamental form Φ one can split the space $\wedge^2(M)$ into orthogonal subspaces: $\wedge^2(M) = \wedge^2_7 \oplus \wedge^2_{21}$. Now we can define a self-dual 2-form on 8-dimension: If a $2-$form on M belongs to $\wedge^2_7$ then we call it self dual $2-$form.

2. Preliminaries

Let us consider $\mathbb{R}^8$ with its standard orientation and inner product and $e_1, \ldots, e_8$ be an oriented orthonormal basis. We take the following $4-$form on $\mathbb{R}^8$

$$
\begin{aligned}
\Phi_0 &= dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} \\
&+ dx_{5678} + dx_{3478} + dx_{3456} + dx_{2468} - dx_{2457} - dx_{2358} - dx_{2367}
\end{aligned} \tag{1}
$$

where $dx_{ijkl} = dx_i \wedge dx_j \wedge dx_k \wedge dx_l$. The subgroup of $Gl(8, \mathbb{R})$ fixing Φ_0 is isomorphic to the group $Spin(7)$ which is a compact, connected, simply-connected, Lie subgroup of $SO(8)$ of dimension 21 (see [3, 9, 10]).

Since the structure group of M is $Spin(7)$ then there exists a nowhere vanishing differential $4-$form Φ on M which can be expressed as (1) locally such a form is called fundamental $4-$form of the manifold M. This fundamental $4-$form determines a metric g on M and this is referred as the metric induced by Φ.

The action of $Spin(7)$ on the tangent bundle induces an action of $Spin(7)$ on $\wedge^k(M)$. This action gives the following orthogonal decompositions of $\wedge^k(M)$

$$
\begin{aligned}
&\wedge^1(M) = \wedge^1_8, \quad \wedge^2(M) = \wedge^2_7 \oplus \wedge^2_{21}, \quad \wedge^3(M) = \wedge^3_8 \oplus \wedge^3_{48} \\
&\wedge^4(M) = \wedge^4_+(M) \oplus \wedge^4_-(M), \quad \wedge^4_+(M) = \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27}, \quad \wedge^4_-(M) = \wedge^4_{35}
\end{aligned}
$$

where $\wedge^4_\pm(M)$ are the $\pm$ eigenspaces of $*$ on $\wedge^4(M)$ and

$$
\begin{aligned}
\wedge^2_7 &= \left\{\alpha \in \wedge^2(M) : *(\alpha \wedge \Phi) = 3\alpha\right\}, \\
\wedge^2_{21} &= \left\{\alpha \in \wedge^2(M) : *(\alpha \wedge \Phi) = -\alpha\right\}, \\
\wedge^3_8 &= \left\{*(\beta \wedge \Phi) : \beta \in \wedge^1(M)\right\}, \\
\wedge^3_{48} &= \left\{\gamma \in \wedge^3(M) : \gamma \wedge \Phi = 0\right\}, \\
\wedge^4_1 &= \left\{f\Phi : f \in C^\infty(M)\right\},
\end{aligned}
$$

$\wedge^k_l$corresponds to an l dimensional $Spin(7)-$irreducible subspace of $\wedge^k(M)$ (see [10]).

Now we give some explanations about Clifford algebras. Let V be a finite dimensional vector space over the field F and $q : V \to F$ a quadratic form on V. The Clifford algebra $Cl(V, q)$ is an associative algebra with unit 1, which contains and generated by V, with $v \cdot v = q(v) \cdot 1$ for all $v \in V$. Formally, one can define Clifford algebra $Cl(V, q)$ as follows: The Clifford algebra $Cl(V, q)$ associated to a real vector space V with quadratic form q can be defined as

$$Cl(V, q) = T(V)/I(q)$$

where $T(V)$ is the tensor algebra $T(V) = F \oplus V \oplus (V \otimes V) \oplus ...$ and $I(q)$ is the two-sided ideal in $T(V)$ generated by elements $v \otimes v - q(v) \cdot 1$.

Just like the tensor algebra and the exterior algebra the Clifford algebra has the following universal property:

Theorem 2.1. *Given an associative unital F-algebra A (with unit 1) and a linear map $f : V \to A$ with $f(v) \cdot f(v) = q(v) \cdot 1$ for all $v \in V$. Then there is a unique homomorphism of algebras $f : Cl(V,q) \to A$ such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "j"'] & Cl(V,q) \arrow[dl, "f"] \\ A & \end{tikzcd}$$

where i_q is natural inclusion.

In particular the algebra $Cl(V,q)$ together with the map $i_q : V \longrightarrow Cl(V,q)$ satisfying $i_q(v)\Delta i_q(v) = q(v)$ is uniquely determined by this property up to isomorphism (see [8]). For the realization of the Clifford algebra $Cl(V,q)$ following lemma is useful.

Lemma 2.2. *Let $e_1, e_2, ..., e_n$ be a orthogonal basis for V. Then the Clifford algebra $Cl(V,q)$ is multiplicatively generated by the elements $e_1, e_2, ..., e_n \in V \subset Cl(V,q)$ which satisfy the relations*

$$e_i^2 = q(e_i), \qquad e_ie_j + e_je_i = 0, \quad i \neq j.$$

A particular basis of the vector space $Cl(V,q)$ is formed by the elements 1 and $e_{i_1}e_{i_2}...e_{i_k}$, $1 \leq i_1 < ... < i_k \leq n$.

An element $x = \sum\limits_I x_I e_I \in Cl(V,q)$ is said to be even degree if $x_I = 0$ unless $|I|$ is odd. The subset of all elements of even degree is a subalgebra of $Cl(V,q)$ and it is denote by $Cl(V,q)^0$.

Notations 2.3.

- *If $V = \mathbb{R}^n$ real vector space with the quadratic form $q : \mathbb{R}^n \to \mathbb{R}$, $q(x_1, x_2, ..., x_n) = -x_1^2 - x_2^2 - ... - x_n^2$ then the corresponding Clifford algebra*

$Cl(V,q)$ is denoted by Cl_n. If we consider the standard basis $e_1, e_2, ..., e_n$ for $\mathbb{R}^n$ then following relations holds.

$$e_i^2 = -1 \text{ and } e_i e_j = -e_j e_i \text{ for } i \neq j.$$

- *If $V = \mathbb{R}^n$ real vector space with the quadratic form $q : \mathbb{R}^n \to \mathbb{R}$, $q(x_1, x_2, ..., x_n) = x_1^2 + x_2^2 + ... + x_n^2$ then the corresponding Clifford algebra $Cl(V,q)$ is denoted by Cl'_n. In that case the product rule for the standard basis $e_1, e_2, ..., e_n$ of $\mathbb{R}^n$ as follows:*

$$e_i^2 = 1 \text{ and } e_i e_j = -e_j e_i \text{ for } i \neq j.$$

- *If $V = \mathbb{C}^n$ complex vector space with the quadratic form $q : \mathbb{C}^n \to \mathbb{C}$, $q(z_1, z_2, ..., z_n) = z_1^2 + z_2^2 + ... + z_n^2$ then the corresponding Clifford algebra $Cl(V,q)$ is denoted by $\mathbb{C}l_n$ and its known that $\mathbb{C}l_n \cong Cl_n \otimes \mathbb{C}$.*

The classification of the complex Clifford algebras is as follows (see [11]).

Proposition 2.4. *If $n = 2k$ then $\mathbb{C}l_n \cong \mathbb{C}\left(2^k\right)$ and if $n = 2k+1$ then $\mathbb{C}l_n \cong \mathbb{C}\left(2^k\right) \oplus \mathbb{C}\left(2^k\right)$.*

There is a natural involution $Cl_n \longrightarrow Cl_n$, $x \longmapsto \widetilde{x}$ defined by

$$\widetilde{x} = \sum_I \varepsilon_I x_I e_I, \qquad \varepsilon_I = (-1)^{k(k+1)/2}, \qquad k = |I|.$$

Now denote by $Spin(n)$ the set of all elements $x \in Cl_n^0$ which satisfy

$$\widetilde{x}x = 1, \qquad x\mathbb{R}^n\widetilde{x} = \mathbb{R}^n, \qquad \widetilde{x} = x.$$

It is known that $Spin(n)$ is a compact, connected, Lie group. The map $\lambda : Spin(n) \longrightarrow SO(n)$ by $\lambda(x)(v) = xv\widetilde{x}$ is a 2:1 homomorphism for $n \geq 3$ where $v \in \mathbb{R}^n$ (see [8]).

The vector space of complex $n-$spinors is $\triangle_n = \mathbb{C}^{2^k}$ for $n = 2k, 2k+1$ and the elements of $\triangle_n$ are called complex spinors. Since $\mathbb{C}\left(2^k\right) \cong End(\triangle_n)$, $\mathbb{C}l_n \cong End(\triangle_n)$ if n is even; $\mathbb{C}l_n \cong End(\triangle_n) \oplus End(\triangle_n)$ if n is odd. Denote by κ_n the so-called spin representation of the complex Clifford algebra $\mathbb{C}l_n$. In the case of an even dimension, $n = 2k$, this is the isomorphism explained above

$$\kappa_n : \mathbb{C}l_n \longrightarrow End(\triangle_n).$$

If n is odd, then κ_n consists of the isomorphism $\mathbb{C}l_n \cong End(\triangle_n) \oplus End(\triangle_n)$ followed by the projection onto the first component,

$$\kappa_n : \mathbb{C}l_n \longrightarrow End(\triangle_n) \oplus End(\triangle_n) \longrightarrow End(\triangle_n).$$

The vector space of complex $n-$spinors is thus turned into a module over the complex Clifford algebra $\mathbb{C}l_n$. Now we consider the subgroup $Spin(n) \subset Cl_n \subset \mathbb{C}l_n$. We obtain a representation κ of the group $Spin(n)$ by restriction:

$$\kappa = \kappa_n|_{Spin(n)} : Spin(n) \longrightarrow Aut(\triangle_n)$$

This representation is faithful.

A vector $x \in \mathbb{R}^n \subset Cl_n \subset \mathbb{C}l_n \longrightarrow End(\triangle_n)$ can be considered as an endomorphism of $\triangle_n$. This called Clifford multiplication of vectors and spinors, this gives a linear map $\mu : \mathbb{R}^n \otimes_{\mathbb{R}} \triangle_n \longrightarrow \triangle_n$ by $\mu\left(x \otimes \psi\right) = \kappa_n\left(x\right)\psi$. Generally simply write $x \cdot \psi$ instead of $\mu\left(x \otimes \psi\right)$. This Clifford multiplication extends to a homomorphism $\mu : \wedge\left(\mathbb{R}^n\right) \otimes_{\mathbb{R}} \triangle_n \longrightarrow \triangle_n$. Then vectors and multi-vectors can be multiplied by spinors. The product is always a spinor, and this multiplication is equivariant for the actions of the group $Spin(n)$. In the case of $n = 2k$ the element $e_1 \dots e_{2k}$ belongs to the center of the algebra Cl_n^0. As $Spin(n) \subset Cl_n^0$ this element commutes with all elements from $Spin(n)$. Hence the endomorphism

$$f = i^k \kappa\left(e_1 \dots e_{2k}\right) : \triangle_{2k} \longrightarrow \triangle_{2k}$$

is an automorphism of the $Spin(n)$ representation. Thus the spin representation $\triangle_{2k}$ decomposes into the eigensubspaces of f :

$$\triangle_{2k} = \triangle_{2k}^+ \oplus \triangle_{2k}^-, \qquad \triangle_{2k}^\pm = \left\{\psi \in \triangle_{2k} : f\left(\psi\right) = \pm\psi\right\}$$

The spinors belonging to the subspaces $\triangle_{2k}^\pm$ are called Weyl spinors. Moreover if $x \in \mathbb{R}^{2k}$ and $\psi^\pm \in \triangle_{2k}^\pm$, then the spinor $x \cdot \psi^\pm$ belongs to $\triangle_{2k}^\mp$. So the Clifford multiplication induces homomorphisms $\mu : \mathbb{R}^{2k} \otimes_{\mathbb{R}} \triangle_{2k}^\pm \longrightarrow \triangle_{2k}^\mp$.

It is also well known that there exists a positive definite Hermitian inner product $\langle , \rangle$ on the spinor space $\triangle_n$ with the invariance property

$$\langle x \cdot \psi, \varphi\rangle + \langle \psi, x \cdot \varphi\rangle = 0$$

where $x \in \mathbb{R}^n, \psi, \varphi \in \triangle_n$. The spin representation $\kappa : Spin(n) \longrightarrow Aut(\triangle_n)$ is a unitary representation with respect to this inner product.

The complex Clifford algebra $\mathbb{C}l_n$ contains the group

$$Spin^c(n) = \left(Spin(n) \times S^1\right)/\mathbb{Z}_2 = Spin(n) \times_{\mathbb{Z}_2} S^1.$$

The elements of $Spin^c(n)$ are thus classes $[g,z]$ of pairs $(g,z) \in Spin(n) \times S^1$ under the equivalance relation $(g,z) \sim (-g,-z)$. One can define several homomorphisms:

- $\lambda : Spin^c(n) \longrightarrow SO(n)$, $\lambda([g,z]) = \lambda(g)$,
- $i : Spin(n) \longrightarrow Spin^c(n)$, $i(g) = [g,1]$,
- $j : S^1 \longrightarrow Spin^c(n)$, $j(z) = [1,z]$,
- $l : Spin^c(n) \longrightarrow S^1$, $l([g,z]) = z^2$,
- $p : Spin^c(n) \longrightarrow SO(n) \times S^1$, $p([g,z]) = (\lambda(g), z^2)$.

Since $Spin(n)$ is contained in the complex Clifford algebra $\mathbb{C}l_n$, the spin representation of the group $Spin(n)$ extends to a $Spin^c(n)-$representation. For an element $[g,z] \in Spin^c(n)$ and any spinor $\psi \in \triangle_n$ we have $\kappa\,[g,z]\,\psi = z\kappa\,(g)\,(\psi)$.

Let (M,g) be an oriented Riemannian $n-$manifold. There is a unique $SO(n)-$principal bundle Q on M which is induced by the metric g and orientation. Under some conditions on M there is another principal bundle P over M with fiber $Spin(n)$ together with a bundle map $\Lambda : P \longrightarrow Q$. More precisely:

Definition 2.5. A *spin* structure on the principal bundle Q is a pair (P,Λ) where

(1) P is a $Spin(n)$-principal bundle over M,
(2) $\Lambda : P \longrightarrow Q$ is a 2-fold covering for which the diagram

$$\begin{array}{ccc} P \times Spin(n) & \longrightarrow & P \\ \downarrow_{\Lambda\times\lambda} & & \downarrow_{\Lambda} \\ Q \times SO(n) & \longrightarrow & Q \end{array}$$

commutes. In diagram the rows contain the action of the respective group on the corresponding principal bundle.

If a manifold M has a *spin* structure then M is called *spin* manifold. In analogy with a *spin* structure, $spin^c$-structure can be defined as follows:

Definition 2.6. A $spin^c-$structure on Q is a pair (P,Λ) consisting of a $Spin^c-$principal bundle P over the space M and a map $\Lambda : P \longrightarrow Q$ such that the diagram

$$\begin{array}{ccc} P\times Spin^c(n) & \longrightarrow & P \\ \downarrow_{\Lambda\times\lambda} & & \downarrow_{\Lambda} \\ Q\times SO(n) & \longrightarrow & Q \end{array}$$

commutes.

If a manifold M has a $spin^c$ structure then M is called $spin^c$ manifold. It is known that each *spin* structure on Q induces a $spin^c$ structure, so every *spin* manifold is a $spin^c$ manifold (see [8, 11]).

The groups $Spin(n)$ and S^1 are subgroups of $Spin^c(n)$ whose elements commute with each other. If (P,Λ) is a $spin^c-$structure then we get the following principal bundles over M

- P/S^1 is a $SO(n)-$bundle isomorphic to Q, that is $Q=P/S^1$,
- $P_1:=P/Spin(n)$ is an S^1-bundle over M.

Hence we have the fiber-product bundle $Q\widetilde{\times}P_1$ over M and the projection $P\longrightarrow Q\widetilde{\times}P_1$ is a 2-fold covering map.

From the map $l:Spin^c(n)\longrightarrow S^1=U(1)$ we get the associated complex line bundle

$$L=P\times_l\mathbb{C}.$$

This complex line bundle is called determinant line bundle. Note that $L=P_1\times_{U(1)}\mathbb{C}$ and $TM=P\times_\lambda\mathbb{R}^n$.

Consider the $SO(n)-$principal bundle Q on M and denoted by

$$TM=Q\times_{SO(n)}\mathbb{R}^n$$

the associated real $n-$dimensional vector bundle (that is the tangent bundle of M). Let (P,Λ) be a $spin^c-$structure on Q. If we consider the $spin^c$ representation

$$\kappa:Spin^c(n)\longrightarrow U(\triangle_n)$$

then we construct a new associated complex vector bundle

$$S=P\times_\kappa\triangle_n$$

This complex vector bundle is called the spinor bundle for a given $spin^c-$structure on Q. Since $T=Q\times_{SO(n)}\mathbb{R}^n=P\times_\lambda\mathbb{R}^n$, the Clifford multiplication μ induces a bundle morphism of the associated bundles

$$\mu : T \otimes S \longrightarrow S \text{ or } \mu : \Lambda(T) \otimes S \longrightarrow S.$$

Let (M, g) be an oriented connected Riemannian manifold and Q be the $SO(n)$-principal bundle of positively oriented orthonormal frames. The Levi-Civita connection $\bigtriangledown$ over M determines an $so(n)-$valued 1-form $Z : TQ \longrightarrow so(n)$ (see [8]). Let $A : TP_1 \longrightarrow i\mathbb{R}$ be a connection 1$-$form on the principal bundle P_1 where Lie algebra of the group S^1 is identified $i\mathbb{R}$. The connections Z and A together define a connection

$$Z \times A : T\left(Q\widetilde{\times}P_1\right) \longrightarrow so(n) \oplus i\mathbb{R}$$

in the fibre-product $Q\widetilde{\times}P_1$. This connection can be lift to a connection $\widetilde{Z \times A}$ in the $spin^c$ principal bundle via the 2-fold covering $\pi : P \longrightarrow Q\widetilde{\times}P_1$ and following diagram commutes:

$$\begin{array}{ccc} T(P) & \overset{\widetilde{Z\times A}}{\longrightarrow} & spin^c(n) = spin(n) \oplus i\mathbb{R} \\ \downarrow_{d\pi} & & \downarrow_{p_*} \\ T\left(Q\widetilde{\times}P_1\right) & \overset{Z\times A}{\longrightarrow} & so(n) \oplus i\mathbb{R} \end{array}$$

where $p_* : spin^c(n) \longrightarrow so(n) \oplus i\mathbb{R}$ is the differential of the 2-fold covering $p : Spin^c(n) \longrightarrow SO(n) \times S^1$. The $spin(n) \oplus i\mathbb{R}-$valued connection 1-form $\widetilde{Z \times A}$ determines a covariant derivative

$$\nabla^A : \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S)$$

in the spinor bundle. The covariant derivative ∇^A satisfies following properties

(1) $\nabla^A_Y (X\psi) = X\left(\nabla^A_Y \psi\right) + (\nabla_Y X)\psi$ for every vector fields X, Y and $\psi \in \Gamma(S)$,

(2) $X\langle\psi, \psi_1\rangle = \left\langle \nabla^A_X \psi, \psi_1\right\rangle + \left\langle \psi, \nabla^A_X \psi_1\right\rangle$ for every vector field X and $\psi, \psi_1 \in \Gamma(S)$.

Now we can describe the curvature 2-form of the connection $\widetilde{Z \times A}$. Let $\Omega^Z : TQ \times TQ \longrightarrow so(n)$ be the curvature 2-form of the Levi-Civita connection with the components

$$\Omega^Z = \sum_{i<j} \Omega_{ij} E_{ij}, \quad \Omega_{ij} : TQ \times TQ \longrightarrow \mathbb{R}.$$

where the matrices E_{ij} $(i<j)$ defined by

$$E_{ij} = \begin{array}{cc} \begin{matrix} & & i & & & j & & & \end{matrix} & \\ \begin{pmatrix} 0 & \cdots & & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & & & & & -1 & & & 0 \\ \vdots & & & & & & & & \vdots \\ \vdots & & 1 & & & & & & \vdots \\ & & & & & & & & 0 \\ 0 & \cdots & & \cdots & & & 0 & \cdots & 0 \end{pmatrix} & \begin{matrix} \\ i \\ \\ \\ j \\ \\ \end{matrix} \end{array}.$$

The curvature 2-form Ω^A of the connection 1-form A on P_1 is simply $\Omega^A = dA$. It is known that the curvature 2-form Ω^A can be considered as a 2-form on base space M with values in $i\mathbb{R}$ (see [8]).

Let (M, g) be a Riemannian manifold with a fixed *spin*^{c} structure and a connection A in the $U(1)-$principal bundle P_1 and ∇^A be the associated covariant derivative in spinor bundle S.

Definition 2.7. (Dirac operator) The composition

$$D_A := \mu \circ \nabla^A : \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S) = \Gamma(TM \otimes S) \longrightarrow \Gamma(S)$$

is called the Dirac operator.

With respect to a (local) orthonormal frame $e_1, \ldots, e_n$ on the manifold M,

$$D_A\psi = \sum_{i=1}^{n} e_i \nabla^A_{e_i}\psi.$$

This operator is a first order differential operator and elliptic.

We know that if the dimension n is even, then the spinor bundle splits into the sum $S = S^+ \oplus S^-$. Since Clifford multiplication by vector interchanges these summands, the Dirac operator decomposes into the sum of two operators, $D_A^{\pm} : \Gamma(S^{\pm}) \longrightarrow \Gamma(S^{\mp})$.

3. Seiberg-Witten Equations in 8-dimension

3.1. **Equations in 4-dimension.** Let M be an oriented, compact 4-dimensional Riemannian manifold. Its known that every compact, orientable 4-dimensional manifold M is a *Spin*^{c} manifold (see [8]). Fix a *spin*$^c-$structure and a connection

A in the $U(1)-$principal bundle P_1 associated to the $spin^c-$structure. The Seiberg-Witten monopole equations on M are

$$D_A^+\psi = 0, \qquad \Omega_A^+ = \sigma(\psi)$$

for $\psi \in \Gamma(S^+)$, where $\sigma : \Gamma(S^+) \longrightarrow \wedge^{2,+}M$, $\sigma(\psi) = -\frac{1}{4}\sum\limits_{i<j} \langle e_i e_j \psi, \psi\rangle e_i \wedge e_j$ and Ω_A^+ is the self dual part of the curvature 2-form Ω_A. The self-dual part Ω_A^+ of Ω_A can be expressed in terms of Hodge star operator by $\Omega_A^+ = \frac{1}{2}(\Omega_A + *\Omega_A)$. It is also possible to write Ω_A^+ in terms of the basis elements f_1, f_2, f_3 of $\wedge^2(M)$ by $\Omega_A^+ = \sum\limits_{i=1}^{3}\langle f_i, \Omega_A\rangle f_i$ where $f_1 = e_1 \wedge e_2 + e_3 \wedge e_4, f_2 = e_1 \wedge e_3 - e_2 \wedge e_4, f_1 = e_1 \wedge e_4 + e_2 \wedge e_3$. Note that the quadratic map also can be given by the formula $\sigma(\psi) = -\frac{1}{4}\sum\limits_{i=1}^{3}\langle f_i \cdot \psi, \psi\rangle f_i$.

3.2. **Equations in 8-dimension.** Note that the first equation $D_A\psi = 0$ is linear and meaningful for any *spin*^{c} manifolds. But the second one is non-linear and there is no natural generalization to higher dimensions. Because self duality notion of 2-forms in Hodge sense is meaningful only for 4-manifolds. Our aim is to write similar equations on 8-dimensional manifolds with topological $Spin(7)-$structure. Because such manifolds are *spin* so *spin*^{c}, this enables us to construct spinor bundle and Dirac operator on it. On the other hand the fundamental 4-form Φ on M gives decomposition $\wedge_7^2 \oplus \wedge_{21}^2$of 2-forms $\wedge^2(M)$.

Let (M, g, Φ) be a $Spin(7)$ manifold. Then M is *spin* and orientable so M has a $spin^c-$structure and denote it by (P, Λ). Let A be a connection 1-form in the $U(1)-$principal bundle P_1 associated to the *spin*^{c} structure (P, Λ). Since M is a 8-dimensional manifold the spinor bundle S associated to (P, Λ) can be split into the sum $S = S^+ \oplus S^-$. So does Dirac operator and we will consider the positive Dirac operator $D_A^+ : \Gamma(S^+) \longrightarrow \Gamma(S^-)$. The 8-dimensional version of first equation in Seiberg-Witten equations is $D_A^+\psi = 0$ for $\psi \in \Gamma(S^+)$. For second equation we need self-duality notion of a 2-form in 8-dimension. So we want to make use of the decomposition of the space $\wedge^2(M) = \wedge_7^2 \oplus \wedge_{21}^2$. Let $\pi_7 : \wedge^2(M) \to \wedge_7^2$ be the orthogonal projection and for $\eta \in \wedge^2(M)$ the self-dual part of η is, by definition, $\pi_7(\eta)$. If $\pi_7(\eta) = \eta$ then η is called self-dual $2-$form. This self-duality definition of a 2-form is consistent with the generally accepted self-duality definition in 8-dimension of CDFN in [5]. Firstly we write a quadratic map $\sigma : \Gamma(S^+) \longrightarrow \wedge_7^2$ by $\sigma(\psi) = -\sum_{i=1} \frac{\langle f_i\psi,\psi\rangle}{\langle f_i,f_i\rangle} f_i$ where $f_1,\ldots,f_7$ is a basis for $\wedge_7^2$ associated to an orthonormal frame $e_1,\ldots,e_8$ on M. Then the 8-dimensional version of second equation in Seiberg-Witten equations is $\pi_7\left(\Omega^A\right) = \sigma(\psi)$. Hence monopole like equations in 8-dimension are

$$D_A^+\psi = 0, \qquad \pi_7(\Omega_A) = \sigma(\psi).$$

where A is a $i\mathbb{R}-$valued connection 1-form on P_1 and $\psi \in \Gamma(S^+)$.

3.3. **Some Local Discussions.** In this case $n = 8$ then the vector space of complex 8$-$spinors is $\triangle_8 = \mathbb{C}^{16}$. Its known that $\mathbb{C}l_8 \cong End(\triangle_8)$. A spin representation κ_8 of the complex Clifford algebra $\mathbb{C}l_8$ can be given with the following matrices:

$$\kappa_8 : \mathbb{C}l_8 \longrightarrow End(\triangle_8)$$

$$\kappa_8(e_1) = \begin{pmatrix} 0 & K_1 \\ -K_1 & 0 \end{pmatrix} \quad \kappa_8(e_2) = \begin{pmatrix} 0 & K_1 \\ K_1 & 0 \end{pmatrix}$$
$$\kappa_8(e_3) = \begin{pmatrix} 0 & K_2 \\ K_2 & 0 \end{pmatrix} \quad \kappa_8(e_4) = \begin{pmatrix} 0 & K_3 \\ K_3 & 0 \end{pmatrix}$$
$$\kappa_8(e_5) = \begin{pmatrix} 0 & K_4 \\ K_4 & 0 \end{pmatrix} \quad \kappa_8(e_6) = \begin{pmatrix} 0 & K_5 \\ K_5 & 0 \end{pmatrix}$$
$$\kappa_8(e_7) = \begin{pmatrix} 0 & K_6 \\ K_6 & 0 \end{pmatrix} \quad \kappa_8(e_8) = \begin{pmatrix} 0 & K_7 \\ K_7 & 0 \end{pmatrix}$$

where K_1 is 8×8 type identity matrix and the matrices K_i as follows:

$$K_2 = \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix}$$

$$K_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$K_4 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \end{pmatrix}$$

$$K_5 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$K_6 = \begin{pmatrix} 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \end{pmatrix}$$

$$K_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $i = 2, \ldots, 8$ these K_i matrices are the images of the generators of $\mathbb{C}l_7$ under an explicit isomorphism $\mathbb{C}l_7 \cong End\left(\mathbb{C}^8\right) \oplus End\left(\mathbb{C}^8\right)$ which is obtained from the periodicity relation $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}$ (see [8]).

This spinor representation $\kappa_8 : \mathbb{C}l_8 \longrightarrow End(\triangle_8)$ decomposes $\triangle_8 = \triangle_8^+ \oplus \triangle_8^-$ where $\triangle_8^+ = \left\{\psi = (\psi_1, \ldots, \psi_{16}) \in \mathbb{C}^{16} | \psi_9 = \psi_{10} = \ldots = \psi_{16} = 0\right\} \cong \mathbb{C}^8$ and $\triangle_8^- = \left\{\psi = (\psi_1, \ldots, \psi_{16}) \in \mathbb{C}^{16} | \psi_1 = \psi_2 = \ldots = \psi_8 = 0\right\} \cong \mathbb{C}^8$. Now we can write Dirac operator $D_A^+ : \Gamma\left(\triangle_8^+\right) \longrightarrow \Gamma\left(\triangle_8^-\right)$, $D_A^+\psi = \sum\limits_{i=1}^{8} e_i \nabla_{e_i}^A \psi$, $\psi \in \Gamma\left(\triangle_8^+\right)$ and A is a $i\mathbb{R}-$valued connection $1-$form. The $i\mathbb{R}-$valued connection $1-$form A locally can be expressed as $A = \sum\limits_{i=1}^{8} A_i dx_i$, where $A_i : \mathbb{R}^8 \longrightarrow i\mathbb{R}$ smooth functions. Then $\Omega_A = dA = \sum\limits_{i<j} F_{ij} dx_i \wedge dx_j$, where $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$

The covariant derivative $\nabla^A \psi$ of a spinor $\psi \in \Gamma\left(\triangle_8^+\right)$ is computed according to the formula

$$\nabla^A \psi = d\psi + \frac{1}{2} \sum_{i<j} w_{ij} e_i e_j \psi + \frac{1}{2} A\psi.$$

where $w_{ij} = g\left(\nabla e_i, e_j\right)$ are forms defining the Levi-Civita connection. Note that on $\mathbb{R}^8$ all w_{ij}s are vanish. Then the local expression of the covariant derivative $\nabla_{e_i}^A \psi$

of ψ in the direction e_i is

$$\nabla^A_{e_i}\psi = \left(d\psi + \frac{1}{2}A\psi\right)(e_i) = d\psi(e_i) + \frac{1}{2}A\psi(e_i)$$

Since $d\psi(e_i) = \begin{pmatrix} d\psi_1 \\ d\psi_2 \\ \vdots \\ d\psi_8 \end{pmatrix}(e_i) = \begin{pmatrix} d\psi_1(e_i) \\ d\psi_2(e_i) \\ \vdots \\ d\psi_8(e_i) \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi_1}{\partial x_i} \\ \frac{\partial\psi_2}{\partial x_i} \\ \vdots \\ \frac{\partial\psi_8}{\partial x_i} \end{pmatrix}$ and

$$\frac{1}{2}A\psi(e_i) = \frac{1}{2}\sum_{j=1}^{8} A_j dx_j(e_i)\psi = \frac{1}{2}A_i\psi,$$

we can write

$$\nabla^A_{e_i}\psi = \begin{pmatrix} \frac{\partial\psi_1}{\partial x_i} \\ \frac{\partial\psi_2}{\partial x_i} \\ \vdots \\ \frac{\partial\psi_8}{\partial x_i} \end{pmatrix} + \frac{1}{2}A_i \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_8 \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi_1}{\partial x_i} + \frac{1}{2}A_i\psi_1 \\ \frac{\partial\psi_2}{\partial x_i} + \frac{1}{2}A_i\psi_2 \\ \vdots \\ \frac{\partial\psi_8}{\partial x_i} + \frac{1}{2}A_i\psi_8 \end{pmatrix}.$$

According to above data we can write Dirac operator $D^+_A : \Gamma\left(\triangle^+_8\right) \longrightarrow \Gamma\left(\triangle^-_8\right)$ by

$$\begin{aligned} D^+_A\psi &= \sum_{i=1}^{8} e_i \cdot \nabla^A_{e_i}\psi = \sum_{i=1}^{8} \kappa_8(e_i)\left(\nabla^A_{e_i}\psi\right) \\ &= \sum_{i=1}^{8} K_i \begin{pmatrix} \frac{\partial\psi_1}{\partial x_i} + \frac{1}{2}A_i\psi_1 \\ \frac{\partial\psi_2}{\partial x_i} + \frac{1}{2}A_i\psi_2 \\ \vdots \\ \frac{\partial\psi_8}{\partial x_i} + \frac{1}{2}A_i\psi_8 \end{pmatrix} \end{aligned}$$

and the explicit form of the first equation $D^+_A\psi = 0$ is

$$\begin{aligned} &\left(\frac{1}{2}A_1 + \partial_1\right)\psi_1 + i\left(\frac{1}{2}A_2 + \partial_2\right)\psi_1 + \left(\frac{1}{2}A_3 + \partial_3\right)\psi_2 - i\left(\frac{1}{2}A_4 + \partial_4\right)\psi_2 \\ &- \left(\frac{1}{2}A_5 + \partial_5\right)\psi_4 + i\left(\frac{1}{2}A_6 + \partial_6\right)\psi_4 + \left(\frac{1}{2}A_7 + \partial_7\right)\psi_8 + i\left(\frac{1}{2}A_8 + \partial_8\right)\psi_8 = 0 \end{aligned}$$

$$\begin{aligned} &\left(\frac{1}{2}A_1 + \partial_1\right)\psi_2 - i\left(\frac{1}{2}A_2 + \partial_2\right)\psi_2 - \left(\frac{1}{2}A_3 + \partial_3\right)\psi_1 - i\left(\frac{1}{2}A_4 + \partial_4\right)\psi_1 \\ &- \left(\frac{1}{2}A_5 + \partial_5\right)\psi_3 + i\left(\frac{1}{2}A_6 + \partial_6\right)\psi_3 + \left(\frac{1}{2}A_7 + \partial_7\right)\psi_7 + i\left(\frac{1}{2}A_8 + \partial_8\right)\psi_7 = 0 \end{aligned}$$

$$\left(\frac{1}{2}A_1+\partial_1\right)\psi_3+i\left(\frac{1}{2}A_2+\partial_2\right)\psi_3+\left(\frac{1}{2}A_3+\partial_3\right)\psi_4+\left(\frac{1}{2}A_5+\partial_5\right)\psi_2$$
$$+i\left(\frac{1}{2}A_4+\partial_4\right)\psi_4+i\left(\frac{1}{2}A_6+\partial_6\right)\psi_2+\left(\frac{1}{2}A_7+\partial_7\right)\psi_6+i\left(\frac{1}{2}A_8+\partial_8\right)\psi_6=0$$

$$\left(\frac{1}{2}A_1+\partial_1\right)\psi_4-i\left(\frac{1}{2}A_2+\partial_2\right)\psi_4-\left(\frac{1}{2}A_3+\partial_3\right)\psi_3+\left(\frac{1}{2}A_5+\partial_5\right)\psi_1$$
$$+i\left(\frac{1}{2}A_4+\partial_4\right)\psi_3+i\left(\frac{1}{2}A_6+\partial_6\right)\psi_1+\left(\frac{1}{2}A_7+\partial_7\right)\psi_5+i\left(\frac{1}{2}A_8+\partial_8\right)\psi_5=0$$

$$\left(\frac{1}{2}A_1+\partial_1\right)\psi_5+i\left(\frac{1}{2}A_2+\partial_2\right)\psi_5+\left(\frac{1}{2}A_3+\partial_3\right)\psi_6-i\left(\frac{1}{2}A_4+\partial_4\right)\psi_6$$
$$-\left(\frac{1}{2}A_7+\partial_7\right)\psi_4-\left(\frac{1}{2}A_5+\partial_5\right)\psi_8+i\left(\frac{1}{2}A_8+\partial_8\right)\psi_4-i\left(\frac{1}{2}A_6+\partial_6\right)\psi_8=0$$

$$\left(\frac{1}{2}A_1+\partial_1\right)\psi_6-i\left(\frac{1}{2}A_2+\partial_2\right)\psi_6-\left(\frac{1}{2}A_3+\partial_3\right)\psi_5-i\left(\frac{1}{2}A_4+\partial_4\right)\psi_5$$
$$-\left(\frac{1}{2}A_7+\partial_7\right)\psi_3-\left(\frac{1}{2}A_5+\partial_5\right)\psi_7+i\left(\frac{1}{2}A_8+\partial_8\right)\psi_3-i\left(\frac{1}{2}A_6+\partial_6\right)\psi_7=0$$

$$\left(\frac{1}{2}A_1+\partial_1\right)\psi_7+i\left(\frac{1}{2}A_2+\partial_2\right)\psi_7+\left(\frac{1}{2}A_3+\partial_3\right)\psi_8-\left(\frac{1}{2}A_7+\partial_7\right)\psi_2$$
$$+\left(\frac{1}{2}A_5+\partial_5\right)\psi_6+i\left(\frac{1}{2}A_4+\partial_4\right)\psi_8+i\left(\frac{1}{2}A_8+\partial_8\right)\psi_2-i\left(\frac{1}{2}A_6+\partial_6\right)\psi_6=0$$

$$\left(\frac{1}{2}A_1+\partial_1\right)\psi_8-i\left(\frac{1}{2}A_2+\partial_2\right)\psi_8-\left(\frac{1}{2}A_3+\partial_3\right)\psi_7-\left(\frac{1}{2}A_7+\partial_7\right)\psi_1$$
$$+\left(\frac{1}{2}A_5+\partial_5\right)\psi_5+i\left(\frac{1}{2}A_4+\partial_4\right)\psi_7+i\left(\frac{1}{2}A_8+\partial_8\right)\psi_1-i\left(\frac{1}{2}A_6+\partial_6\right)\psi_5=0$$

Now we deal with the second equation $\pi_7(\Omega_A)=\sigma(\psi)$. Firstly let us consider following basis $f_1,\ldots,f_7$ for $\wedge_7^2$ associated to the orthonormal frame $e_1,\cdots,e_8$ of

$\mathbb{R}^8$

$$\begin{aligned} f_1 &= dx_{12}+dx_{34}+dx_{56}+dx_{78} \\ f_2 &= dx_{13}-dx_{24}+dx_{57}-dx_{68} \\ f_3 &= dx_{14}+dx_{23}-dx_{58}-dx_{67} \\ f_4 &= dx_{15}-dx_{26}-dx_{37}+dx_{48} \\ f_5 &= dx_{16}+dx_{25}+dx_{38}+dx_{47} \\ f_6 &= dx_{17}-dx_{28}+dx_{35}-dx_{46} \\ f_7 &= -dx_{18}-dx_{27}+dx_{36}+dx_{45}. \end{aligned}$$

Since $\langle f_i, f_i\rangle = 4$ the orthogonal projection $\pi_7 : \wedge^2\left(\mathbb{R}^8\right) \to \wedge^2_7$ can be written as $\pi_7\left(\Omega_A\right) = \sum\limits_{i=1}^{8} \frac{\langle \Omega_A, f_i\rangle}{\langle f_i, f_i\rangle} f_i = \frac{1}{4}\sum\limits_{i=1}^{8}\langle \Omega_A, f_i\rangle f_i$ and the quadratic map $\sigma(\psi) = -\sum\limits_{i=1}^{8} \frac{\langle f_i\cdot\psi,\psi\rangle}{\langle f_i, f_i\rangle} f_i = -\frac{1}{4}\sum\limits_{i=1}^{8}\langle f_i\cdot\psi,\psi\rangle f_i$. Then the equation $\pi_7\left(\Omega_A\right) = \sigma(\psi)$ turns to the following set of equations

$$\langle \Omega_A, f_i\rangle = \langle f_i\cdot\psi,\psi\rangle\,,$$

$i = 1, \ldots 7$. Hence the more explicit form of the equation $\pi_7\left(\Omega_A\right) = \sigma(\psi)$ is

$$F_{12}+F_{34}+F_{56}+F_{78} = -4i\left(\left|\psi_8\right|^2 - \left|\psi_3\right|^2\right)$$

$$F_{13}-F_{24}+F_{57}-F_{68} = \\ 2\left(\psi_4\overline{\psi}_3 - \psi_3\overline{\psi}_4 + \psi_7\overline{\psi}_3 - \psi_3\overline{\psi}_7 + \psi_8\overline{\psi}_4 - \psi_4\overline{\psi}_8 + \psi_8\overline{\psi}_7 - \psi_7\overline{\psi}_8\right)$$

$$F_{14}+F_{23}-F_{58}-F_{67} = \\ 2i\left(\psi_4\overline{\psi}_3 + \psi_3\overline{\psi}_4 - \psi_7\overline{\psi}_3 - \psi_3\overline{\psi}_7 - \psi_8\overline{\psi}_4 - \psi_4\overline{\psi}_8 + \psi_8\overline{\psi}_7 + \psi_7\overline{\psi}_8\right)$$

$$F_{15}-F_{26}-F_{37}+F_{48} = \\ 2\left(\psi_2\overline{\psi}_3 - \psi_3\overline{\psi}_2 + \psi_8\overline{\psi}_2 - \psi_2\overline{\psi}_8 + \psi_3\overline{\psi}_5 - \psi_5\overline{\psi}_3 + \psi_5\overline{\psi}_8 - \psi_8\overline{\psi}_5\right)$$

$$F_{16}+F_{25}+F_{38}+F_{47} = \\ 2i\left(\psi_2\overline{\psi}_3 + \psi_3\overline{\psi}_2 - \psi_8\overline{\psi}_2 - \psi_2\overline{\psi}_8 + \psi_3\overline{\psi}_5 + \psi_5\overline{\psi}_3 - \psi_5\overline{\psi}_8 - \psi_8\overline{\psi}_5\right)$$

$$F_{17} - F_{28} + F_{35} - F_{46} =$$
$$2\left(\psi_1\overline{\psi}_3 - \psi_3\overline{\psi}_1 + \psi_8\overline{\psi}_1 - \psi_1\overline{\psi}_8 + \psi_6\overline{\psi}_3 - \psi_3\overline{\psi}_6 + \psi_8\overline{\psi}_6 - \psi_6\overline{\psi}_8\right)$$

$$F_{18} + F_{27} - F_{36} - F_{45} =$$
$$2i\left(-\psi_1\overline{\psi}_3 - \psi_3\overline{\psi}_1 + \psi_8\overline{\psi}_1 + \psi_1\overline{\psi}_8 + \psi_6\overline{\psi}_3 + \psi_3\overline{\psi}_6 - \psi_8\overline{\psi}_6 - \psi_6\overline{\psi}_8\right).$$

Note that the last system is non-linear with respect to the pair (A, Ψ). If we put $\psi_3 = \psi_8$ then this non-linear system turns to the following linear homogenous system:

$$\begin{aligned}
F_{12} + F_{34} + F_{56} + F_{78} &= 0 \\
F_{13} - F_{24} + F_{57} - F_{68} &= 0 \\
F_{14} + F_{23} - F_{58} - F_{67} &= 0 \\
F_{15} - F_{26} - F_{37} + F_{48} &= 0 \\
F_{16} + F_{25} + F_{38} + F_{47} &= 0 \\
F_{17} - F_{28} + F_{35} - F_{46} &= 0 \\
F_{18} + F_{27} - F_{36} - F_{45} &= 0
\end{aligned}$$

Non-trivial Solutions:

If we take $A_1 = i\,(4x_1x_2)$ and $A_2 = 2i\left(x_1^2 + x_2^2\right)$, $A_3 = A_4 = A_5 = A_6 = A_7 = A_8 = 0$ and $\psi_1 = \psi_2 = e^{-i\left(x_2x_1^2 + x_2^3/3\right)}$ and $\psi_3 = \psi_4 = \psi_5 = \psi_6 = \psi_7 = \psi_8 = 0$ then the pair (A, Ψ) is a non-trivial solution for local equations. Although this solution is non-trivial it is flat as $F_A = 0$. It is also possible to give a non-flat solution for these equations. Namely if we take $A_1 = ix_2$ and $A_4 = ix_3$, $A_2 = A_3 = A_5 = A_6 = A_7 = A_8 = 0$ and $\psi_4 = e^{\left(x_2^2 + x_3^2\right)/4}$ and $\psi_1 = \psi_2 = \psi_3 = \psi_5 = \psi_6 = \psi_7 = \psi_8 = 0$ then the pair (A, Ψ).is a non-trivial and non-flat.

From these solutions it is possible to generate infinitely many solutions to these equations. If the pair (A, Ψ) is a solution to the system $D_A(\Psi) = 0$, $\pi_7(\Omega_A) = \sigma(\psi)$ then the pair $\left(A + id\theta, e^{-i\theta}\psi\right)$ is also a solution to these equations where $\theta : \mathbb{R}^8 \longrightarrow \mathbb{R}$ is a smooth function.

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