

ON THE DEFINITION OF a -UNIFORMLY NONSQUARE AND A GENERALIZED GAO CONSTANT

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ABSTRACT. We introduce a generalized concept of uniformly nonsquare which of course contains usual uniformly nonsquare, and a generalized Gao constant $E(a, X)$. We give the relations between this definition and some generalized constants, including $C_{NJ}(a, X)$, $J(a, X)$, and $E(a, X)$. We also give some properties of the generalized Gao constant $E(a, X)$. Finally, we compute $E(a, H)$ for Hilbert space and give some examples.

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1. INTRODUCTION AND PRELIMINARY

It is well known that the notion of normal structure and uniform normal structure play important role in metric fixed point theory(see [11]). And we have known that uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings(see [5]).

In this paper, we consider some generalized constants and give a more generalized definition of uniformly nonsquare which might imply some more generalized fixed point properties. Next we introduce a new coefficient $E(a, X)$ generalizing the Gao

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constant(see [7]) and give some properties, then we give the relations between the generalized uniformly nonsquare and $E(a, X)$, $C_{NJ}(a, X)$, $J(a, X)$. For a Banach space X , we show that if $E(a, X) < 2(1+a)^2$ for some $a \in [0, 1]$ then X possesses uniform normal structure. As an example, we compute $E(a, H)$ for all $a \in [0, 2]$ when H is a Hilbert space.

Throughout the paper we let X be a Banach space. By B_X and S_X we denote the closed unit ball and the unit sphere of X , resp. We shall say that a nonempty weakly compact convex subset C of X has the fixed point property(fpp for short) if every nonexpansive mapping $T : C \rightarrow C$ has a fixed point(that is, there exists $x \in C$ such that $T(x) = x$). Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. We shall say that X has the fixed point property(fpp) if every closed convex subset of X has the fpp. Let A be a nonempty bounded set in X . The number $r(A) = \inf\{\sup_{y \in A} \|x - y\| : x \in A\}$ is called the Chebyshev radius of A . The number $diam A = \sup\{\|x - y\| : x, y \in A\}$ is called the diameter of A . A Banach space X has normal structure if

$$r(A) < diam A. \quad (1.1)$$

for every bounded convex closed subset A of X with $diam A > 0$. When (1.1) holds for every weakly compact convex subset A of X with $diam A > 0$, we say X has weak normal structure. Normal structure and weak normal structure coincide if X is reflexive. A space X is said to have uniform normal structure if $\inf\{(diam A)/(r(A))\} > 1$, where the infimum is taken over all bounded convex closed subsets A of X with $diam A > 0$. Weak normal structure, as well as many other properties imply the fpp. The relevant papers are [4],[6],[13],[15],[16].

Some sufficient properties for a Banach space X to have uniformly normal structure are:

- (i) $J(X) < 3/2$ (see[9]) which is extended to $\frac{1+\sqrt{5}}{2}$ (see[8])
- (ii) $R(X) > 0$ (see[8])
- (iii) $C_{NJ}(X) < 5/4$ (see[12])
- (iv) X is an u-space, a class of spaces including uniformly convex spaces and uniformly smooth spaces. (see[9])
- (v) $C_{NJ}(a, X) < ((1+a)^2 + (3a-1)^2)/2(1+a^2)$ for some $a \in (1/2, 1]$ or $C_{NJ}(0, X) < (3 + \sqrt{5})/4$. (see[3])

(vi) $J(a, X) < (3 + a)/2$ for some $a \in [0, 1]$. (see[8])

A Banach space X is called uniformly nonsquare provided that there exists $\delta > 0$ such that if $x, y \in S_X$, then $\|x + y\|/2 \leq 1 - \delta$ or $\|x - y\|/2 \leq 1 - \delta$. In [5], Falset et al. prove that uniformly nonsquare Banach spaces have fpp for nonexpansive mappings.

In[7], J.Gao introduced the definition of the Gao constant, $E(X) = \sup\{\|x + y\|^2 + \|x - y\|^2 : x, y \in S_X\}$, and in[10], he proved an inequality among $J(X)$, $C_{NJ}(X)$ and $E(X)$, that is

$$(J(X))^2/2 \leq E(X)/4 \leq C_{NJ}(X). \quad (1.2)$$

In[8] Gao proved that $E(X) < 5$ implies X has normal structure, and Wang extended 5 to $3 + \sqrt{5}$ in [17].

One technique used in this paper is the “ultraproduct” technique. We refer to Askoy and Khamsi[1] and Sims[15] for a complete discussion on the topic. However, let us briefly recall the construction of an ultrapower of a Banach space X . As a first step we consider the space $l_\infty(X)$ consisting of all bounded sequences (x_n) of elements of X . The norm in $l_\infty(X)$ is given by the formula $\|(x_n)\| = \sup_{n \in N} \|x_n\|$, where N is the set of positive integers. Now, let U be an ultrafilter on N . The set

$$N = \{(x_n) \in l_\infty(X) : \lim_U (x_n) = 0\}$$

is a closed linear subspace of $l_\infty(X)$. Here, $\lim_U$ stands for the limit over the ultrafilter U . The ultrapower $\tilde{X}$ of X with respect to U is defined to be the quotient space $l_\infty(X)/N$. By $\tilde{x}$ we denote the equivalent class of $x = (x_n)$. From the definition of the quotient norm, we can derive the following canonical formula $\|\tilde{x}\| = \lim_U \|x_n\|$. Identifying an element $x \in X$ with the equivalence class of the constant sequence $(x, x, \dots)$, we can treat X as a subspace of $\tilde{X}$. In the following, we shall consider only non-trivial ultrafilters on the set of positive integers. Under this setting, the ultrapower $\tilde{X}$ is finitely representable in X . Consequently, $\tilde{X}$ inherits all finite-dimensional geometrical properties of X .

Definition 1.2 Let P be a Banach space property. We say a Banach space X has the property super- P if each Banach space finitely representable in X has property

P .

Theorem1.3 (See[1,Theorem 3.5]) Let X and Y be Banach spaces and suppose that Y is finitely representable in X . Then there is an ultrafilter U on the set N such that Y is isometrically isomorphic to a subspace of $\tilde{X}$.

We remark that when the property P is hereditary:that is, any subspace of a space with P also has P , one has the following stronger conclusion.

Corollary1.4 (See[1]) Let P be a Banach space property which is inherited by subspaces. Then a Banach space X has super- P if and only if every ultrapower $\tilde{X}$ of X has P .

Theorem1.5 (See[1]) Let X be a Banach space. If X has super-normal structure, then X has uniform normal structure.

2. a -UNIFORMLY NONSQUARE

We now extend the definition of uniformly nonsquare by taking any x, y, z instead of x, y analogically to $C_{NJ}(a, X)$ and $J(a, X)$.

Definition2.1 A Banach space X is called a -uniformly nonsquare if there exists $\delta > 0$, such that for any $x, y, z \in B_X$ with $\|y - z\| \leq a\|x\|$, either $\|x + y\|/2 < 1 - \delta$ or $\|x - z\|/2 < 1 - \delta$ holds, here $a \in [0, 2]$.

It is clear that a Banach space X is uniformly nonsquare if it is a -uniformly nonsquare for some $a \in [0, 2]$, so a -uniformly nonsquare Banach spaces also have fpp.

Theorem2.2 For any fixed a , a Banach space X is a -uniformly nonsquare iff it has one of next conditions.

- (i) $C_{NJ}(a, X) < 2$
- (ii) $J(a, X) < 2$

The proof is clear.

3. GENERALIZED GAO CONSTANT

We now introduce the definition and some properties of $E(a, X)$.

Definition3.1 Let X be a Banach space,
 $E(a, X) = \sup\{\|x + y\|^2 + \|x - z\|^2, \|y - z\| \leq a\|x\|, x, y, z \in B_X\}$, here $a \geq 0$.

First let's show some clear properties of $E(a, X)$:

Proposition3.2

- (i) $E(0, X) = E(X)$
- (ii) $4 \leq E(a, X) \leq 8$
- (iii) $E(a, X) = 8$ for $a \geq 2$
- (iv) $E(a, X)$ is a nondecreasing function with respect to a
- (v) For any fixed a, X is a -uniformly nonsquare iff $E(a, X) < 8$
- (vi) $E(a, X) = \sup\{\|x + y\|^2 + \|x - z\|^2, \|y - z\| \leq a\|x\|, x, y, z \in B_X \text{ and at least one belongs to } S_X\}$
- (vii) $(J(a, X))^2/2 \leq E(a, X)/4 \leq C_{NJ}(a, X)$

Example3.3 ($l_\infty - l_1$ norm) Let $X = R^2$ be equipped with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x_1 x_2 \geq 0 \\ \|x\|_1 & \text{if } x_1 x_2 \leq 0 \end{cases}$$

Take $x = (1, 1), y = (0, 1), z = (-1, 0)$. Then we have $y - z = (1, 1) = x$ and $\|x + y\| = 2, \|x - z\| = 2$. So $8 = \|x + y\|^2 + \|x - z\|^2 \leq E(1, X) \leq 8$. Hence $E(1, X) = 8$.

Remark: The above example is example 3.2 in [3], the authors note that $C_{NJ}(0, X)$ in the example is not exactly computed yet. However, if we take $x = (1, 1)$ and $y = (-1, 1)$, we can get $E(0, X) = 8$ directly.

Now we show that $E(\cdot, X)$ is a continuous function on $[0, \infty)$. The proof is analogous to proposition 3.3 in [3].

Proposition 3.4 $E(\cdot, X)$ is a continuous function on $[0, \infty)$.

Proof. We have noted that $E(\cdot, X)$ is nondecreasing, which we prove is continuous on $(0, \infty)$ first. Now suppose that for some $a > 0$,

$$\sup\{E(b, X), b < a\} = \alpha < \beta < \gamma = \inf\{E(b, X), b > a\}.$$

We choose $\gamma_n \downarrow a$ with $\gamma_n \leq a\sqrt{\beta/\alpha}$ and $x_n, y_n, z_n \in B_X$ such that $\|y_n - z_n\| = \gamma_n\|x_n\|$ and $g(x_n, y_n, z_n) \geq \beta$ for all $n \in N$. Here $g(x, y, z) = \|x + y\|^2 + \|x - z\|^2$. Choose $\eta_n \downarrow 1$ such that $\gamma_n/\eta_n < a$ and $\beta - \eta_n^2\alpha \geq 0$ for all n . Thus

$$g(\eta_n x_n, y_n, z_n) = \eta_n^2 g(x_n, y_n/\eta_n, z_n/\eta_n) \leq \eta_n^2 \alpha \quad \text{for all } n \in N.$$

Take a subsequence (n') of (n) such that all the sequences $\|x_{n'} + y_{n'}\|, \|x_{n'} - z_{n'}\|, \|x_{n'}\|, \|y_{n'}\|$ and $\|z_{n'}\|$ converge. As

$$\|x_n + \omega\| - (\eta_n - 1)\|x_n\| \leq \eta_n\|x_n + \omega\| \leq \|x_n + \omega\| + (\eta_n - 1)\|x_n\|.$$

for any $\omega \in X$, we have

$$\lim_{n'} \|\eta_{n'} x_{n'} + y_{n'}\| = \lim_{n'} \|x_{n'} + y_{n'}\|$$

and

$$\lim_{n'} \|\eta_{n'} x_{n'} - z_{n'}\| = \lim_{n'} \|x_{n'} - z_{n'}\|.$$

Consequently,

$$0 \leq \beta - \eta_n^2 \alpha \leq g(x_{n'}, y_{n'}, z_{n'}) - g(\eta_{n'} x_{n'}, y_{n'}, z_{n'}) \rightarrow 0$$

that is $\beta - \alpha = 0$, a contradiction.

Now we show that $E(\cdot, X)$ is continuous at 0. For $a = 0$, given $\varepsilon > 0$, we take $x_n, y_n, z_n \in B_X$ and

$$\|y_n - z_n\| = \alpha_n \|x_n\|, \quad \alpha_n \downarrow 0,$$

and

$$E(0+, X) - \varepsilon := \inf_{a>0} E(a, X) - \varepsilon < \lim_n g(x_n, y_n, z_n).$$

Putting $\varepsilon_n = 4\alpha_n + \alpha_n^2$, we have

$$g(x_n, y_n, z_n) \leq \|x_n + y_n\|^2 + \|x_n - y_n\|^2 + \varepsilon_n \leq E(0, X) + \varepsilon_n.$$

So $E(0+, X) - \varepsilon < E(0, X) \leq E(0+, X)$ and thus $E(0+, X) = E(0, X)$. Hence the proof is completed. $\square$

We are now ready to obtain an important tool. And in the proof of next theorem we still use the function $g(x, y, z)$ as above.

Theorem 3.5 $E(a, X) = E(a, \tilde{X})$.

Proof. Clearly $E(a, X) \leq E(a, \tilde{X})$. We now prove that $E(a, X) \geq E(a, \tilde{X})$.

Let $\delta > 0, \alpha \in [0, a]$ and suppose $\tilde{x}, \tilde{y}, \tilde{z} \in B_{\tilde{X}}$ and for which

$$\|\tilde{y} - \tilde{z}\| \leq a\|\tilde{x}\|.$$

If $\|\tilde{x}\| = 0$, then

$$g(\tilde{x}, \tilde{y}, \tilde{z}) \leq 4 \leq E(a, X).$$

Now if $\|\tilde{x}\| > 0$, take $\varepsilon > 0$, such that $\varepsilon < \delta\|\tilde{x}\|$. Since

$$C := \|\tilde{x} + \tilde{y}\|^2 + \|\tilde{x} - \tilde{z}\|^2 = \lim_U \|x_n + y_n\|^2 + \|x_n - z_n\|^2 = \lim_U C_n$$

the set

$$A_1 := \{n \in N : |C_n - C| < \delta \quad \text{and} \quad \|y_n - z_n\| \leq a\|x_n\| + \varepsilon\}$$

belongs to U . And for each $r > 0$,

$$\{n : \delta\|x_n\| > \delta\|\tilde{x}\| - r\} \in U,$$

take r such that $\delta\|\tilde{x}\| - r > \varepsilon$ and we have the set $A_2 := \{n : \delta\|x_n\| > \varepsilon\}$ belongs to U

and thus

$$A_2 \cap A_1 = \{n : \|y_n - z_n\| \leq a\|x_n\| + \varepsilon < (a + \delta)\|x_n\|\} \in U.$$

In particular,

$$C < g(x_n, y_n, z_n) + \delta \leq E(a + \delta, X) + \delta$$

for some n . So we have $E(a, \tilde{X}) \leq E(a, X)$ by the arbitrariness of δ and the continuity of $E(\cdot, X)$. $\square$

Now we consider the relation between $E(a, X)$ and uniform normal structure. We start from a lemma.

Lemma3.6(See[17]) Let X be a Banach space without weak normal structure, then for any $0 < \varepsilon < 1$ and each $0 \leq r \leq 1$, there exist $x_1 \in S_X$ and $x_2, x_3 \in rS_X$ satisfying

- (i) $x_2 - x_3 = ax_1$ with $|a - r| < \varepsilon$
- (ii) $\|x_1 - x_2\| > 1 - 3\varepsilon$
- (iii) $\|x_1 + x_2\| > (1 + r) - 3\varepsilon, \|x_3 - x_1\| > (1 + r) - 3\varepsilon$.

We now obtain sufficient conditions for X to have uniform normal structure.

Theorem3.7 Let X be a Banach space, if $E(r, X) < 2(1+r)^2$ for some $r \in [0, 1]$, then X has uniform normal structure.

Proof. It suffices to show that this condition implies X has normal structure. As then by Theorem3.5, it follows that $\tilde{X}$ also has normal structure, so X has super-normal structure, by Corollary1.4, and hence X has uniform normal structure by Theorem1.5.

The case $E(r, X) < 2(1+r)^2 \leq 4$ implies X is a -uniformly nonsquare, so X is uniformly nonsquare and so reflexive. Thus, normal structure and weak normal structure coincide. It then suffices to prove that X has weak normal structure.

By the continuity of $E(\cdot, X)$, $E(r', X) < 2(1+r)^2$ for some $r' > r$. Choose $m \in \mathbb{N}$ such that $r + 1/m < r'$.

Suppose now X does not have weak normal structure, by lemma 3.6 there exist $x_n \in S_X$ and $y_n, z_n \in rS_X$ such that, for each $n \in \mathbb{N}$, $y_n - z_n = a_n x_n$ with $|a_n - r| < \frac{1}{n+m}$,

$$\|x_n - y_n\|^2 > (1 - \frac{3}{n+m})^2, \|x_n + y_n\|^2 > (1 + r - \frac{3}{n+m})^2,$$

and

$$\|x_n - z_n\|^2 > (1 + r - \frac{3}{n+m})^2.$$

Observe that

$$\|y_n - z_n\| = a_n < r + \frac{1}{n+m} \leq r'$$

and

$$\liminf_n \|x_n + y_n\|^2 \geq (1+r)^2, \quad \liminf_n \|x_n - y_n\|^2 \geq (1+r)^2.$$

Thus

$$2(1+r)^2 \leq \liminf_n \|x_n + y_n\|^2 + \|x_n - y_n\|^2 \leq E(r', X) < 2(1+r)^2$$

a contradiction. $\square$

Remark: Since it is proved in [2] that $J(a, X) < (3+a)/2$ implies uniformly normal structure and $2(J(a, X))^2 \leq E(a, X)$, $E(a, X) < 2(1+a)^2$ implies $J(a, X) < (3+a)/2$. This result is a corollary of a result in reference 2.

Corollary 3.8 If $E(\cdot, X)$ is concave and $E(a, X) < (3 + \sqrt{5})(1-a) + 8a$ for some $a \in [0, 1]$, then X has normal structure.

Proof. If $E(1, X) < 8$, we have done by theorem 3.7. Let $E(1, X) = 8$ and suppose that X does not have normal structure. Therefore $E(X) = E(0, X) \geq 3 + \sqrt{5}$. By the concavity of $E(\cdot, X)$, we have for all $a \in [0, 1]$,

$$E(a, X) \geq (1-a)E(0, X) + aE(1, X) \geq (3 + \sqrt{5})(1-a) + 8a$$

a contradiction. $\square$

Now we show the value of $E(a, H)$ when H is a Hilbert space as an example.

Theorem 3.9 Let H be a Hilbert space, $E(a, H) = 4 + 2a$ for any $a \in [0, 2]$.

Proof. For any $x, y, z \in B_H$ with $\|y - z\| \leq a\|x\|$,

$$\|x + y\|^2 + \|x - z\|^2 = 2\|x\|^2 + \|y\|^2 + \|z\|^2 + 2\|x\|\|y - z\| \leq 4 + 2\|x\|\|y - z\| \leq 4 + 2a.$$

On the other hand, take e_1 and e_2 be orthonormal elements of S_H . Put

$$x = e_1, y = \frac{a}{2}e_1 + \sqrt{1 - \frac{a^2}{4}}e_2, z = \frac{-a}{2}e_1 + \sqrt{1 - \frac{a^2}{4}}e_2.$$

Thus we have $\|y - z\| = a\|x\|$ and $\|x + y\|^2 + \|x - z\|^2 = 4 + 2a$. And we complete the proof. $\square$

When we take $a = 1$, and we have $E(1, H) = 6 < 8 = 2(1 + 1)^2$, so we get:

Corollary 3.10 A Hilbert space has uniform normal structure.

Finally we use the theorem 3.15 in [3] and vii of proposition 3.2 as another example.

Theorem 3.11 If X is an u-space, then it has $E(a, X) < 8$ for all $0 < a < 2$.

Proof. We have known that If X is an u-space, then it has $C_{NJ}(a, X) < 2$ for all $0 < a < 2$, and vii of proposition 3.2 shows that $E(a, X)/4 \leq C_{NJ}(a, X)$, so we complete the proof. □

Finally, we would like to post some open questions:

Question 1: What is the best boundary for constant a which implies uniformly normal structure?

Question 2: What is relationship between a -uniform nonsquare and uniformly normal structure?

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