

# MEASURE-THEORETICAL SENSITIVITY AND SCRAMBLED SETS VIA FURSTENBERG FAMILIES

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ABSTRACT. For an invariant measure  $\mu$  in a topological dynamic system, notions of  $\mathcal{F}$ - $\mu$ -pairwise sensitivity and  $\mathcal{F}$ - $\mu$ -sensitivity are introduced and investigated, where  $\mathcal{F}$  is a Furstenberg family. It is shown that  $\mathcal{F}$ - $\mu$ -pairwise sensitivity is equivalent to  $\mathcal{F}$ - $\mu$ -sensitivity, when  $\mathcal{F}$  is a filterdual and is extensively compatible to the 2-fold product system. Moreover, it turns out that if  $(X, \mathcal{B}, \mu, f)$  is a measure-preserving system of weakly mixing and the support of  $\mu$  is  $X$ , then there exists a positive  $c$  such that  $(X, f)$  has an  $(M_*(c), M_*(0))$ -scrambled set with full induced outer measure by  $\mu$ .  $M_*(c)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose lower density is not less than  $c$ , and  $M_*(0)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose lower density is non-zero.

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## 1. INTRODUCTION

Throughout this paper a topological dynamical system is a pair  $(X, f)$  (TDS for short), where  $X$  is a compact metric space with a metric  $d$  and  $f$  is a continuous surjective map from  $X$  to itself. Let  $\mathcal{B}$  be the  $\sigma$ -algebra of Borel subsets of  $X$ . We use  $M(X, f)$  to denote the collection of all  $f$ -invariant Borel probability measures.

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Let  $\mathbb{Z}_+$  be the set of non-negative integers, and let  $\mathcal{P}$  be the collection of all subsets of  $\mathbb{Z}_+$ . A subset  $\mathcal{F}$  of  $\mathcal{P}$  is called a Furstenberg family [1] if it is hereditary upwards, i.e.  $F_1 \subset F_2$  and  $F_1 \in \mathcal{F}$  imply  $F_2 \in \mathcal{F}$ . Let  $\mathcal{F}_{inf}$  be the family of all infinite subsets of  $\mathbb{Z}_+$ . It is easy to see that  $\mathcal{F}_{inf}$  is a Furstenberg family.

The sensitivity ([4][8]) is widely understood to be one of central ideas of chaos. In recent years, the sensitivity has attracted much attention ([2][5][16]). People have tried to give various definitions of sensitivity according to their understanding, for instance, family  $\mathcal{F}$ -sensitivity ([1][13]); pairwise sensitivity [3]; measure-theoretical sensitivity [6].

Now we recall some notions of sensitivity.

A TDS  $(X, f)$  is weakly  $\mathcal{F}$ -sensitive ([1][13]) if there is a positive  $\varepsilon$ —a weakly  $\mathcal{F}$ -sensitive constant—such that for every nonempty open subset  $U$  of  $X$  there exist  $x$  and  $y$  of  $U$  such that

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F},$$

where  $\mathcal{F}$  is a Furstenberg family.

A TDS  $(X, f)$  is  $\mathcal{F}$ -sensitive [13] if there is a positive  $\varepsilon$ —a  $\mathcal{F}$ -sensitive constant—such that for every  $x \in X$  and every open neighborhood  $U$  of  $x$  there exists  $y \in U$  such that

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F},$$

where  $\mathcal{F}$  is a Furstenberg family.

Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\mathcal{B}_X^+$  be the collection of Borel sets of  $X$  with positive measures.

A TDS  $(X, f)$  is pairwise sensitive for  $\mu$  or  $\mu$ -pairwise sensitive [3] if there is  $\varepsilon > 0$ —a  $\mu$ -pairwise sensitive constant—such that for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  one can find  $n \geq 0$  with  $d(f^n(x), f^n(y)) > \varepsilon$ .

A TDS  $(X, f)$  is  $\mu$ -sensitive [6] if there is  $\varepsilon > 0$ —a  $\mu$ -sensitive constant—such that for any  $A \in \mathcal{B}_X^+$  one can find  $x, y \in A$  and  $n \geq 0$  with

$$d(f^n(x), f^n(y)) > \varepsilon.$$

In [3] it is proved that any measure-preserving system of weakly mixing is  $\mu$ -pairwise sensitive. Moreover, it turns out that  $\mu$ -sensitivity is equivalent to  $\mu$ -pairwise sensitivity in [6].

Following these ideas which come from ([3][6][13]), we have:

**Definition 1.1** Let  $(X, f)$  be a TDS and  $\mu \in M(X, f)$ . Suppose that  $\mathcal{F}$  is a Furstenberg family. A TDS  $(X, f)$  is said to be  $\mathcal{F}$ - $\mu$ -sensitive if there is  $\varepsilon > 0$ —an  $\mathcal{F}$ - $\mu$ -sensitive constant—such that for any  $A \in \mathcal{B}_X^+$  there exist  $x, y \in A$  with

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}.$$

**Definition 1.2** Let  $(X, f)$  be a TDS and  $\mu \in M(X, f)$ . Suppose that  $\mathcal{F}$  is a Furstenberg family. A TDS  $(X, f)$  is said to be  $\mathcal{F}$ - $\mu$ -pairwise sensitive if there is  $\varepsilon > 0$ —an  $\mathcal{F}$ - $\mu$ -pairwise sensitive constant—such that for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  with

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}.$$

Now we recall the notion of  $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled set [11].

Let  $(X, f)$  be a TDS. Suppose that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are two Furstenberg families. A point pair  $(x, y)$  is said to be an  $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled pair, if there is  $\varepsilon > 0$  such that  $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}_1$  and  $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}_2$  for any  $\delta > 0$ . A  $(\mathcal{F}_{inf}, \mathcal{F}_{inf})$ -scrambled pair is usually Li-Yorke scrambled pair. An  $(\overline{M}(1), \overline{M}(1))$ -scrambled pair is usually distributively scrambled pair [9], where  $\overline{M}(1)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose upper density is 1.

A subset  $K$  of  $X$  is said to be  $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled set, if  $(x, y)$  is an  $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled pair for any two distinct points  $x, y$  of  $K$ . A TDS  $(X, f)$  is said to be  $(\mathcal{F}_1, \mathcal{F}_2)$ -chaotic, if it contains an uncountable  $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled set. In [11], a TDS is constructed which is  $(M^*(1), M^*(\frac{1}{\sqrt{e}}))$ -chaotic but not distributively chaotic [9], where  $M^*(a)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose upper density is not less than  $a$ .

In this article we investigate the  $\mathcal{F}$ - $\mu$ -sensitivity,  $\mathcal{F}$ - $\mu$ -pairwise sensitivity and the measure of  $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled sets. This article is organized as follows. In section 2, some basic notions related to Furstenberg families are introduced. In section 3, we study  $\mathcal{F}$ - $\mu$ -sensitivity and  $\mathcal{F}$ - $\mu$ -pairwise sensitivity. It is shown that  $\mathcal{F}$ - $\mu$ -pairwise sensitivity is equivalent to  $\mathcal{F}$ - $\mu$ -sensitivity, when  $\mathcal{F}$  is a filterdual and is extensively compatible to the 2-fold product system (see Corollary 3.8 ). Finally, the existence of scrambled sets with full measure is discussed. It is proved that if  $(X, \mathcal{B}, \mu, f)$  is a measure-preserving system of weakly mixing, and the support of  $\mu$  is  $X$ , then there exists a positive  $c$  such that  $(X, f)$  has an  $(M_*(c), M_*(0))$ -scrambled set with full

induced outer measure by  $\mu$ .  $M_*(c)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose lower density is not less than  $c$ , and  $M_*(0)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose lower density is non-zero ( see Theorem 4.6).

## 2. PRELIMINARIES

In this section, we introduce some basic notions related to Furstenberg families ( for details see [1] ).

For a Furstenberg family  $\mathcal{F}$ , its dual family is

$$k\mathcal{F} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset, \forall F' \in \mathcal{F}\} = \{F \in \mathcal{P} : \mathbb{Z}_+ - F \in \mathcal{F}\}.$$

Clearly, if  $\mathcal{F}$  is a Furstenberg family then so is  $k\mathcal{F}$ . It is easy to see that  $k\mathcal{P} = \emptyset$ ,  $k\emptyset = \mathcal{P}$ . Clearly,  $k(k\mathcal{F}) = \mathcal{F}$  and  $\mathcal{F}_1 \subset \mathcal{F}_2$  implies  $k\mathcal{F}_2 \subset k\mathcal{F}_1$ . It is easy to see that  $k\mathcal{F}_{inf}$  is the family of all cofinite subsets of  $\mathbb{Z}_+$ .

A Furstenberg family  $\mathcal{F}$  is proper if it is a proper subset of  $\mathcal{P}$ . It is easy to see that a Furstenberg family  $\mathcal{F}$  is proper if and only if  $\mathbb{Z}_+ \in \mathcal{F}$  and  $\emptyset \notin \mathcal{F}$ .

For Furstenberg families  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , let  $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$ . A Furstenberg family  $\mathcal{F}$  is filter if it is proper and  $\mathcal{F} \supset \mathcal{F} \cdot \mathcal{F}$ . A Furstenberg family  $\mathcal{F}$  is filterdual if  $k\mathcal{F}$  is filter. Moreover, A Furstenberg family  $\mathcal{F}$  is filterdual if and only if  $\mathcal{F}$  has Ramsey property, i.e. if  $F_1 \cup F_2 \in \mathcal{F}$ , then  $F_1 \in \mathcal{F}$  or  $F_2 \in \mathcal{F}$ .

Any subset  $\mathcal{A}$  of  $\mathcal{P}$  can generate a Furstenberg family  $[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$ . It is the smallest Furstenberg family which contains  $\mathcal{A}$ . It is easy to see that  $\mathcal{A} = \emptyset$  if and only if  $[\mathcal{A}] = \emptyset$ , and  $\emptyset \in \mathcal{A}$  if and only if  $[\mathcal{A}] = \mathcal{P}$ . So  $[\mathcal{A}]$  is a proper Furstenberg family if and only if  $\mathcal{A} \neq \emptyset$  and  $\emptyset \notin \mathcal{A}$ . A Furstenberg family  $\mathcal{F}$  is countably generated ([1][14]) if there exists a countable subset  $\mathcal{A}$  of  $\mathcal{P}$  such that  $[\mathcal{A}] = \mathcal{F}$ . Clearly,  $k\mathcal{F}_{inf}$  is a countably generated proper family.

Let  $J \subset \mathbb{Z}_+$ . The upper density of  $J$  is

$$\bar{d}(J) = \limsup_{n \rightarrow \infty} \frac{\sharp(J \cap \{0, 1, \dots, n-1\})}{n}$$

where  $\sharp$  denotes the cardinality of the set. The lower density of  $J$  is

$$\underline{d}(J) = \liminf_{n \rightarrow \infty} \frac{\sharp(J \cap \{0, 1, \dots, n-1\})}{n}.$$

If  $\bar{d}(J) = \underline{d}(J)$ , then we say  $J$  has density  $d(J)$ .

Let  $(X, f)$  be a TDS and  $U, V \subset X$ . We define the meeting time set

$$N(U, V) = \{n \in \mathbb{Z}_+ : f^n(U) \cap V \neq \emptyset\}.$$

Especially,  $N(x, V) = \{n \in \mathbb{Z}_+ : f^n(x) \in V\}$  for  $x \in X$ .

Let  $A \subset X$  and  $x \in X$ . If  $N(x, A) \in \mathcal{F}$ ,  $x$  is called an  $\mathcal{F}$ -attaching point of  $A$ . The set of all  $\mathcal{F}$ -attaching points of  $A$  is called the set of  $\mathcal{F}$ -attaching of  $A$ , denoted by  $\mathcal{F}(A)$ . Clearly,

$$\mathcal{F}(A) = \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} f^{-n}(A) = \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} f^{-n}(A).$$

Let  $(X, f)$  be a TDS and  $F \subset \mathbb{Z}_+$ . A pair  $(x, y) \in X \times X$  is called  $F$ -proximal if

$$\liminf_{F \ni n \rightarrow \infty} d(f^n(x), f^n(y)) = 0.$$

We denote the set of all  $F$ -proximal pairs by  $P_F$ .

We shall use following some relations on  $X$ .

$$\overline{V}_\varepsilon = \{(x, y) : d(x, y) \leq \varepsilon\}, \quad V_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$$

Let  $\mathcal{F}$  be a Furstenberg family. A pair  $(x, y) \in X \times X$  is called  $\mathcal{F}$ -proximal if  $(x, y) \in \mathcal{F}(V_\varepsilon)$  for any  $\varepsilon > 0$ . We denote the set of all  $\mathcal{F}$ -proximal pairs by  $P_{\mathcal{F}}$ .

Note ([10]):

$$\begin{aligned} P_{\mathcal{F}} &= \bigcap_{\varepsilon > 0} \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} (f \times f)^{-n}(V_\varepsilon) \\ &= \bigcap_{k=1}^{\infty} \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} (f \times f)^{-n}(V_{\frac{1}{k}}) \\ &= \bigcap_{k=1}^{\infty} \mathcal{F}(V_{\frac{1}{k}}) \\ &= \bigcap_{F \in k\mathcal{F}} P_F \end{aligned}$$

### 3. MEASURE-THEORETICAL SENSITIVITY VIA FURSTENBERG FAMILIES

In this section, we research the concepts of  $\mathcal{F}$ - $\mu$ -sensitivity and  $\mathcal{F}$ - $\mu$ -pairwise sensitivity via Furstenberg families.

By concepts of  $\mathcal{F}$ - $\mu$ -sensitivity and weakly  $\mathcal{F}$ -sensitivity, it is clear that following proposition holds.

**Proposition 3.1** Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\text{supp}(\mu) = X$ . Suppose that  $\mathcal{F}$  is a Furstenberg family. If  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -sensitive, then  $(X, f)$  is weakly  $\mathcal{F}$ -sensitive.

**Definition 3.2** Let  $(X, f)$  be a TDS. A Furstenberg family  $\mathcal{F}$  is extensively compatible to the system  $(X, f)$  if the set of  $\mathcal{F}$ -attaching of  $U$  is a Borel set of  $X$  for each open set  $U$  of  $X$ .

**Proposition 3.3** Let  $(X, f)$  be a TDS. A Furstenberg family  $\mathcal{F}$  is extensively compatible to the system  $(X, f)$  if and only if the set of  $k\mathcal{F}$ -attaching of  $B$  is a Borel set of  $X$  for each closed set  $B$  of  $X$ .

**Proof.** Suppose that  $B$  is a closed subset of  $X$ . Then

$$\begin{aligned} x \in k\mathcal{F}(B) &\Leftrightarrow N(x, B) \in k\mathcal{F} \\ &\Leftrightarrow \mathbb{Z}_+ \sim N(x, X \sim B) \in k\mathcal{F} \\ &\Leftrightarrow N(x, X \sim B) \in \overline{\mathcal{F}} \\ &\Leftrightarrow x \in \overline{\mathcal{F}}(X \sim B) \end{aligned}$$

Hence,  $k\mathcal{F}(B) = X \sim \mathcal{F}(X \sim B)$ . Since  $\mathcal{F}(X \sim B)$  is a Borel set of  $X$ , so is  $k\mathcal{F}(B)$ .

**Remark** In [14], the notion of a Furstenberg family  $\mathcal{F}$  is compatible to a system  $(X, f)$  is introduced. A Furstenberg family  $\mathcal{F}$  is compatible to a system  $(X, f)$  if the set of  $\mathcal{F}$ -attaching of  $U$  is a  $G_\delta$  set of  $X$  for each open set  $U$  of  $X$ . Clearly, if a Furstenberg family  $\mathcal{F}$  is compatible to a system  $(X, f)$ , then the Furstenberg family  $\mathcal{F}$  is extensively compatible to the system  $(X, f)$ .

**Proposition 3.4** Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\text{supp}(\mu) = X$ . Suppose that  $\mathcal{F}$  is a filterdual. If  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -pairwise sensitive, then  $(X, f)$  is  $\mathcal{F}$ -sensitive.

**Proof.** Assume that  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -pairwise sensitive, then there is  $\varepsilon > 0$  such that for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  with  $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}$ . Since  $\text{supp}(\mu) = X$ , then for every  $x$  of  $X$  and every neighborhood  $U$  of  $x$ , we have  $\mu \times \mu(U \times U) > 0$ . Thus there exist  $y, z \in U \times U$  such that  $\{n \in \mathbb{Z}_+ : d(f^n(y), f^n(z)) > \varepsilon\} \in \mathcal{F}$ .

if either of  $y, z$  is  $x$ , then the proposition is true.

if neither of  $y, z$  is  $x$ . Since  $\mathcal{F}$  is filterdual, by the triangle inequality, then at least one of  $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon/2\} \in \mathcal{F}$  and  $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(z)) > \varepsilon/2\} \in \mathcal{F}$  is true. Thus  $(X, f)$  is  $\mathcal{F}$ -sensitive.

Let  $(X, f)$  be a TDS and  $\mu \in M(X, f)$ . Suppose that Furstenberg family  $\mathcal{F}$  is extensively compatible to the system  $(X \times X, f \times f)$ .

For  $x \in X$  and  $\varepsilon > 0$ , let

$$E_\varepsilon(x) = \{y \in X : (x, y) \in k\mathcal{F}(\overline{V}_\varepsilon)\},$$

and consider the map  $\Phi_{\mu,\varepsilon} : X \rightarrow R_+$  defined by

$$\Phi_{\mu,\varepsilon}(x) = \mu(E_\varepsilon(x)) \text{ for any } x \in X.$$

It is clear that  $\Phi_{\mu,\varepsilon}$  is measurable and according to Fubini's Theorem,

$$\mu \times \mu(k\mathcal{F}(\overline{V}_\varepsilon)) = \int_X \Phi_{\mu,\varepsilon}(x) d\mu(x). \quad (*)$$

**Proposition 3.5** Let  $(X, f)$  be a TDS and  $\mu \in M(X, f)$ . Suppose that Furstenberg family  $\mathcal{F}$  is extensively compatible to the system  $(X \times X, f \times f)$ . Then  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -pairwise sensitive with  $\mathcal{F}$ - $\mu$ -pairwise sensitive constant  $\varepsilon$  if and only if  $\Phi_{\mu,\varepsilon}(x) = 0, \mu - a.e. x \in X$ .

**Proof.** By  $(*)$ ,  $\mu \times \mu(k\mathcal{F}(\overline{V}_\varepsilon)) = \int_X \Phi_{\mu,\varepsilon}(x) d\mu(x)$ . So  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -pairwise sensitive  $\Leftrightarrow \mu \times \mu(k\mathcal{F}(\overline{V}_\varepsilon)) = 0 \Leftrightarrow \Phi_{\mu,\varepsilon}(x) = 0, \mu - a.e. x \in X$ .

**Proposition 3.6** Let  $(X, f)$  be a TDS and  $\mu \in M(X, f)$ . Suppose that  $\mathcal{F}$  is a Furstenberg family. If  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -pairwise sensitive, then  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -sensitive.

**Proof.** Suppose that  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -pairwise sensitive, then there is  $\varepsilon > 0$  such that for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  with

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}.$$

Since  $\mu \times \mu(A \times A) > 0$  for every  $A \in \mathcal{B}_X^+$ , there exists  $(x, y) \in A \times A$  such that  $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}$ . So  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -sensitive.

**Theorem 3.7** Let  $(X, f)$  be a TDS and  $\mu \in M(X, f)$ . Suppose that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are Furstenberg families,  $\mathcal{F}_1$  is extensively compatible to the system  $(X \times X, f \times f)$  and  $k\mathcal{F}_1 \cdot k\mathcal{F}_1 \subset k\mathcal{F}_2$ . If  $(X, f)$  is  $\mathcal{F}_2$ - $\mu$ -sensitive, then  $(X, f)$  is  $\mathcal{F}_1$ - $\mu$ -pairwise sensitive.

**Proof.** Assume that  $(X, f)$  is  $\mathcal{F}_2$ - $\mu$ -sensitive. Then there is  $\varepsilon > 0$  such that for any  $A \in \mathcal{B}_X^+$  there exist  $x, y \in A$  with  $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}_2$ . Next we shall show that  $(X, f)$  is  $\mathcal{F}_1$ - $\mu$ -pairwise sensitive with  $\mathcal{F}_1$ - $\mu$ -pairwise sensitive constant  $\varepsilon/2$ . By Proposition 3.5 assume the contrary that there exists  $x \in X$  with  $\Phi_{\mu,\varepsilon/2}(x) > 0$ . That is  $\mu(A) > 0$ , where  $A = E_{\varepsilon/2}(x) = \{y \in X : (x, y) \in k\mathcal{F}_1(\overline{V}_{\varepsilon/2})\}$  and  $\overline{V}_{\varepsilon/2} = \{(x, y) : d(x, y) \leq \varepsilon/2\}$ . Then for any  $y, z \in A$  we have

$(x, y) \in k\mathcal{F}_1(\overline{V}_{\varepsilon/2})$ ,  $(x, z) \in k\mathcal{F}_1(\overline{V}_{\varepsilon/2})$ . Assume that

$$F_1 = \{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq \varepsilon/2\} \in k\mathcal{F}_1,$$

$$F_2 = \{n \in \mathbb{Z}_+ : d(f^n(x), f^n(z)) \leq \varepsilon/2\} \in k\mathcal{F}_1.$$

Since  $k\mathcal{F}_1 \cdot k\mathcal{F}_1 \subset k\mathcal{F}_2$ , then  $F_1 \cap F_2 \in k\mathcal{F}_2$ . So  $\{n \in \mathbb{Z}_+ : d(f^n(y), f^n(z)) \leq \varepsilon\} \in k\mathcal{F}_2$ . Thus  $\{n \in \mathbb{Z}_+ : d(f^n(y), f^n(z)) > \varepsilon\} \notin \mathcal{F}_2$ , which contradicts the assumption that  $(X, f)$  is  $\mathcal{F}_2$ - $\mu$ -sensitive with  $\varepsilon$  being an  $\mathcal{F}_2$ - $\mu$ -sensitive constant. Therefore  $(X, f)$  is  $\mathcal{F}_1$ - $\mu$ -pairwise sensitive.

**Corollary 3.8** Let  $(X, f)$  be a TDS and  $\mu \in M(X, f)$ . Suppose that  $\mathcal{F}$  is a filterdual, and is extensively compatible to the system  $(X \times X, f \times f)$ . Then  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -sensitive if and only if it is  $\mathcal{F}$ - $\mu$ -pairwise sensitive.

Let  $(X, \mathcal{B}, \mu, f)$  be a measure-preserving system and  $\mathcal{F}$  be a Furstenberg family. A measure-preserving system  $(X, \mathcal{B}, \mu, f)$  is  $\mathcal{F}$ -ergodic [7] if any  $A \in \mathcal{B}_X^+$  and any  $F \in k\mathcal{F}$  we have  $\mu(\bigcup_{i \in F} f^{-i}(A)) = 1$ . A measure-preserving system  $(X, \mathcal{B}, \mu, f)$  is  $\mathcal{F}$ -mixing [7] if  $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, f \times f)$  is  $\mathcal{F}$ -ergodic.

**Proposition 3.9** Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\text{supp}(\mu) = X$ . Suppose that  $\mathcal{F}$  is a Furstenberg family, and  $k\mathcal{F}$  is a countably generated proper family. If  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is  $\mathcal{F}$ -mixing, then there exists  $\varepsilon > 0$  such that for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  with  $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}$ , i.e.,  $(X, f)$  is  $\mathcal{F}$ - $\mu$ -pairwise sensitive.

**Proof.** Choose  $\varepsilon > 0$  such that  $X \times X - \overline{V}_\varepsilon \neq \emptyset$ , where  $\overline{V}_\varepsilon = \{(x, y) : d(x, y) \leq \varepsilon\}$ . We note that

$$\mathcal{F}(X \times X - \overline{V}_\varepsilon) = \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} (f \times f)^{-n}(X \times X - \overline{V}_\varepsilon).$$

As  $f$  is  $\mathcal{F}$ -mixing, then we have  $\mu \times \mu(\bigcup_{n \in F} (f \times f)^{-n}(X \times X - \overline{V}_\varepsilon)) = 1$  for any  $F \in k\mathcal{F}$ . Let  $\mathcal{A}$  be a countable generated set of  $k\mathcal{F}$ . Hence

$$\mu \times \mu(\mathcal{F}(X \times X - \overline{V}_\varepsilon)) = \mu \times \mu\left(\bigcap_{F \in \mathcal{A}} \bigcup_{n \in F} (f \times f)^{-n}(X \times X - \overline{V}_\varepsilon)\right) = 1.$$

**Proposition 3.10** Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\text{supp}(\mu) = X$ . If  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is weakly mixing, then there exist  $\varepsilon > 0$  and  $c > 0$  such that for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  with  $\underline{d}(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\}) = c$ , where  $\underline{d}(J)$  denotes the lower density of the set  $J$ .

**Proof.** Choose  $\varepsilon > 0$  such that  $E = X \times X - \overline{V_\varepsilon} \neq \emptyset$ , where  $\overline{V_\varepsilon} = \{(x, y) : d(x, y) \leq \varepsilon\}$ . Since  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is weakly mixing, then  $f \times f : (X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu) \rightarrow (X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$  is ergodic. By Birkhoff Ergodic Theorem we have  $\frac{1}{n} \sum_{i=0}^{n-1} (\chi_E(f \times f)^i(x, y)) \rightarrow f^* \in L^1(\mu \times \mu)$  for  $\mu \times \mu - a.e.$   $(x, y) \in X \times X$ . Since  $f^* = \int \chi_E(x, y) d\mu \times \mu = \mu \times \mu(E) = c > 0$  for  $\mu \times \mu - a.e.$   $(x, y) \in X \times X$ , then  $\underline{d}(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\}) = c$  for  $\mu \times \mu - a.e.$   $(x, y) \in X \times X$ .

**Definition 3.11** Let  $(X, \mathcal{B}_X, \mu_1)$  and  $(Y, \mathcal{B}_Y, \mu_2)$  be two measure-preserving systems. A transformation  $\phi : (X, \mathcal{B}_X, \mu_1) \rightarrow (Y, \mathcal{B}_Y, \mu_2)$  is positive-measure-preserving, if  $\phi(A) \in \mathcal{B}_Y^+$  for any  $A \in \mathcal{B}_X^+$ .

**Proposition 3.12** Let  $(X, f)$  be a TDS and  $\mu_1 \in M(X, f)$ . And let  $(Y, g)$  be a TDS and  $\mu_2 \in M(Y, g)$ . Suppose that  $\phi : (X, f) \rightarrow (Y, g)$  is a factor map, and  $\phi : (X, \mathcal{B}_X, \mu_1) \rightarrow (Y, \mathcal{B}_Y, \mu_2)$  is a positive-measure-preserving transformation. If  $(Y, g)$  is  $\mathcal{F}$ - $\mu_2$ -sensitive, then  $(X, f)$  is  $\mathcal{F}$ - $\mu_1$ -sensitive, where  $\mathcal{F}$  is a Furstenberg family.

**Proof.** Suppose that  $\varepsilon > 0$  is  $\mathcal{F}$ - $\mu_2$ -sensitive constant. Since factor map  $\phi$  is continuous, then there exists  $\delta > 0$  such that  $d_1(x, y) > \delta$  when  $d_2(\phi(x), \phi(y)) > \varepsilon$ . Since  $\phi$  is positive-measure-preserving, then  $\phi(A) = B \in \mathcal{B}_Y^+$  for any  $A \in \mathcal{B}_X^+$ . Since  $(Y, g)$  is  $\mathcal{F}$ - $\mu_2$ -sensitive, then there are  $y_1, y_2 \in B$  such that  $\{n \in \mathbb{Z}_+ : d_2(g^n(y_1), g^n(y_2)) > \varepsilon\} \in \mathcal{F}$ . Put  $x_1, x_2 \in A$  such that  $\phi(x_1) = y_1, \phi(x_2) = y_2$ , then  $\{n \in \mathbb{Z}_+ : d_1(f^n(x_1), f^n(x_2)) > \delta\} \in \mathcal{F}$ , this implies  $(X, f)$  is  $\mathcal{F}$ - $\mu_1$ -sensitive.

#### 4. MEASURE OF $(\mathcal{F}_1, \mathcal{F}_2)$ -SCRAMBLED SETS

This section deals with the measure of  $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled sets of some measure-preserving systems.

Suppose that  $(X, \mathcal{S}, \mu)$  is a measure space. For any set  $A \subset X$  let  $\mu^*(A) = \inf\{\mu(B) : B \in \mathcal{S}, B \supset A\}$ , then  $\mu^*$  is an outer measure on  $X$ , and the restriction of  $\mu^*$  on  $\mathcal{S}$  coincides with  $\mu$ . The  $\mu^*$  is said to be an induced outer measure by  $\mu$ .

Suppose that  $(X, \mathcal{S}, \mu)$  is a probability space.  $\mathcal{D} \subset \mathcal{S}$  is said to be a regular base of the measure  $\mu$ , if for any  $\varepsilon > 0$  and any  $B \in \mathcal{S}$ , there exists  $F \in \mathcal{D}$  such that  $F \supset B$  and  $\mu(F - B) < \varepsilon$ .

We need the following lemmas that come from [12] and [15] respectively.

**Lemma 4.1** [12] Suppose that  $\mu$  is Borel probability measure on a metric space  $X$ , i.e.  $(X, \mathcal{B}, \mu)$  is a probability space, where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra. Then the topology of the metric space  $X$  (i.e. the collection of all open subsets of  $X$ ) is a regular base of the measure  $\mu$ .

If a metric space  $X$  is separable, then the cardinality of its topology is not greater than the cardinality of the continuum. Therefore, each Borel probability space on a separable metric space has a regular base whose cardinality is not greater than the cardinality of the continuum.

**Lemma 4.2** [15] Suppose that  $(X, \mathcal{S}, \mu)$  is a measure space with a regular base  $\mathcal{D}$  whose cardinality is not greater than the cardinality of the continuum. Suppose that  $\alpha = \{\mathcal{R}_\gamma\}_{\gamma \in \Gamma}$  is a countable family of relations of the full measure on  $(X, \mathcal{S}, \mu)$  (i.e. for each  $\gamma \in \Gamma$ , there exists some  $s_\gamma \geq 1$  such that  $\mathcal{R}_\gamma \subset X^{s_\gamma}$  and  $\mu^{s_\gamma}(\mathcal{R}_{s_\gamma}) = 1$ , where  $\mu^{s_\gamma}$  is the product measure of  $s_\gamma$  measures  $\mu$  and  $X^{s_\gamma}$  is the product space of  $s_\gamma$  spaces  $X$ ). Let  $\mu^*$  be the outer measure induced by  $\mu$ . Then there exists an  $\alpha$ -dependent set  $K$  of the full outer measure (i.e.  $K \subset X$  satisfies that  $\mu^*(K) = 1$  such that for each  $\gamma \in \Gamma$  and any  $s_\gamma$  distinct points  $x_1, x_2, \dots, x_{s_\gamma}$  in  $K$ , we have  $(x_1, x_2, \dots, x_{s_\gamma}) \in \mathcal{R}_\gamma$ ).

**Proposition 4.3** Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\text{supp}(\mu) = X$ . If  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is weakly mixing, then  $\mu \times \mu(P_{M_*(0)}) = 1$ .  $P_{M_*(0)}$  denotes the set of all  $M_*(0)$ -proximal pairs, and  $M_*(0)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose lower density is non-zero.

**Proof.** For any  $k > 0$  let  $V_{\frac{1}{k}} = \{(x, y) : d(x, y) < \frac{1}{k}\}$ . Since  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is weakly mixing, then  $f \times f : (X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu) \rightarrow (X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$  is ergodic. By Birkhoff Ergodic Theorem we have  $\frac{1}{n} \sum_{i=0}^{n-1} (\chi_{V_{\frac{1}{k}}}(f \times f)^i(x, y)) \rightarrow f^* \in L^1(\mu \times \mu)$  for  $\mu \times \mu - a.e. (x, y) \in X \times X$ . Since  $f^* = \int \chi_{V_{\frac{1}{k}}}(x, y) d\mu \times \mu = \mu \times \mu(V_{\frac{1}{k}}) > 0$  for  $\mu \times \mu - a.e. (x, y) \in X \times X$ . Thus  $d_*\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \frac{1}{k}\} > 0$  for  $\mu \times \mu - a.e. (x, y) \in X \times X$ , i.e.  $\mu \times \mu(M_*(0)(V_{\frac{1}{k}})) = 1$ . Hence  $\mu \times \mu(P_{M_*(0)}) = \mu \times \mu(\bigcap_{k=1}^{\infty} M_*(0)(V_{\frac{1}{k}})) = 1$ .

**Proposition 4.4** Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\text{supp}(\mu) = X$ . Suppose that  $\mathcal{F}$  is a Furstenberg family and  $k\mathcal{F}$  is a countably generated proper family. If  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is  $\mathcal{F}$ -mixing, then the set of all  $(\mathcal{F}, \mathcal{F})$ -scrambled pairs is a full measure set of  $X \times X$ .

**Proof.** Suppose that  $k\mathcal{F}$  is a countably generated proper family by  $\mathcal{A}$ , where  $\mathcal{A}$  is a countable set. Then

$$P_{\mathcal{F}} = \bigcap_{k=1}^{\infty} \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} (f \times f)^{-n}(V_{\frac{1}{k}}) = \bigcap_{k=1}^{\infty} \bigcap_{F \in \text{mathcal{A}}} \bigcup_{n \in F} (f \times f)^{-n}(V_{\frac{1}{k}}).$$

Since  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is  $\mathcal{F}$ -mixing, then we have  $\mu \times \mu(\bigcup_{i \in F} (f \times f)^{-i}(V_{\frac{1}{k}})) = 1$  for any  $F \in k\mathcal{F}$ . So  $\mu \times \mu(P_{\mathcal{F}}) = 1$ . By Proposition 3.9 there is  $\varepsilon > 0$  such that  $\mu \times \mu(\mathcal{F}(X \times X - \overline{V}_{\varepsilon})) = 1$ . Hence the proposition holds.

**Proposition 4.5** Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\text{supp}(\mu) = X$ . Suppose that  $\mathcal{F}$  is a Furstenberg family and  $k\mathcal{F}$  is a countably generated proper family. Let  $\mu^*$  be an outer measure induced by  $\mu$ . If  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is  $\mathcal{F}$ -mixing, then  $(X, f)$  has an  $(\mathcal{F}, \mathcal{F})$ -scrambled set  $K \subset X$  with  $\mu^*(K) = 1$ .

**Proof.** As  $X$  is a compact metric space, then  $(X, \mathcal{B}, \mu)$  has a regular base  $\mathcal{D}$  whose cardinality is not greater than the cardinality of the continuum. Let  $\mathcal{R}_2 = \{(x, y) : (x, y) \text{ is an } (\mathcal{F}, \mathcal{F})\text{-scrambled pair}\}$ ,  $\mathcal{R}_n = X^n$  when  $n \geq 3$ . By Proposition 4.4,  $\mu^2(\mathcal{R}_2) = 1$ . Clearly,  $\mu^n(\mathcal{R}_n) = 1$  for any  $n \geq 3$ . By Lemma 4.2, there exists an  $(\mathcal{F}, \mathcal{F})$ -scrambled set  $K \subset X$  such that  $\mu^*(K) = 1$ .

By Proposition 3.10, Proposition 4.3 and Lemma 4.2 we have also:

**Theorem 4.6** Let  $(X, f)$  be a TDS,  $\mu \in M(X, f)$  and  $\text{supp}(\mu) = X$ . Suppose that  $\mu^*$  is an outer measure induced by  $\mu$ . If  $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  is weakly mixing, then there exists a positive  $c$  such that  $(X, f)$  has an  $(M_*(c), M_*(0))$ -scrambled set  $K \subset X$  with  $\mu^*(K) = 1$ .  $M_*(c)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose lower density is not less than  $c$ , and  $M_*(0)$  denotes the family of all subsets of  $\mathbb{Z}_+$  whose lower density is non-zero.

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