

AN EXTENSION OF ALMOST SURE CENTRAL LIMIT THEORY FOR THE PRODUCT OF PARTIAL SUMS

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ABSTRACT. An almost sure central limit theorem for the product of the partial sums of i.i.d. positive random variables is shown, which is an extension of Gonchigdanzan and Rempała [5]. Moreover, a more general almost sure central limit theorem for a triangular array is proved.

AMS Classification: 60F15, 60F05

Keywords: Almost sure central limit theorem, Partial sums, Product.

1. INTRODUCTION AND MAIN RESULT

It is well known that the products of independent identically distributed (i.i.d), positive, square integrable random variables (r.v.) are asymptotically log-normal. This fact is an immediate consequence of the classical central limit theorem (CLT). This point, up to the knowledge of the author, was first argued by Arnold and Villaseñor [1], who considered the limiting properties of the sums of records. In their paper Arnold and Villaseñor obtained the following version of the CLT for a sequence of i.i.d. exponential r.v.'s $(X_n)_{n \geq 1}$ with the mean equal to one:

$$\frac{\sum_{k=1}^n \log S_k - n \log n + n}{\sqrt{2n}} \xrightarrow{\mathcal{L}} \Phi, \text{ as } n \rightarrow \infty,$$

where $S_k = \sum_{j=1}^k X_j$, $1 \leq k \leq n$, and Φ is a standard normal r.v. Rempała and Wośowski [9] have noted that this limit behavior of a product of partial sums has a universal character and holds for any sequence of square integrable, positive i.i.d. random variables. Namely, they have proved the following.

Theorem RW Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu$, $\text{Var}X_1 = \sigma^2 > 0$ and the coefficient of variation $\gamma = \sigma/\mu$. Then

$$\left(\frac{\prod_{k=1}^n S_k}{n! \mu^n} \right)^{1/(\gamma \sqrt{n})} \xrightarrow{\mathcal{L}} e^{\sqrt{2}\Phi}.$$

Recently, Gonchigdanzan and Rempała [5] discussed an almost sure limit theorem for the product of the partial sums of i.i.d. positive random variables as follows.

Theorem GR Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu > 0$, $\text{Var}X_1 = \sigma^2$. Denote $\gamma = \sigma/\mu$ the coefficient of variation. Then for any real x ,

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I \left(\left(\frac{\prod_{k=1}^n S_k}{n! \mu^n} \right)^{1/(\gamma \sqrt{n})} \leq x \right) = F(x), \quad a.s.$$

where $F(\cdot)$ is the distribution function of the r.v. $e^{\sqrt{2}\Phi}$.

In the present paper, we are interested in the following which is an extension of Theorem GR.

Theorem 1. *Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu > 0$, $\text{Var}X_1 = \sigma^2$. Denote $\gamma = \sigma/\mu$ the coefficient of variation. Then for every bounded Lipschitz 1 function f on the real line or the indicator function of a Borel set $A \subset \mathbb{R}$ with $\lambda(\partial A) = 0$, we have*

$$(1.1) \quad \lim_{N \rightarrow \infty} \frac{1}{D_N} \sum_{n=1}^N d_n f \left(\frac{1}{\gamma \sqrt{2n}} \sum_{k=1}^n \log \frac{S_k}{k\mu} \right) = \int_{-\infty}^{\infty} f(x) d\Phi(x) \quad a.s.$$

Here $D_n = \sum_{k=1}^n d_k$ and d_k satisfies the following assumptions: $kd_k = O(1)$ and $d_k k^\alpha$ is eventually non-increasing for some $0 < \alpha < 1$ and that for some $\rho > 0$

$$(1.2) \quad d_k = O \left(\frac{D_k}{k(\log D_k)^\rho} \right).$$

In particular,

$$\lim_{N \rightarrow \infty} \frac{1}{D_N} \sum_{n=1}^N d_n f \left(\left(\frac{\prod_{k=1}^n S_k}{n! \mu^n} \right)^{1/(\gamma \sqrt{n})} \right) = \int_{-\infty}^{\infty} f(x) dF(x) \quad a.s.$$

where $F(\cdot)$ is the same with one in Theorem GR.

Remark 1. Let $d_k = 1/k$ and $D_k = \log k$ in Theorem 1 and

$$f \left(\frac{1}{\gamma \sqrt{2n}} \sum_{k=1}^n \log \frac{S_k}{k\mu} \right) = I \left(\frac{1}{\gamma \sqrt{2n}} \sum_{k=1}^n \log \frac{S_k}{k\mu} \leq x \right),$$

then we have

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I \left(\frac{1}{\gamma \sqrt{2n}} \sum_{k=1}^n \log \frac{S_k}{k\mu} \leq x \right) = \int_{-\infty}^{\infty} f(x) d\Phi(x), \quad a.s. \quad \forall x.$$

That is to say that

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I \left(\left(\frac{\prod_{k=1}^n S_k}{n! \mu^n} \right)^{1/(\gamma \sqrt{n})} \right) = \int_{-\infty}^{\infty} f(x) dF(x), \quad a.s. \quad \forall x$$

which is Theorem GR.

2. PRELIMINARY AND PROOF OF MAIN RESULTS

2.1. Assumptions and auxiliary results. In order to prove Theorem 1 we first establish the almost sure central limit theorem (ASCLT) for the triangular array $(X_{k,n})_{1 \leq k \leq n, n \geq 1}$. It is well known that for i.i.d. r.v's ASCLT holds under the same assumptions as the CLT, thus the investigation of a general ASCLT seems somewhat more challenging. We refer to the survey papers [2, 3, 4, 6, 7, 8]. In the sequel, for every $n \geq 1$, let $(X_{k,n})_{1 \leq k \leq n}$ be a sequence of independent random variables with partial sums $S_{k,n} = \sum_{i=1}^k X_{i,n}$, ($1 \leq k \leq n$). Now we state our conditions as follows:

Conditions (C)

(C₁) Assume that for some numerical sequences $a_n > 0$ and b_n we have

$$(2.1) \quad \frac{S_{n,n}}{a_n} - b_n \xrightarrow{\mathcal{L}} H,$$

with some (possibly degenerate) distribution function H .

(C₂) For $1 \leq k \leq l \leq n$, there exists a random variable $\Delta_{k,l}$, which depends on $\{X_{i,k}, X_{j,l}, 1 \leq i \leq k, 1 \leq j \leq l\}$, such that $S_{k,k}$ is independent on $S_{l,l} - \Delta_{k,l}$ and

$$(2.2) \quad \mathbb{E} \left(\frac{|\Delta_{k,l}|}{a_l} \right) \leq C(k/l)^\beta$$

for some positive constants C, β .

(C₃) Assume that

$$(2.3) \quad kd_k = O(1) \text{ and } d_k k^\alpha \text{ is eventually non-increasing for some } 0 < \alpha < 1$$

and that for some $\rho > 0$

$$(2.4) \quad d_k = O \left(\frac{D_k}{k(\log D_k)^\rho} \right).$$

Theorem 2. *Under the conditions (C), then for any bounded Lipschitz 1 function f on the real line or the indicator function of a Borel set $A \subset \mathbb{R}$ with $\lambda(\partial A) = 0$, we have*

$$(2.5) \quad \lim_{N \rightarrow \infty} D_N^{-1} \sum_{k=1}^N d_k f \left(\frac{S_{k,k}}{a_k} - b_k \right) = \int_{-\infty}^{\infty} f(x) dH(x), \quad a.s.$$

Remark 2. *In fact, if the triangular array $(X_{k,n})_{1 \leq k \leq n, n \geq 1}$ becomes a sequence of random variables $(X_n)_{n \geq 1}$ with partial sums $S_n = \sum_{k=1}^n X_k$, our condition (C₂) is weaker than (7) and (8) in Hörmann [6]. In some sense Theorem 2 is an extensive result of Theorem 1 in Hörmann [6].*

2.2. Proof of Theorem 2. Without loss of generality we may assume $|f| \leq 1$.

Lemma 1. *(See Lemma 1 in [6]) Let (D_N) be a summation procedure, then the condition (C₃) implies that $D_N = o(N^\varepsilon)$ for any $\varepsilon > 0$.*

Lemma 2. *Let $\{X_{k,n}\}_{1 \leq k \leq n}$ be a sequence of independent random variables with partial sums $S_{k,n} = \sum_{i=1}^k X_{i,n}$, ($1 \leq k \leq n$). Assume that condition (C₂) is satisfied. Further let $b_n = 0$ and f be a bounded Lipschitz 1 function. Then there exists constant c , such that*

$$(2.6) \quad \left| \text{Cov} \left(f \left(\frac{S_{k,k}}{a_k} \right), f \left(\frac{S_{l,l}}{a_l} \right) \right) \right| \leq c(k/l)^\beta \quad (1 \leq k \leq l).$$

Proof. By using $|f| \leq 1$ and the independence of $S_{k,k}$ and $S_{l,l} - \Delta_{k,l}$,

$$\begin{aligned} & \left| Cov\left(f\left(\frac{S_{k,k}}{a_k}\right), f\left(\frac{S_{l,l}}{a_l}\right)\right) \right| \\ &= \left| Cov\left(f\left(\frac{S_{k,k}}{a_k}\right), f\left(\frac{S_{l,l}}{a_l}\right) - f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right)\right) \right| \\ &\leq 2\mathbb{E} \left| f\left(\frac{S_{l,l}}{a_l}\right) - f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right) \right| \\ &\leq const \cdot \mathbb{E} \left(\left| \frac{\Delta_{k,l}}{a_l} \right| \wedge 2 \right). \end{aligned}$$

From (2.2), we complete the proof of the lemma. □

Remark 3. It is obvious to see that if we replace β by any $0 < \beta' < \beta$, Lemma 2 also holds. Thus we may assume, without loss of generality, that β in the last lemma is the same as in (2.3).

Next we will use the following notations,

$$(2.7) \quad \xi_l := f\left(\frac{S_{l,l}}{a_l}\right) - \mathbb{E}f\left(\frac{S_{l,l}}{a_l}\right), \quad \xi_{k,l} := f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right) - \mathbb{E}f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right).$$

Lemma 3. Assume that condition (C_2) of Theorem 2 is satisfied and $b_n = 0$, and define ξ_l and $\xi_{k,l}$ as in (2.7). If $\{d_k, k \geq 1\}$ are arbitrary positive weights, then we have for any $k \leq m \leq n$ and $p \in \mathbb{N}$,

$$\mathbb{E} \left| \sum_{l=m}^n d_l (\xi_l - \xi_{k,l}) \right|^p \leq E_p \left(\sum_{l=m}^n d_l^2 \right)^{p/2},$$

where $E_p = const \cdot 4^p (p/\beta)^{p/2}$.

Proof. Let $Q(l) = Q(k, l) = \xi_l - \xi_{k,l}$, then

$$\begin{aligned} \mathbb{E}|Q(l)|^p &= \mathbb{E} \left| f\left(\frac{S_{l,l}}{a_l}\right) - \mathbb{E}f\left(\frac{S_{l,l}}{a_l}\right) - \left[f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right) - \mathbb{E}f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right) \right] \right|^p \\ &\leq 4^{p-1} \mathbb{E} \left| f\left(\frac{S_{l,l}}{a_l}\right) - \mathbb{E}f\left(\frac{S_{l,l}}{a_l}\right) - \left[f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right) - \mathbb{E}f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right) \right] \right|^p \\ &\leq 2 \cdot 4^{p-1} \mathbb{E} \left| f\left(\frac{S_{l,l}}{a_l}\right) - f\left(\frac{S_{l,l} - \Delta_{k,l}}{a_l}\right) \right|^p \\ &\leq c \cdot 4^{p-1} (k/l)^\beta, \end{aligned}$$

where the last inequality is due to the proof of Lemma 2. Thus by the Hölder's inequality, we get

$$\begin{aligned}
\mathbb{E} \left| \sum_{l=m}^n d_l (\xi_l - \xi_{k,l}) \right|^p &\leq \sum_{l_1=m}^n \cdots \sum_{l_p=m}^n d_{l_1} \cdots d_{l_p} (\mathbb{E} |Q(l_1)|^p \cdots \mathbb{E} |Q(l_p)|^p)^{1/p} \\
&\leq c \cdot 4^{p-1} k^\beta \sum_{l_1=m}^n \cdots \sum_{l_p=m}^n d_{l_1} \cdots d_{l_p} l_1^{-\beta/p} \cdots l_p^{-\beta/p} \\
&= c \cdot 4^{p-1} k^\beta \left(\sum_{l=m}^n d_l l^{-\beta/p} \right)^p \\
&\leq c \cdot 4^{p-1} m^\beta \left(\sum_{l=m}^n d_l^2 l \right)^{p/2} \left(\sum_{l=m}^n l^{-2\beta-1} \right)^{p/2}.
\end{aligned}$$

For $m \geq 2$, it is easy to see that

$$\begin{aligned}
m^\beta \left(\sum_{l=m}^n l^{-(2\beta/p)-1} \right)^{p/2} &\leq m^\beta \left(\int_{m-1}^{\infty} l^{-(2\beta/p)-1} dl \right)^{p/2} \\
&\leq \left(\frac{m}{m-1} \right)^\beta \left(\frac{p}{2\beta} \right)^{p/2} \leq \left(\frac{p}{\beta} \right)^{p/2},
\end{aligned}$$

and similarly, we get for $m = 1$ and $p \geq 2\beta$,

$$\left(\sum_{l=1}^n l^{-(2\beta/p)-1} \right)^{p/2} \leq \left(1 + \frac{p}{2\beta} \right)^{p/2} \leq \left(\frac{p}{\beta} \right)^{p/2}.$$

This completes the proof of our result. $\square$

The crucial step of the proof of Theorem 2 is the following moment inequality, which stems from Hörmann [6], but for completeness we give the simple proof.

Lemma 4. *Let $\{X_{k,n}\}_{1 \leq k \leq n}$ be a sequence of independent random variables with partial sums $S_{k,n} = \sum_{i=1}^k X_{i,n}$, ($1 \leq k \leq n$). Assume that conditions (C_2) , (C_3) are satisfied. Further let $b_n = 0$ and f be a bounded Lipschitz 1 function. Then for every $p \in \mathbb{N}$ we have*

$$(2.8) \quad \mathbb{E} \left| \sum_{k=1}^N d_k \left(f\left(\frac{S_{k,k}}{a_k}\right) - \mathbb{E} f\left(\frac{S_{k,k}}{a_k}\right) \right) \right|^p \leq C_p \left(\sum_{1 \leq k \leq l \leq N} d_k d_l \left(\frac{k}{l} \right)^\beta \right)^{p/2},$$

where $C_p > 0$ is a constant which depends on the parameter p and β is from (2.6).

Proof. At first, we set $C_p = (4\gamma)^{p^2}$ and

$$V_{m,n} := \sum_{l=m}^n d_l l^{-\beta} \left(\sum_{k=1}^l d_k k^\beta \right), \quad (1 \leq m \leq n).$$

For obtaining our result, it is enough to show that the following claim,

”if the number γ is chosen large enough, then

$$(2.9) \quad \mathbb{E} \left| \sum_{k=m}^n d_k \xi_k \right|^p \leq C_p (V_{m,n})^{p/2}, \quad \text{for all } 1 \leq m \leq n,$$

where ξ_k is defined as in (2.7).”

We will use the induction on p to show (2.9). By Lemma 2, we have

$$\mathbb{E} \left| \sum_{k=m}^n d_k \xi_k \right|^2 \leq 2 \sum_{m \leq k \leq l \leq n} d_k d_l |\mathbb{E} \xi_k \xi_l| \leq 2c \sum_{m \leq k \leq l \leq n} d_k d_l (k/l)^\beta \leq 2c V_{m,n}.$$

Hence if we choose γ so large that $(4\gamma)^4 \geq 2c$, then (2.9) holds for $p = 2$.

Assume now that (2.9) is true for $p-1 \geq 2$. From $k d_k = O(1)$ it follows that there is a positive constant A such that $\sum_{k=1}^l d_k k^\beta \geq A l^\beta$. Then we get for $V_{m,n} \leq \gamma$ as the proof of Lemma 3,

$$\left| \sum_{k=m}^n d_k \xi_k \right| \leq (2/A) \sum_{k=m}^n d_k k^{-\beta} \left(\sum_{l=1}^k d_l l^\beta \right) \leq (2/A) \gamma^{1/2} V_{m,n}^{1/2}.$$

In the case of $V_{m,n} \leq \gamma$, we have shown (2.9) is valid.

We now want to show that if for any given $X \geq \gamma$ and the inequality (2.9) holds for $V_{m,n} \leq X$, then it will also hold for $V_{m,n} \leq 3X/2$ and this will show that (2.9) holds for any value of $V_{m,n}$, i.e. complete the induction step.

Assume $V_{m,n} \leq 3X/2$ and set

$$S_1 + S_2 := \sum_{k=m}^w d_k \xi_k + \sum_{k=w+1}^n d_k \xi_k, \quad T_2 := \sum_{k=w+1}^n d_k \xi_{w,k}, \quad (m \leq w \leq n).$$

From Lemma 1, for a fixed m and n we choose w in such a way that

$$V_{m,w} \leq X, \quad V_{w+1,n} \leq X \quad \text{and} \quad \frac{V_{w+1,n}}{V_{m,w}} = \lambda \in [1/2, 1].$$

From the mean value theorem we get

$$(2.10) \quad |S_2^j - T_2^j| \leq j |S_2 - T_2| (|S_2|^{j-1} + |T_2|^{j-1}) \quad (j \geq 1).$$

Since condition (C_2) and Remarks 3, there exists a constant $B > 0$ such that for all $l \geq 1$,

$$B \sum_{k=1}^l d_k k^\beta \geq l^{1+\beta} d_l.$$

This also shows that

$$\sum_{l=m}^n l d_l^2 \leq B V_{m,n}, \quad \text{for all } 1 \leq m \leq n.$$

By Lemma 3, we get for all $j \geq 1$,

$$\mathbb{E}|S_2 - T_2|^j \leq F_j(V_{w+1,n})^{j/2},$$

where $F_j = B^{j/2} E_j$ and E_j is the constant in Lemma 3.

From the induction hypothesis in the case of $1 \leq j \leq p-1$ and from the validity of (2.9) for $V_{m,n} \leq X$ in the case of $j = p$, we have

$$(2.11) \quad \mathbb{E}|S_1|^j \leq C_j(V_{m,w})^{j/2}, \quad (1 \leq j \leq p)$$

and

$$(2.12) \quad \mathbb{E}|S_2|^j \leq C_j(V_{w+1,n})^{j/2} \leq C_j \lambda^{j/2}(V_{m,w})^{j/2}, \quad (1 \leq j \leq p).$$

By Minkowski's inequality, we have

$$(2.13) \quad \mathbb{E}|T_2|^j \leq 2^j C_j \lambda^{j/2}(V_{m,w})^{j/2}, \quad (1 \leq j \leq p).$$

Furthermore, from the Hölder's inequality the following result is easy,

$$(2.14) \quad \begin{aligned} \mathbb{E}|S_1|^j |S_2 - T_2| |S_2|^{p-j-1} &\leq (\mathbb{E}|S_1|^p)^{j/p} (\mathbb{E}|S_2 - T_2|^p)^{1/p} (\mathbb{E}|S_2|^p)^{(p-j-1)/p} \\ &\leq C_p^{(p-1)/p} F_p^{1/p} \lambda^{(p-j)/2} (V_{m,w})^{p/2}. \end{aligned}$$

The last inequality remains valid, with an extra factor 2^{p-j-1} on the right hand side, if $|S_2|^{p-j-1}$ on the left hand side is replaced by $|T_2|^{p-j-1}$. Since S_1 and T_2 are independent, we get

$$\mathbb{E}|S_1 + S_2|^p \leq \mathbb{E}|S_1|^p + \mathbb{E}|S_2|^p + \sum_{j=1}^{p-1} G_p^j (\mathbb{E}|S_1|^j |S_2^{p-j} - T_2^{p-j}| + \mathbb{E}|S_1|^j \mathbb{E}|T_2|^{p-j}),$$

where G_p^j denotes the combination, i.e., $G_p^j = p![j!(p-j)!]^{-1}$. Substituting (2.10) – (2.14) (using also the analogue of (2.14) with $|T_2|^{p-j-1}$) in the above inequality and get

$$\begin{aligned} \mathbb{E}|S_1 + S_2|^p &\leq C_p(V_{m,w})^{p/2} \left(1 + \lambda^{p/2} + C_p^{-1/p} F_p^{1/p} \sum_{j=1}^{p-1} 2^{p-j} G_p^j (p-j) \lambda^{(p-j)/2} \right. \\ &\quad \left. + C_p^{-1} \sum_{j=1}^{p-1} 2^{p-j} G_p^j \lambda^{(p-j)/2} C_j C_{p-j} \right). \end{aligned}$$

Note that

$$C_p^{-1/p} F_p^{1/p} \leq \text{const} \cdot p^{1/2} (4\gamma)^{-p}$$

and

$$C_j C_{p-j} / C_p \leq (4\gamma)^{-p}, \quad (1 \leq j \leq p-1),$$

thus, by $\lambda \leq 1$, we have

$$C_p^{-1/p} F_p^{1/p} \sum_{j=1}^{p-1} 2^{p-j} G_p^j (p-j) \lambda^{(p-j)/2} \leq \text{const} \cdot p^{3/2} \gamma^{-p}$$

and

$$C_p^{-1} \sum_{j=1}^{p-1} 2^{p-j} G_p^j \lambda^{(p-j)/2} C_j C_{p-j} \leq \text{const} \cdot \gamma^{-p}.$$

Since $\lambda \geq 1/2$ we have shown that for a large γ the relation $\mathbb{E}|S_1 + S_2|^p \leq C_p(1 + \lambda)^{p/2} (V_{m,w})^{p/2} = C_p(V_{m,n})^{p/2}$ is true, i.e., for $V_{m,n} \leq 3X/2$, (2.9) is valid. $\square$

Lemma 5. (See Lemma 5 in [6]) Assume the condition (C_3) of Theorem 2 is satisfied. Then for any $\alpha > 0$ and any $\eta < \rho$, we have

$$\sum_{1 \leq k \leq l \leq N} d_k d_l \left(\frac{k}{l} \right)^\alpha = O\left(\frac{D_N^2}{(\log D_N)^\eta} \right).$$

Proof of Theorem 2. Without loss of generality, from Lemma 4 and Lemma 5, we have, for any $\varepsilon > 0$, and $p \in \mathbb{N}$,

$$\mathbb{P}\left(\left| \sum_{k=1}^N d_k \xi_k \right| > \varepsilon D_N \right) \leq c(p, \varepsilon) (\log D_N)^{-p\eta/2}, \quad \text{for } N \geq N_0.$$

By (C_3) , we have $D_{N+1}/D_N \rightarrow 1$, thus we can choose $p > 4/\eta$ and (N_j) such that $D_{N_j} \sim \exp\{\sqrt{j}\}$. Applying Borel-Cantelli lemma, we get

$$\lim_{j \rightarrow \infty} D_{N_j}^{-1} \sum_{k=1}^{N_j} d_k \xi_k = 0 \quad a.s..$$

For $N_j \leq N \leq N_{j+1}$, we have

$$D_N^{-1} \left| \sum_{k=1}^N d_k \xi_k \right| \leq D_{N_j}^{-1} \left| \sum_{k=1}^{N_j} d_k \xi_k \right| + 2(D_{N_{j+1}}/D_{N_j} - 1) \quad a.s..$$

The convergence of the subsequence implies that whole sequence converges a.s., since $D_{N_{j+1}}/D_{N_j} \rightarrow 1$. This complete the proof of the theorem. For indicator functions, by routine approximation arguments, it is easy to see (cf. Lacey and Philipp [7]). $\square$

2.3. Proof of Theorem 1. At first, we check the conditions (C). Let $Y_i = S_i/(i\mu)$. Observing

$$\frac{1}{\gamma\sqrt{2k}} \sum_{i=1}^k (Y_i - 1) = \frac{1}{\gamma\sqrt{2k}} \sum_{i=1}^k \left(\frac{S_i}{i\mu} - 1 \right) =: \frac{1}{\sqrt{2k}} \sum_{i=1}^k X_{i,k} = \frac{1}{\sqrt{2k}} S_{k,k},$$

from Lemma 1 in [9] we know that

$$\frac{1}{\sqrt{2k}} S_{k,k} \xrightarrow{\mathcal{L}} \Phi,$$

then taking $a_k = \sqrt{2k}$ and $b_k = 0$, (C_1) in conditions (C) is satisfied. From the proof of Theorem 2, the aim of the condition (C_2) is to prove Lemma 2. However, from the proof of Theorem 3 in [5], we know our Lemma 2 holds.

Therefore we see that (2.5) is equivalent to

$$(2.15) \quad \lim_{N \rightarrow \infty} D_N^{-1} \sum_{k=1}^N d_k f \left(\frac{1}{\gamma\sqrt{2k}} \sum_{i=1}^k (Y_i - 1) \right) = \int_{-\infty}^{\infty} f(x) d\Phi(x), \quad a.s.$$

By the law of the iterated logarithm we have for $k \rightarrow \infty$

$$|Y_k - 1| = O \left(\left(\frac{\log \log k}{k} \right)^{1/2} \right) \quad a.s.$$

Since for $|x| < 1$ we have $\log(1+x) = x - R(x)$ with $\lim_{x \rightarrow 0} \frac{R(x)}{x^2} = \frac{1}{2}$, thus

$$\begin{aligned} \left| \sum_{i=1}^k \log Y_i - \sum_{i=1}^k (Y_i - 1) \right| &= O \left(\sum_{i=1}^k (Y_i - 1)^2 \right) \\ &= O \left(\sum_{i=1}^k \frac{\log \log i}{i} \right) = O((\log \log k) \log k) \quad a.s. \end{aligned}$$

Note that in order to prove (1.1) it is sufficient to show that

$$(2.16) \quad \lim_{N \rightarrow \infty} \frac{1}{D_N} \sum_{k=1}^N d_k \left| f \left(\frac{1}{\gamma \sqrt{2k}} \sum_{i=1}^k \log Y_i \right) - f \left(\frac{1}{\gamma \sqrt{2k}} \sum_{i=1}^k (Y_i - 1) \right) \right| = 0 \quad a.s.$$

However, by the property of f and (C_3) , it is easy to see that

$$\begin{aligned} & \frac{1}{D_N} \sum_{k=1}^N d_k \left| f \left(\frac{1}{\gamma \sqrt{2k}} \sum_{i=1}^k \log Y_i \right) - f \left(\frac{1}{\gamma \sqrt{2k}} \sum_{i=1}^k (Y_i - 1) \right) \right| \\ & \leq \frac{1}{D_N} \sum_{k=1}^N d_k \left| \frac{1}{\gamma \sqrt{2k}} \sum_{i=1}^k \log Y_i - (Y_i - 1) \right| \\ & = O \left(\frac{1}{D_N} \sum_{k=1}^N \frac{D_k}{k(\log D_k)^\rho} \frac{(\log \log k) \log k}{\sqrt{k}} \right) \quad a.s. \end{aligned}$$

From Lemma 1, we know that $D_N = o(N^\varepsilon)$ for any $\varepsilon > 0$. Thus the following series is convergent,

$$\sum_{k=1}^N \frac{D_k}{k(\log D_k)^\rho} \frac{(\log \log k) \log k}{\sqrt{k}}$$

and (2.16) holds by the following fact

$$\frac{1}{D_N} \sum_{k=1}^N \frac{D_k}{k(\log D_k)^\rho} \frac{(\log \log k) \log k}{\sqrt{k}} \rightarrow 0.$$

3. ACKNOWLEDGEMENT

The author is supported by the Natural Science Foundation of Henan Education Department (No. 2009B110010).

REFERENCES

- [1] B.C. Arnold and J.A. Villaseñor, The asymptotic distribution of sums of records, *Extremes*, 1(3) (1998) 351-363.
- [2] I. Berkes and E. Csáki, A universal result in almost sure central limit theory. *Stoch. Model. Appl.*, 94 (2001) 105-134.
- [3] I. Berkes and L. Horváth, The logarithmic average of sample extremes is asymptotically normal. *Stoch. Model. Appl.*, 91(1) (2001) 77-98.
- [4] E. Csáki and K. Gonchigdanzan, Almost sure limit theorems for the maximum of stationary Gaussian sequences. *Statist. Probab. Lett.*, 58(2) (2002) 195-203.
- [5] K. Gonchigdanzan and G. A. Rempala, A note on the almost sure limit theorem for the product of partial sums. *Appl. Math. Lett.*, 19 (2006) 191-196.
- [6] S. Hörmann, An extension of almost sure central limit theory. *Statist. Probab. Lett.*, 76 (2006) 191-202.

- [7] M.T. Lacey and W. Philipp, A note on the almost sure central limit theorem. *Statist. Probab. Lett.*, 9 (1990) 201-205.
- [8] Y. Miao, Central limit theorem and almost sure central limit theorem for the product of some partial sums. *Proc. Indian Acad. Sci. Math. Sci.* 118(2) (2008) 289-294.
- [9] G.A. Rempala and J. Wesolowski, Asymptotics for products of sums and U -statistics. *Electron. Commun. Probab.*, 7 (2002) 47-54.