

FRAME OPERATOR AND HILBERT-SCHMIDT OPERATOR IN TENSOR PRODUCT OF HILBERT SPACES

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ABSTRACT. The tensor product of frames in Hilbert spaces is introduced. It was shown that the tensor product of two frames is a frame for the tensor product of Hilbert spaces. The concept of tensor frame operator on tensor product of Hilbert space is given and results of it are presented. Inequalities between tensor product of frames and the Hilbert-Schmidt operators are established.

AMS Classification: 41A58, 47B10, 42C99

Keywords: Tensor product, Frames, Frame operator, Tensor frame operator.

1. INTRODUCTION

Frames for Hilbert spaces were introduced by R.J. Duffin and A.C. Schaeffer in 1952, while discussing some problems in the theory of non-harmonic Fourier series. Later in 1986, I. Daubechies, A. Grossman and Y.M. Meyer found applications of frames to wavelets and Gabor transforms. Again in the year 1998, this theory was developed by Peter G. Casazza, D.Han and D.R. Larson.

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 7, NUMBER 1 (2009) 61-70.
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Peter G. Casazza [4] developed many results on frame theory and he suggested the major directions of research in frame theory. S. Rabinson [6] presented the basic concepts of Tensor product of Hilbert spaces. Amir Khosravi and Behrooz Khosravi [1] are studied the tensor product frames for Hilbert C - modulus. Peter Balazs in [3] presented a relation between frames and Hilbert- Schmidt operators. J.C. Amson and N. Gopal Reddy [2] are introduced the concept of Hilbert-Schmidt quadratic operator which is of course entirely different direction from our present work in which the concept of Hilbert-Schmidt operator is useful.

In this paper we generalized the results of [1, 3] to the tensor product of Hilbert spaces. The tensor product of frames in Hilbert spaces is introduced. We have seen the tensor product of two frames is a frame for the tensor product of Hilbert spaces. Results on tensor frame operator are presented. Inequalities between tensor product of frames and the Hilbert-Schmidt operators are established.

2. NOTATION AND PRELIMINARIES

Throughout this paper H will denote a Hilbert space and J is a countable index set.

Definition 2.1. Let $\{f_j\}$ is an orthonormal basis for H , then $f \in H$,

$$\sum_{j \in J} |\langle f, f_j \rangle|^2 \leq \|f\|^2$$

is called a Bessel's inequality and the constant M is called Bessel bound of the sequence $\{f_j\}$.

Definition 2.2. A sequence $\{f_j\}$ of vectors in a Hilbert space H is called a frame if there exists two constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B \|f\|^2 \text{ for all } f \in H.$$

The above inequality is called a frame inequality. The numbers A and B are called the lower and upper frame bounds respectively. Here the following from [4] are useful.

Definition 2.3. Let l_2 be the space of all sequences of scalars $x = (a_n)$ for which $\|x\|_2 = \left(\sum |a_n|^2 \right)^{1/2} < \infty$ and $\{f_j\}$ is a frame for Hilbert space H . A synthesis operator $T : l_2 \rightarrow H$ is defined as $Te_j = f_j$ where $\{e_j\}$ is an orthonormal basis for l_2 .

Definition 2.4. Let $\{f_j\}$ be a frame for H and $\{e_j\}$ be an orthonormal basis for l_2 . Then the analysis operator $T : H \rightarrow l_2$ is the adjoint of synthesis operator T^* and is defined as $Tf = \sum \langle f, f_j \rangle e_j$ for all $f \in H$.

Definition 2.5. Let $\{f_j\}$ be a frame for Hilbert space H . A frame operator $S = TT^* : H \rightarrow H$ is defined as

$$Sf = \sum_j \langle f, f_j \rangle f_j \text{ for all } f \in H.$$

The following few theorems from [4, 5] are useful in our discussion.

Theorem 2.6. For an orthonormal system $\{e_i\}_{i=1}^\infty$, the following are equivalent.

- (i) $\{e_i\}_{i=1}^\infty$ is an orthonormal basis. (ii) $f = \sum_{i=1}^\infty \langle f, e_i \rangle e_i$ for all $f \in H$.
- (iii) $\langle f, g \rangle = \sum_{i=1}^\infty \langle f, e_i \rangle \langle e_i, g \rangle$, for all $f, g \in H$. (iv) $\sum_{i=1}^\infty |\langle f, e_i \rangle|^2 = \|f\|^2$ for all $f \in H$.
- (v) $\text{Span } \{e_i\}_{i=1}^\infty = H$. (vi) If $\langle f, e_i \rangle = 0$ for all i , then $f = 0$.

Theorem 2.7. Suppose $\{f_j\}$ is a frame for H if and only if $AI_{0p} \leq S \leq BI_{0p}$ and $\{f_j\}$ is normalized tight frame for H if and only if $S = I_{0p}$, where I_{0p} is an identity operator on H .

Theorem 2.8. Let S be a frame operator for the frame $\{f_j\}$ with frame bounds A and B in the Hilbert space H . Then $\{S^{-1}f_j\}$ is a frame for H with frame bounds $\frac{1}{B}$ and $\frac{1}{A}$.

The following theorem gives the existence of inverse of frame operator.

Theorem 2.9. Let S be a frame operator for the frame $\{f_j\}$ with frame bounds A and B in the Hilbert space H . Then S^{-1} exists, positive and $B^{-1}I_{0p} \leq S^{-1} \leq AI_{0p}$.

3. TENSOR FRAMES AND TENSOR FRAME OPERATOR

Let H_1 and H_2 be two Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ and norms $\|\cdot\|_1$, $\|\cdot\|_2$ respectively. The tensor product of H_1 and H_2 is denoted by $H_1 \otimes H_2$ and is an inner product space with respect to the inner product

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle_1 \langle x_2, y_2 \rangle_2 \quad (1)$$

for all $x_1, y_1 \in H_1$ and $x_2, y_2 \in H_2$. The norm on $H_1 \otimes H_2$ is defined by

$$\|x \otimes y\| = \|x\|_1 \|y\|_2 \quad \forall x \in H_1, y \in H_2 \quad (2)$$

The space $H_1 \otimes H_2$ is clearly completion with the above inner product. Therefore the space $H_1 \otimes H_2$ is a Hilbert space. We denote $L(H_1, H_2)$ be the space of all linear operators from $H_1 \rightarrow H_2$. Let $M \in L(H_1)$ and $N \in L(H_2)$ be two operators, then the tensor product of operator $M \otimes N$ acts on $H_1 \otimes H_2$ as $(M \otimes N)(xy) = Mx \otimes Ny$ for every $x \in H_1, y \in H_2$ and $x \otimes y \in H_1 \otimes H_2$. We note that if $M_1, M_2 \in L(H_1), N_1, N_2 \in L(H_2)$ and $M_1 \otimes N_1, M_2 \otimes N_2 \in L(H_1 \otimes H_2)$ then $(M_1 \otimes N_1)(M_2 \otimes N_2) = M_1 M_2 \otimes N_1 N_2$. In this paper we denote I_{H_1} is the identity operator on H_1 and I_{H_2} is the identity operator on H_2 then $I_{H_1} \otimes I_{H_2} = I_{H_1 \otimes H_2}$ is the identity operator on $H_1 \otimes H_2$. The following is the extension of (2.2) to the sequence $\{x_i \otimes y_j\}$

Definition 3.1. Let $\{x_i\}$ and $\{y_j\}$ be the sequences of vectors in Hilbert spaces H_1 and H_2 respectively. Then the sequence of vectors $\{x_i \otimes y_j\}$ is said to be a frame for the tensor product of Hilbert spaces $H_1 \otimes H_2$ if there exists two constants $0 < A \leq B < \infty$ such that

$$A \|x \otimes y\|^2 \leq \sum_{i,j} |\langle x \otimes y, x_i \otimes y_j \rangle|^2 \leq B \|x \otimes y\|^2 \quad \text{for all } x \otimes y \in H_1 \otimes H_2.$$

The numbers A and B are called lower and upper frame bounds of the frame respectively.

Theorem 3.2. Let $\{x_i\}$ and $\{y_j\}$ be two sequences in Hilbert spaces H_1 and H_2 respectively. Then the sequence $\{x_i \otimes y_j\}$ is a frame for $H_1 \otimes H_2$ if and only if $\{x_i\}$ and $\{y_j\}$ are the frames for H_1 and H_2 respectively.

Proof. Suppose that $\{x_i \otimes y_j\}$ is a frame for $H_1 \otimes H_2$ with frame bounds A and B . Then for each $x \otimes y \in H_1 \otimes H_2 - \{0 \otimes 0\}$ for all $x \in H_1, y \in H_2$

$$A \|x \otimes y\|^2 \leq \sum_{i,j} |\langle x \otimes y, x_i \otimes y_j \rangle|^2 \leq B \|x \otimes y\|^2$$

On using (1) and (2) the above equation becomes

$$A \|x\|_1^2 \|y\|_2^2 \leq \left(\sum_i |\langle x, x_i \rangle_1|^2 \right) \left(\sum_j |\langle y, y_j \rangle_2|^2 \right) \leq B \|x\|_1^2 \|y\|_2^2$$

Consider $x \otimes y$ is a non zero vector i.e. x and y are non zero vectors, therefore

$$\left(\sum_i |\langle x, x_i \rangle_1|^2 \right), \left(\sum_j |\langle y, y_j \rangle_2|^2 \right) \text{ and } \|x \otimes y\|^2 \text{ are non zero .}$$

$$\Rightarrow \frac{A \|y\|_2^2}{\sum_j |\langle y, y_j \rangle_2|^2} \|x\|_1^2 \leq \left(\sum_i |\langle x, x_i \rangle_1|^2 \right) \leq \frac{B \|y\|_2^2}{(\sum_j |\langle y, y_j \rangle_2|^2)^2} \|x\|_1^2$$

$$\Rightarrow A_1 \|x\|_1^2 \leq \left(\sum_i |\langle x, x_i \rangle_1|^2 \right) \leq B_1 \|x\|_1^2.$$

where

$$A_1 = \frac{A \|y\|_2^2}{(\sum_j |\langle y, y_j \rangle_2|^2)} \text{ and } B_1 = \frac{B \|y\|_2^2}{(\sum_j |\langle y, y_j \rangle_2|^2)}$$

which shows that $\{x_i\}$ is a frame for H_1 . Similarly we can prove that $\{y_i\}$ is a frame for H_2 . Conversely assume that if $\{x_i\}$ is a frame for H_1 with frame bounds A_1, B_1 and $\{y_i\}$ is a frame for H_2 with frame bounds A_2, B_2 , therefore, we have

$$A_1 \|x\|_1^2 \leq \left(\sum_i |\langle x, x_i \rangle_1|^2 \right) \leq B_1 \|x\|_1^2 \quad (3)$$

and

$$A_2 \|y\|_2^2 \leq \left(\sum_j |\langle y, y_j \rangle_2|^2 \right) \leq B_2 \|y\|_2^2 \quad (4)$$

multiplying the equations (3) and (4) we get

$$A_1 A_2 \|x\|_1^2 \|y\|_2^2 \leq \left(\sum_i |\langle x, x_i \rangle_1|^2 \right) \left(\sum_j |\langle y, y_j \rangle_2|^2 \right) B_1 B_2 \|x\|_1^2 \|y\|_2^2$$

$$\Rightarrow A_1 A_2 \|x \otimes y\|^2 \leq \sum_{i,j} |\langle x \otimes y, x_i \otimes y_j \rangle|^2 \leq B_1 B_2 \|x \otimes y\|^2$$

$\Rightarrow \{x_i \otimes y_j\}$ is a frame for $H_1 \otimes H_2$. Hence we can have the following remark.

Remark 3.3. If the sequences $\{x_i\}$, $\{y_j\}$ and $\{x_i \otimes y_j\}$ are the frames for the Hilbert spaces H_1 and H_2 and $H_1 \otimes H_2$ respectively and S, S' and S'' are the frame operators respectively of above frames, then from 2.5, we have the following.

$$Sx = \sum_i \langle x, x_i \rangle_1 x_i, \quad S'y = \sum_j \langle y, y_j \rangle_2 y_j$$

$$S''(x \otimes y) = \sum_{i,j} \langle x \otimes y, x_i \otimes y_j \rangle (x_i \otimes y_j)$$

for all $x \in H_1, y \in H_2$ and $x \otimes y \in H_1 \otimes H_2$.

Theorem 3.4. If $\{x_i\}$, $\{y_j\}$ and $\{x_i \otimes y_j\}$ are the frames for the Hilbert spaces H_1 , H_2 and $H_1 \otimes H_2$ with the frame operators S, S' and S'' respectively, then

$$S'' = S \otimes S'.$$

Proof. For $x \otimes y \in H_1 \otimes H_2$, we have

$$\begin{aligned} S''(x \otimes y) &= \sum_{i,j} \langle x \otimes y, x_i \otimes y_j \rangle (x_i \otimes y_j) \\ &= \sum_{i,j} \langle x, x_i \rangle_1 x_i \otimes \langle y, y_j \rangle_2 y_j \\ &= (S \otimes S')(x \otimes y) \\ \Rightarrow S'' &= S \otimes S'. \end{aligned}$$

Lemma 3.5. If S, S' are the frame operators for the frames $\{x_i\}$ and $\{y_j\}$ respectively, then $I_{H_1 \otimes H_2} \leq S \otimes S' \leq I_{H_1 \otimes H_2}$, where $A = A_1 A_2, B = B_1 B_2$ and $I_{H_1 \otimes H_2}$ is the identity operator on $H_1 \otimes H_2$.

Proof. The prove follows easily by using theorem 2.7.

The following lemma is useful in our discussion and proof of it is obvious, so left to reader.

Lemma 3.6. If M and N are two invertible operators in $L(H_1)$ and $L(H_2)$ respectively. Then $M \otimes N$ is invertible in $L(H_1 \otimes H_2)$ and $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$.

The frame operators S and S' are invertible operators in $L(H_1)$ and $L(H_2)$ respectively. So by the lemma 3.6, we have $S \otimes S'$ is invertible and $(S \otimes S')^{-1} = S^{-1} \otimes S'^{-1}$.

Theorem 3.7. Let S be the frame operator for the frame $\{x_i\}$ with frame bounds A_1, B_1 in H_1 . Let S' be the frame operator for the frame $\{y_j\}$ with frame bounds A_2, B_2 in H_2 . Then $\left\{ (S \otimes S')^{-1} (x_i \otimes y_j) \right\}$ is a frame for $H_1 \otimes H_2$ with frame bounds $(B_2 B_1)^{-1}$ and $(A_2 A_1)^{-1}$.

Proof. Consider

$$\begin{aligned} (S \otimes S')^{-1} (x \otimes y) &= (S^{-1} \otimes S'^{-1}) (x \otimes y) \\ &= S^{-1} x \otimes S'^{-1} y \\ &= \sum_i \langle x, S^{-1} x_i \rangle S^{-1} x_i \otimes \sum_j \langle y, S'^{-1} y_j \rangle S'^{-1} y_j \\ &= \sum_{i,j} \langle x \otimes y, S^{-1} x_i \otimes S'^{-1} y_j \rangle (S^{-1} x_i \otimes S'^{-1} y_j) \end{aligned}$$

$$= \sum_{i,j} \left\langle x \otimes y, \left(S^{-1} \otimes S'^{-1} \right) (x_i \otimes y_j) \right\rangle \left(S^{-1} \otimes S'^{-1} \right) (x_i \otimes y_j)$$

$$(S \otimes S')^{-1}(x \otimes y) = \sum_{i,j} \langle x \otimes y, (S \otimes S')^{-1}(x_i \otimes y_j) \rangle (S \otimes S')^{-1}(x_i \otimes y_j) \quad (5).$$

Given $\{x_i\}$ and $\{y_j\}$ are the frames with the frame operators S, S' respectively. By the theorem 2.9, we have $B_1^{-1}I_{H_1} \leq S^{-1} \leq A_1^{-1}I_{H_1}$ and $B_2^{-1}I_{H_2} \leq S'^{-1} \leq A_2^{-1}I_{H_2}$

By taking the tensor product of above two inequalities we get

$$(B_2B_1)^{-1} \langle I_{H_1 \otimes H_2}(x \otimes y), x \otimes y \rangle \leq \left\langle \left(S \otimes S' \right)^{-1}(x \otimes y), x \otimes y \right\rangle$$

$$\leq (A_2A_1)^{-1} \langle I_{H_1 \otimes H_2}(x \otimes y), x \otimes y \rangle$$

on using (5), the above equation becomes

$$(B_2B_1)^{-1} \|x \otimes y\|^2 \leq \sum_{i,j} \left| \left\langle x \otimes y, \left(S \otimes S' \right)^{-1}(x_i \otimes y_j) \right\rangle \right|^2 \leq (A_2A_1)^{-1} \|x \otimes y\|^2$$

$$\Rightarrow \left\{ \left(S \otimes S' \right)^{-1}(x_i \otimes y_j) \right\} \text{ is a frame for } H_1 \otimes H_2 \text{ with frame bounds } (B_2B_1)^{-1}$$

$$\text{and } (A_2A_1)^{-1}.$$

4. TENSOR FRAMES CLASSIFYING HILBERT-SCHMIDT OPERATORS

The following definition is from [2]. Definition 4.1. Let H be a Hilbert space. A bounded operator M on a Hilbert space H is said be Hilbert-Schmidt operator (HS) if there exists an orthonormal basis $\{e_i\}$ of H such that $\sum_i \|Me_i\|^2 < \infty$. If M and N are any two Hilbert-Schmidt operators, then the Hilbert-Schmidt inner product can be defined as $\langle M, N \rangle_{HS} = \sum_i \langle Me_i, Ne_i \rangle$. The induced norm is called the Hilbert-Schmidt norm and is defined as $\|M\|_{HS} = \sqrt{\sum_i \|Me_i\|^2}$. Note that if M is a Hilbert-Schmidt operator then $\|M\|_{HS} = \|M^*\|_{HS}$. To prove the main result of this section we require the following lemmas.

Lemma 4.2. If $\{e_k\}$ and $\{e_i\}$ are the orthonormal basis for the Hilbert spaces H_1 and H_2 respectively. Then $\{e_k \otimes e_i\}$ is an orthonormal basis for $H_1 \otimes H_2$.

Proof. Since $\{e_k\}$ and $\{e_i\}$ are the orthonormal basis for H_1 and H_2 respectively, then by the theorem 2.6, we have $\|x\|^2 = \sum_k |\langle x, e_k \rangle|^2$ for every $x \in H_1$ and $\|y\|^2 = \sum_l |\langle y, e_l \rangle|^2$ for every $y \in H_2$. For every $x \otimes y \in H_1 \otimes H_2$, we have,

$$\begin{aligned}
\|x \otimes y\|^2 &= \|x\|^2 \|y\|^2 \\
&= \sum_k \sum_l |\langle x, e_k \rangle_1 \langle y, e_l \rangle_2|^2 \\
&= \sum_{k,l} |\langle x \otimes y, e_k \otimes e_l \rangle|^2
\end{aligned}$$

$\Rightarrow \{e_k \otimes e_l\}$ is an orthonormal basis for $H_1 \otimes H_2$. We note that if M and N are Hilbert-Schmidt operators on H_1 and H_2 respectively. Then $M \otimes N$ is also Hilbert-Schmidt operator on $H_1 \otimes H_2$. And if $M \otimes N$ is Hilbert-Schmidt operator on $H_1 \otimes H_2$, then $\|M \otimes N\|_{HS} = \|(M \otimes N)^*\|_{HS}$.

Theorem 4.3. Let $\{x_i \otimes y_j\}$ be a frame and $\{e_k \otimes e_l\}$ be an orthonormal basis for $H_1 \otimes H_2$. If $M \otimes N$ is an operator from $H_1 \otimes H_2 \rightarrow H_1 \otimes H_2$ then

$$\begin{aligned}
A \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2 &\leq A \sum_{i,j} \|(M \otimes N)(x_i \otimes y_j)\|^2 \\
&\leq B \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2.
\end{aligned}$$

Proof . Since $\{x_i \otimes y_j\}$ is a frame for $H_1 \otimes H_2$, hence by 3.1, we have

$$A \|x \otimes y\|^2 \leq \sum_{i,j} |\langle x \otimes y, x_i \otimes y_j \rangle|^2 \leq B \|x \otimes y\|^2 \quad (6)$$

Since $\{e_k \otimes e_l\}$ is an orthonormal basis for $H_1 \otimes H_2$. By 2.6, we have

$$\|x \otimes y\|^2 = \sum_{k,l} |\langle x \otimes y, e_k \otimes e_l \rangle|^2 \quad (7)$$

Consider

$$\begin{aligned}
\sum_{i,j} \|(M \otimes N)(x_i \otimes y_j)\|^2 &= \sum_{i,j} \sum_{k,l} |\langle (M \otimes N)(x_i \otimes y_j), e_k \otimes e_l \rangle|^2 \quad (\text{by (7)}) \\
&= \sum_{k,l} \sum_{i,j} |\langle (M \otimes N)^*(e_k \otimes e_l), x_i \otimes y_j \rangle|^2 \quad (8)
\end{aligned}$$

The constants A and B are chosen such that

$$\begin{aligned}
\text{RHS of equation (8)} &\geq A \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2 \\
\text{RHS of equation (8)} &\leq B \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2
\end{aligned}$$

Hence we have

$$\begin{aligned} A \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2 &\leq \sum_{i,j} \|(M \otimes N)(x_i \otimes x_j)\|^2 \\ &\leq B \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2 \end{aligned}$$

The following corollary follows easily.

Corollary 4.4. Let $\{x_i \otimes y_j\}$ be a Bessel sequence and $\{e_k \otimes e_l\}$ be an orthonormal basis for $H_1 \otimes H_2$. If $M \otimes N$ is an operator from $H_1 \otimes H_2 \rightarrow H_1 \otimes H_2$ then

$$\sum_{i,j} \|(M \otimes N)(x_i \otimes y_j)\|^2 \leq B \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2$$

where B is Bessel bound. The following theorem is the main result of this section.

Theorem 4.5. Let $\{x_i \otimes y_j\}$ be a frame and $\{e_k \otimes e_l\}$ be an orthonormal basis for $H_1 \otimes H_2$. If $M \otimes N : H_1 \otimes H_2 \rightarrow H_1 \otimes H_2$ is a Hilbert-Schmidt operator then $\sqrt{A} \|M \otimes N\|_{HS} \leq \sqrt{\sum_{i,j} \|(M \otimes N)(x_i \otimes y_j)\|^2} \leq \sqrt{A} \|M \otimes N\|_{HS}$.

Proof. Since $\{x_i \otimes y_j\}$ is a frame for $H_1 \otimes H_2$, by the theorem 4.3, we have

$$\begin{aligned} A \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2 &\leq \sum_{i,j} \|(M \otimes N)(x_i \otimes x_j)\|^2 \\ &\leq B \sum_{k,l} \|(M \otimes N)^*(e_k \otimes e_l)\|^2 \end{aligned} \tag{9}$$

Since $M \otimes N$ is Hilbert-Schmidt operator on $H_1 \otimes H_2$, therefore we have

$$\begin{aligned} \|M \otimes N\|_{HS} &= \sqrt{\sum_{k,l} \|(M \otimes N)(e_k \otimes e_l)\|^2} \text{ and} \\ \|(M \otimes N)^*\|_{HS}^2 &= \sqrt{\sum_{k,l} \|(M^* \otimes N^*)(e_k \otimes e_l)\|^2} \end{aligned}$$

By using the theorem 4.3 the equation (9) becomes

$$\sqrt{A} \|M \otimes N\|_{HS} \leq \sum_{k,l} \|(M \otimes N)(x_i \otimes y_i)\|^2 \leq \sqrt{B} \|M \otimes N\|_{HS}.$$

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