

MEASURE-THEORETICAL SENSITIVITY AND SCRAMBLED SETS VIA FURSTENBERG FAMILIES

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ABSTRACT. For an invariant measure μ in a topological dynamic system, notions of $\mathcal{F}$ - μ -pairwise sensitivity and $\mathcal{F}$ - μ -sensitivity are introduced and investigated, where $\mathcal{F}$ is a Furstenberg family. It is shown that $\mathcal{F}$ - μ -pairwise sensitivity is equivalent to $\mathcal{F}$ - μ -sensitivity, when $\mathcal{F}$ is a filterdual and is extensively compatible to the 2-fold product system. Moreover, it turns out that if $(X, \mathcal{B}, \mu, f)$ is a measure-preserving system of weakly mixing and the support of μ is X , then there exists a positive c such that (X, f) has an $(M_*(c), M_*(0))$ -scrambled set with full induced outer measure by μ . $M_*(c)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose lower density is not less than c , and $M_*(0)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose lower density is non-zero.

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1. INTRODUCTION

Throughout this paper a topological dynamical system is a pair (X, f) (TDS for short), where X is a compact metric space with a metric d and f is a continuous surjective map from X to itself. Let $\mathcal{B}$ be the σ -algebra of Borel subsets of X . We

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use $M(X, f)$ to denote the collection of all f -invariant Borel probability measures.

Let $\mathbb{Z}_+$ be the set of non-negative integers, and let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is called a Furstenberg family [1] if it is hereditary upwards, i.e. $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. Let $\mathcal{F}_{inf}$ be the family of all infinite subsets of $\mathbb{Z}_+$. It is easy to see that $\mathcal{F}_{inf}$ is a Furstenberg family.

The sensitivity ([2][3]) is widely understood to be one of central ideas of chaos. In recent years, the sensitivity has attracted much attention ([4][5][6]). People have tried to give various definitions of sensitivity according to their understanding, for instance, family $\mathcal{F}$ -sensitivity ([1][7]); pairwise sensitivity [8]; measure-theoretical sensitivity [9].

Now we recall some notions of sensitivity.

A TDS (X, f) is weakly $\mathcal{F}$ -sensitive ([1][7]) if there is a positive ε —a weakly $\mathcal{F}$ -sensitive constant—such that for every nonempty open subset U of X there exist x and y of U such that

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F},$$

where $\mathcal{F}$ is a Furstenberg family.

A TDS (X, f) is $\mathcal{F}$ -sensitive [7] if there is a positive ε —a $\mathcal{F}$ -sensitive constant—such that for every $x \in X$ and every open neighborhood U of x there exists $y \in U$ such that

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F},$$

where $\mathcal{F}$ is a Furstenberg family.

Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\mathcal{B}_X^+$ be the collection of Borel sets of X with positive measures.

A TDS (X, f) is pairwise sensitive for μ or μ -pairwise sensitive [8] if there is $\varepsilon > 0$ —a μ -pairwise sensitive constant—such that for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$ one can find $n \geq 0$ with $d(f^n(x), f^n(y)) > \varepsilon$.

A TDS (X, f) is μ -sensitive [9] if there is $\varepsilon > 0$ — a μ -sensitive constant— such that for any $A \in \mathcal{B}_X^+$ one can find $x, y \in A$ and $n \geq 0$ with

$$d(f^n(x), f^n(y)) > \varepsilon.$$

In [8] it is proved that any measure-preserving system of weakly mixing is μ -pairwise sensitive. Moreover, it turns out that μ -sensitivity is equivalent to μ -pairwise sensitivity in [9].

Following these ideas which come from ([7] [8] [9]), we have:

Definition 1.1 Let (X, f) be a TDS and $\mu \in M(X, f)$. Suppose that $\mathcal{F}$ is a Furstenberg family. A TDS (X, f) is said to be $\mathcal{F}$ - μ -sensitive if there is $\varepsilon > 0$ —an $\mathcal{F}$ - μ -sensitive constant—such that for any $A \in \mathcal{B}_X^+$ there exist $x, y \in A$ with

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}.$$

Definition 1.2 Let (X, f) be a TDS and $\mu \in M(X, f)$. Suppose that $\mathcal{F}$ is a Furstenberg family. A TDS (X, f) is said to be $\mathcal{F}$ - μ -pairwise sensitive if there is $\varepsilon > 0$ —an $\mathcal{F}$ - μ -pairwise sensitive constant—such that for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$ with

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}.$$

Now we recall the notion of $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled set [10].

Let (X, f) be a TDS. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are two Furstenberg families. A point pair (x, y) is said to be an $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled pair, if there is $\varepsilon > 0$ such that $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}_1$ and $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \delta\} \in \mathcal{F}_2$ for any $\delta > 0$. A $(\mathcal{F}_{inf}, \mathcal{F}_{inf})$ -scrambled pair is usually Li-Yorke scrambled pair. An $(\overline{M}(1), \overline{M}(1))$ -scrambled pair is usually distributively scrambled pair [11], where $\overline{M}(1)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose upper density is 1.

A subset K of X is said to be $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled set, if (x, y) is an $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled pair for any two distinct points x, y of K . A TDS (X, f) is said to be $(\mathcal{F}_1, \mathcal{F}_2)$ -chaotic, if it contains an uncountable $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled set. In [10], a TDS is constructed which is $(M^*(1), M^*(\frac{1}{\sqrt{e}}))$ -chaotic but not distributively chaotic [11], where $M^*(a)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose upper density is

not less than a .

In this article we investigate the $\mathcal{F}$ - μ -sensitivity, $\mathcal{F}$ - μ -pairwise sensitivity and the measure of $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled sets. This article is organized as follows. In section 2, some basic notions related to Furstenberg families are introduced. In section 3, we study $\mathcal{F}$ - μ -sensitivity and $\mathcal{F}$ - μ -pairwise sensitivity. It is shown that $\mathcal{F}$ - μ -pairwise sensitivity is equivalent to $\mathcal{F}$ - μ -sensitivity, when $\mathcal{F}$ is a filterdual and is extensively compatible to the 2-fold product system (see Corollary 3.8). Finally, the existence of scrambled sets with full measure is discussed. It is proved that if $(X, \mathcal{B}, \mu, f)$ is a measure-preserving system of weakly mixing, and the support of μ is X , then there exists a positive c such that (X, f) has an $(M_*(c), M_*(0))$ -scrambled set with full induced outer measure by μ . $M_*(c)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose lower density is not less than c , and $M_*(0)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose lower density is non-zero (see Theorem 4.6).

2. PRELIMINARIES

In this section, we introduce some basic notions related to Furstenberg families (for details see [1]).

For a Furstenberg family $\mathcal{F}$, its dual family is

$$k\mathcal{F} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset, \forall F' \in \mathcal{F}\} = \{F \in \mathcal{P} : \mathbb{Z}_+ - F \notin \mathcal{F}\}.$$

Clearly, if $\mathcal{F}$ is a Furstenberg family then so is $k\mathcal{F}$. It is easy to see that $k\mathcal{P} = \emptyset$, $k\emptyset = \mathcal{P}$. Clearly, $k(k\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}_1 \subset \mathcal{F}_2$ implies $k\mathcal{F}_2 \subset k\mathcal{F}_1$. It is easy to see that $k\mathcal{F}_{inf}$ is the family of all cofinite subsets of $\mathbb{Z}_+$.

A Furstenberg family $\mathcal{F}$ is proper if it is a proper subset of $\mathcal{P}$. It is easy to see that a Furstenberg family $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.

For Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$, let $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. A Furstenberg family $\mathcal{F}$ is filter if it is proper and $\mathcal{F} \supset \mathcal{F} \cdot \mathcal{F}$. A Furstenberg family $\mathcal{F}$ is filterdual if $k\mathcal{F}$ is filter. Moreover, A Furstenberg family $\mathcal{F}$ is filterdual if and only if $\mathcal{F}$ has Ramsey property, i.e. if $F_1 \cup F_2 \in \mathcal{F}$, then $F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$.

Any subset $\mathcal{A}$ of $\mathcal{P}$ can generate a Furstenberg family $[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$. It is the smallest Furstenberg family which contains $\mathcal{A}$. It is easy to see that $\mathcal{A} = \emptyset$ if and only if $[\mathcal{A}] = \emptyset$, and $\emptyset \in \mathcal{A}$ if and only if $[\mathcal{A}] = \mathcal{P}$. So $[\mathcal{A}]$ is a proper Furstenberg family if and only if $\mathcal{A} \neq \emptyset$ and $\emptyset \notin \mathcal{A}$. A Furstenberg family $\mathcal{F}$ is countably generated ([1][12]) if there exists a countable subset $\mathcal{A}$ of $\mathcal{P}$ such that $[\mathcal{A}] = \mathcal{F}$. Clearly, $k\mathcal{F}_{inf}$ is a countably generated proper family.

Let $J \subset \mathbb{Z}_+$. The upper density of J is

$$\overline{d}(J) = \limsup_{n \rightarrow \infty} \frac{\sharp(J \cap \{0, 1, \dots, n-1\})}{n}$$

where $\sharp$ denotes the cardinality of the set. The lower density of J is

$$\underline{d}(J) = \liminf_{n \rightarrow \infty} \frac{\sharp(J \cap \{0, 1, \dots, n-1\})}{n}.$$

If $\overline{d}(J) = \underline{d}(J)$, then we say J has density $d(J)$.

Let (X, f) be a TDS and $U, V \subset X$. We define the meeting time set

$$N(U, V) = \{n \in \mathbb{Z}_+ : f^n(U) \cap V \neq \emptyset\}.$$

Especially, $N(x, V) = \{n \in \mathbb{Z}_+ : f^n(x) \in V\}$ for $x \in X$.

Let $A \subset X$ and $x \in X$. If $N(x, A) \in \mathcal{F}$, x is called an $\mathcal{F}$ -attaching point of A . The set of all $\mathcal{F}$ -attaching points of A is called the set of $\mathcal{F}$ -attaching of A , denoted by $\mathcal{F}(A)$. Clearly,

$$\mathcal{F}(A) = \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} f^{-n}(A) = \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} f^{-n}(A).$$

Let (X, f) be a TDS and $F \subset \mathbb{Z}_+$. A pair $(x, y) \in X \times X$ is called F -proximal if

$$\liminf_{F \ni n \rightarrow \infty} d(f^n(x), f^n(y)) = 0.$$

We denote the set of all F -proximal pairs by P_F .

We shall use following some relations on X .

$$\overline{V}_\varepsilon = \{(x, y) : d(x, y) \leq \varepsilon\}, \quad V_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$$

Let $\mathcal{F}$ be a Furstenberg family. A pair $(x, y) \in X \times X$ is called $\mathcal{F}$ -proximal if $(x, y) \in \mathcal{F}(V_\varepsilon)$ for any $\varepsilon > 0$. We denote the set of all $\mathcal{F}$ -proximal pairs by $P_{\mathcal{F}}$.

Note ([13]):

$$\begin{aligned}
 P_{\mathcal{F}} &= \bigcap_{\varepsilon > 0} \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} (f \times f)^{-n}(V_\varepsilon) \\
 &= \bigcap_{k=1}^{\infty} \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} (f \times f)^{-n}(V_{\frac{1}{k}}) \\
 &= \bigcap_{k=1}^{\infty} \mathcal{F}(V_{\frac{1}{k}}) \\
 &= \bigcap_{F \in k\mathcal{F}} P_F
 \end{aligned}$$

3. MEASURE-THEORETICAL SENSITIVITY VIA FURSTENBERG FAMILIES

In this section, we research the concepts of $\mathcal{F}$ - μ -sensitivity and $\mathcal{F}$ - μ -pairwise sensitivity via Furstenberg families.

By concepts of $\mathcal{F}$ - μ -sensitivity and weakly $\mathcal{F}$ -sensitivity, it is clear that following proposition holds.

Proposition 3.1 Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\text{supp}(\mu) = X$. Suppose that $\mathcal{F}$ is a Furstenberg family. If (X, f) is $\mathcal{F}$ - μ -sensitive, then (X, f) is weakly $\mathcal{F}$ -sensitive.

Definition 3.2 Let (X, f) be a TDS. A Furstenberg family $\mathcal{F}$ is extensively compatible to the system (X, f) if the set of $\mathcal{F}$ -attaching of U is a Borel set of X for each open set U of X .

Proposition 3.3 Let (X, f) be a TDS. A Furstenberg family $\mathcal{F}$ is extensively compatible to the system (X, f) if and only if the set of $k\mathcal{F}$ -attaching of B is a Borel set of X for each closed set B of X .

Proof. Suppose that B is a closed subset of X . Then

$$\begin{aligned}
 x \in k\mathcal{F}(B) &\Leftrightarrow N(x, B) \in k\mathcal{F} \\
 &\Leftrightarrow \mathbb{Z}_+ \sim N(x, X \sim B) \in k\mathcal{F} \\
 &\Leftrightarrow N(x, X \sim B) \in \mathcal{F} \\
 &\Leftrightarrow x \in \mathcal{F}(X \sim B)
 \end{aligned}$$

Hence, $k\mathcal{F}(B) = X \sim \mathcal{F}(X \sim B)$. Since $\mathcal{F}(X \sim B)$ is a Borel set of X , so is $k\mathcal{F}(B)$.

Remark In [12], the notion of a Furstenberg family $\mathcal{F}$ is compatible to a system (X, f) is introduced. A Furstenberg family $\mathcal{F}$ is compatible to a system (X, f) if the set of $\mathcal{F}$ -attaching of U is a G_δ set of X for each open set U of X . Clearly, if a Furstenberg family $\mathcal{F}$ is compatible to a system (X, f) , then the Furstenberg family $\mathcal{F}$ is extensively compatible to the system (X, f) .

Proposition 3.4 Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\text{supp}(\mu) = X$. Suppose that $\mathcal{F}$ is a filterdual. If (X, f) is $\mathcal{F}$ - μ -pairwise sensitive, then (X, f) is $\mathcal{F}$ -sensitive.

Proof. Assume that (X, f) is $\mathcal{F}$ - μ -pairwise sensitive, then there is $\varepsilon > 0$ such that for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$ with $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}$. Since $\text{supp}(\mu) = X$, then for every x of X and every neighborhood U of x , we have $\mu \times \mu(U \times U) > 0$. Thus there exist $y, z \in U \times U$ such that $\{n \in \mathbb{Z}_+ : d(f^n(y), f^n(z)) > \varepsilon\} \in \mathcal{F}$.

if either of y, z is x , then the proposition is true.

if neither of y, z is x . Since $\mathcal{F}$ is filterdual, by the triangle inequality, then at least one of $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon/2\} \in \mathcal{F}$ and $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(z)) > \varepsilon/2\} \in \mathcal{F}$ is true. Thus (X, f) is $\mathcal{F}$ -sensitive.

Let (X, f) be a TDS and $\mu \in M(X, f)$. Suppose that Furstenberg family $\mathcal{F}$ is extensively compatible to the system $(X \times X, f \times f)$.

For $x \in X$ and $\varepsilon > 0$, let

$$E_\varepsilon(x) = \{y \in X : (x, y) \in k\mathcal{F}(\overline{V}_\varepsilon)\},$$

and consider the map $\Phi_{\mu,\varepsilon} : X \rightarrow R_+$ defined by

$$\Phi_{\mu,\varepsilon}(x) = \mu(E_\varepsilon(x)) \text{ for any } x \in X.$$

It is clear that $\Phi_{\mu,\varepsilon}$ is measurable and according to Fubini's Theorem,

$$\mu \times \mu(k\mathcal{F}(\overline{V}_\varepsilon)) = \int_X \Phi_{\mu,\varepsilon}(x) d\mu(x). \quad (*)$$

Proposition 3.5 Let (X, f) be a TDS and $\mu \in M(X, f)$. Suppose that Furstenberg family $\mathcal{F}$ is extensively compatible to the system $(X \times X, f \times f)$. Then (X, f) is $\mathcal{F}$ - μ -pairwise sensitive with $\mathcal{F}$ - μ -pairwise sensitive constant ε if and only if $\Phi_{\mu,\varepsilon}(x) = 0, \mu - a.e. x \in X$.

Proof. By $(*)$, $\mu \times \mu(k\mathcal{F}(\overline{V}_\varepsilon)) = \int_X \Phi_{\mu,\varepsilon}(x) d\mu(x)$. So (X, f) is $\mathcal{F}$ - μ -pairwise sensitive $\Leftrightarrow \mu \times \mu(k\mathcal{F}(\overline{V}_\varepsilon)) = 0 \Leftrightarrow \Phi_{\mu,\varepsilon}(x) = 0, \mu - a.e. x \in X$.

Proposition 3.6 Let (X, f) be a TDS and $\mu \in M(X, f)$. Suppose that $\mathcal{F}$ is a Furstenberg family. If (X, f) is $\mathcal{F}$ - μ -pairwise sensitive, then (X, f) is $\mathcal{F}$ - μ -sensitive.

Proof. Suppose that (X, f) is $\mathcal{F}$ - μ -pairwise sensitive, then there is $\varepsilon > 0$ such that for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$ with

$$\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}.$$

Since $\mu \times \mu(A \times A) > 0$ for every $A \in \mathcal{B}_X^+$, there exists $(x, y) \in A \times A$ such that $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}$. So (X, f) is $\mathcal{F}$ - μ -sensitive.

Theorem 3.7 Let (X, f) be a TDS and $\mu \in M(X, f)$. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are Furstenberg families, $\mathcal{F}_1$ is extensively compatible to the system $(X \times X, f \times f)$ and $k\mathcal{F}_1 \cdot k\mathcal{F}_1 \subset k\mathcal{F}_2$. If (X, f) is $\mathcal{F}_2$ - μ -sensitive, then (X, f) is $\mathcal{F}_1$ - μ -pairwise sensitive.

Proof. Assume that (X, f) is $\mathcal{F}_2$ - μ -sensitive. Then there is $\varepsilon > 0$ such that for any $A \in \mathcal{B}_X^+$ there exist $x, y \in A$ with $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}_2$. Next we shall show that (X, f) is $\mathcal{F}_1$ - μ -pairwise sensitive with $\mathcal{F}_1$ - μ -pairwise sensitive

constant $\varepsilon/2$. By Proposition 3.5 assume the contrary that there exists $x \in X$ with $\Phi_{\mu, \varepsilon/2}(x) > 0$. That is $\mu(A) > 0$, where $A = E_{\varepsilon/2}(x) = \{y \in X : (x, y) \in k\mathcal{F}_1(\overline{V}_{\varepsilon/2})\}$ and $\overline{V}_{\varepsilon/2} = \{(x, y) : d(x, y) \leq \varepsilon/2\}$. Then for any $y, z \in A$ we have $(x, y) \in k\mathcal{F}_1(\overline{V}_{\varepsilon/2})$, $(x, z) \in k\mathcal{F}_1(\overline{V}_{\varepsilon/2})$. Assume that

$$F_1 = \{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) \leq \varepsilon/2\} \in k\mathcal{F}_1,$$

$$F_2 = \{n \in \mathbb{Z}_+ : d(f^n(x), f^n(z)) \leq \varepsilon/2\} \in k\mathcal{F}_1.$$

Since $k\mathcal{F}_1 \cdot k\mathcal{F}_1 \subset k\mathcal{F}_2$, then $F_1 \cap F_2 \in k\mathcal{F}_2$. So $\{n \in \mathbb{Z}_+ : d(f^n(y), f^n(z)) \leq \varepsilon\} \in k\mathcal{F}_2$. Thus $\{n \in \mathbb{Z}_+ : d(f^n(y), f^n(z)) > \varepsilon\} \notin \mathcal{F}_2$, which contradicts the assumption that (X, f) is $\mathcal{F}_2$ - μ -sensitive with ε being an $\mathcal{F}_2$ - μ -sensitive constant. Therefore (X, f) is $\mathcal{F}_1$ - μ -pairwise sensitive.

Corollary 3.8 Let (X, f) be a TDS and $\mu \in M(X, f)$. Suppose that $\mathcal{F}$ is a filterdual, and is extensively compatible to the system $(X \times X, f \times f)$. Then (X, f) is $\mathcal{F}$ - μ -sensitive if and only if it is $\mathcal{F}$ - μ -pairwise sensitive.

Let $(X, \mathcal{B}, \mu, f)$ be a measure-preserving system and $\mathcal{F}$ be a Furstenberg family. A measure-preserving system $(X, \mathcal{B}, \mu, f)$ is $\mathcal{F}$ -ergodic [14] if any $A \in \mathcal{B}_X^+$ and any $F \in k\mathcal{F}$ we have $\mu(\bigcup_{i \in F} f^{-i}(A)) = 1$. A measure-preserving system $(X, \mathcal{B}, \mu, f)$ is $\mathcal{F}$ -mixing [14] if $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, f \times f)$ is $\mathcal{F}$ -ergodic.

Proposition 3.9 Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\text{supp}(\mu) = X$. Suppose that $\mathcal{F}$ is a Furstenberg family, and $k\mathcal{F}$ is a countably generated proper family. If $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is $\mathcal{F}$ -mixing, then there exists $\varepsilon > 0$ such that for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$ with $\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}$, i.e., (X, f) is $\mathcal{F}$ - μ -pairwise sensitive.

Proof. Choose $\varepsilon > 0$ such that $X \times X - \overline{V}_\varepsilon \neq \emptyset$, where $\overline{V}_\varepsilon = \{(x, y) : d(x, y) \leq \varepsilon\}$. We note that

$$\mathcal{F}(X \times X - \overline{V}_\varepsilon) = \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} (f \times f)^{-n}(X \times X - \overline{V}_\varepsilon).$$

As f is $\mathcal{F}$ -mixing, then we have $\mu \times \mu(\bigcup_{n \in F} (f \times f)^{-n}(X \times X - \overline{V}_\varepsilon)) = 1$ for any $F \in k\mathcal{F}$. Let $\mathcal{A}$ be a countable generated set of $k\mathcal{F}$. Hence

$$\mu \times \mu(\mathcal{F}(X \times X - \overline{V}_\varepsilon)) = \mu \times \mu\left(\bigcap_{F \in \mathcal{A}} \bigcup_{n \in F} (f \times f)^{-n}(X \times X - \overline{V}_\varepsilon)\right) = 1.$$

Proposition 3.10 Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\text{supp}(\mu) = X$. If $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is weakly mixing, then there exist $\varepsilon > 0$ and $c > 0$ such that for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$ with $\underline{d}(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\}) = c$, where $\underline{d}(J)$ denotes the lower density of the set J .

Proof. Choose $\varepsilon > 0$ such that $E = X \times X - \overline{V}_\varepsilon \neq \emptyset$, where $\overline{V}_\varepsilon = \{(x, y) : d(x, y) \leq \varepsilon\}$. Since $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is weakly mixing, then $f \times f : (X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu) \rightarrow (X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$ is ergodic. By Birkhoff Ergodic Theorem we have $\frac{1}{n} \sum_{i=0}^{n-1} (\chi_E(f \times f)^i(x, y)) \rightarrow f^* \in L^1(\mu \times \mu)$ for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$. Since $f^* = \int \chi_E(x, y) d\mu \times \mu = \mu \times \mu(E) = c > 0$ for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$, then $\underline{d}(\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) > \varepsilon\}) = c$ for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$.

Definition 3.11 Let $(X, \mathcal{B}_X, \mu_1)$ and $(Y, \mathcal{B}_Y, \mu_2)$ be two measure-preserving systems. A transformation $\phi : (X, \mathcal{B}_X, \mu_1) \rightarrow (Y, \mathcal{B}_Y, \mu_2)$ is positive-measure-preserving, if $\phi(A) \in \mathcal{B}_Y^+$ for any $A \in \mathcal{B}_X^+$.

Proposition 3.12 Let (X, f) be a TDS and $\mu_1 \in M(X, f)$. And let (Y, g) be a TDS and $\mu_2 \in M(Y, g)$. Suppose that $\phi : (X, f) \rightarrow (Y, g)$ is a factor map, and $\phi : (X, \mathcal{B}_X, \mu_1) \rightarrow (Y, \mathcal{B}_Y, \mu_2)$ is a positive-measure-preserving transformation. If (Y, g) is $\mathcal{F}$ - μ_2 -sensitive, then (X, f) is $\mathcal{F}$ - μ_1 -sensitive, where $\mathcal{F}$ is a Furstenberg family.

Proof. Suppose that $\varepsilon > 0$ is $\mathcal{F}$ - μ_2 -sensitive constant. Since factor map ϕ is continuous, then there exists $\delta > 0$ such that $d_1(x, y) > \delta$ when $d_2(\phi(x), \phi(y)) > \varepsilon$. Since ϕ is positive-measure-preserving, then $\phi(A) = B \in \mathcal{B}_Y^+$ for any $A \in \mathcal{B}_X^+$. Since (Y, g) is $\mathcal{F}$ - μ_2 -sensitive, then there are $y_1, y_2 \in B$ such that $\{n \in \mathbb{Z}_+ : d_2(g^n(y_1), g^n(y_2)) > \varepsilon\} \in \mathcal{F}$. Put $x_1, x_2 \in A$ such that $\phi(x_1) = y_1, \phi(x_2) = y_2$,

then $\{n \in \mathbb{Z}_+ : d_1(f^n(x_1), f^n(x_2)) > \delta\} \in \mathcal{F}$, this implies (X, f) is $\mathcal{F}$ - μ_1 -sensitive.

4. MEASURE OF $(\mathcal{F}_1, \mathcal{F}_2)$ -SCRAMBLED SETS

This section deals with the measure of $(\mathcal{F}_1, \mathcal{F}_2)$ -scrambled sets of some measure-preserving systems.

Suppose that $(X, \mathcal{S}, \mu)$ is a measure space. For any set $A \subset X$ let $\mu^*(A) = \inf\{\mu(B) : B \in \mathcal{S}, B \supset A\}$, then μ^* is an outer measure on X , and the restriction of μ^* on $\mathcal{S}$ coincides with μ . The μ^* is said to be an induced outer measure by μ .

Suppose that $(X, \mathcal{S}, \mu)$ is a probability space. $\mathcal{D} \subset \mathcal{S}$ is said to be a regular base of the measure μ , if for any $\varepsilon > 0$ and any $B \in \mathcal{S}$, there exists $F \in \mathcal{D}$ such that $F \supset B$ and $\mu(F - B) < \varepsilon$.

We need the following lemmas that come from [15] and [16] respectively.

Lemma 4.1 [15] Suppose that μ is Borel probability measure on a metric space X , i.e. $(X, \mathcal{B}, \mu)$ is a probability space, where $\mathcal{B}$ is the Borel σ -algebra. Then the topology of the metric space X (i.e. the collection of all open subsets of X) is a regular base of the measure μ .

If a metric space X is separable, then the cardinality of its topology is not greater than the cardinality of the continuum. Therefore, each Borel probability space on a separable metric space has a regular base whose cardinality is not greater than the cardinality of the continuum.

Lemma 4.2 [16] Suppose that $(X, \mathcal{S}, \mu)$ is a measure space with a regular base $\mathcal{D}$ whose cardinality is not greater than the cardinality of the continuum. Suppose that $\alpha = \{\mathcal{R}_\gamma\}_{\gamma \in \Gamma}$ is a countable family of relations of the full measure on $(X, \mathcal{S}, \mu)$ (i.e. for each $\gamma \in \Gamma$, there exists some $s_\gamma \geq 1$ such that $\mathcal{R}_\gamma \subset X^{s_\gamma}$ and $\mu^{s_\gamma}(\mathcal{R}_{s_\gamma}) = 1$, where μ^{s_γ} is the product measure of s_γ measures μ and X^{s_γ} is the product space of s_γ spaces X). Let μ^* be the outer measure induced by μ . Then

there exists an α -dependent set K of the full outer measure (i.e. $K \subset X$ satisfies that $\mu^*(K) = 1$ such that for each $\gamma \in \Gamma$ and any s_γ distinct points $x_1, x_2, \dots, x_{s_\gamma}$ in K , we have $(x_1, x_2, \dots, x_{s_\gamma}) \in \mathcal{R}_\gamma$).

Proposition 4.3 Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\text{supp}(\mu) = X$. If $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is weakly mixing, then $\mu \times \mu(P_{M_*(0)}) = 1$. $P_{M_*(0)}$ denotes the set of all $M_*(0)$ -proximal pairs, and $M_*(0)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose lower density is non-zero.

Proof. For any $k > 0$ let $V_{\frac{1}{k}} = \{(x, y) : d(x, y) < \frac{1}{k}\}$. Since $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is weakly mixing, then $f \times f : (X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu) \rightarrow (X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$ is ergodic. By Birkhoff Ergodic Theorem we have $\frac{1}{n} \sum_{i=0}^{n-1} (\chi_{V_{\frac{1}{k}}}(f \times f)^i(x, y)) \rightarrow f^* \in L^1(\mu \times \mu)$ for $\mu \times \mu - a.e. (x, y) \in X \times X$. Since $f^* = \int \chi_{V_{\frac{1}{k}}}(x, y) d\mu \times \mu = \mu \times \mu(V_{\frac{1}{k}}) > 0$ for $\mu \times \mu - a.e. (x, y) \in X \times X$. Thus $d_*\{n \in \mathbb{Z}_+ : d(f^n(x), f^n(y)) < \frac{1}{k}\} > 0$ for $\mu \times \mu - a.e. (x, y) \in X \times X$, i.e. $\mu \times \mu(M_*(0)(V_{\frac{1}{k}})) = 1$. Hence $\mu \times \mu(P_{M_*(0)}) = \mu \times \mu(\bigcap_{k=1}^{\infty} M_*(0)(V_{\frac{1}{k}})) = 1$.

Proposition 4.4 Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\text{supp}(\mu) = X$. Suppose that $\mathcal{F}$ is a Furstenberg family and $k\mathcal{F}$ is a countably generated proper family. If $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is $\mathcal{F}$ -mixing, then the set of all $(\mathcal{F}, \mathcal{F})$ -scrambled pairs is a full measure set of $X \times X$.

Proof. Suppose that $k\mathcal{F}$ is a countably generated proper family by $\mathcal{A}$, where $\mathcal{A}$ is a countable set. Then

$$P_{\mathcal{F}} = \bigcap_{k=1}^{\infty} \bigcap_{F \in k\mathcal{F}} \bigcup_{n \in F} (f \times f)^{-n}(V_{\frac{1}{k}}) = \bigcap_{k=1}^{\infty} \bigcap_{F \in \mathcal{A}} \bigcup_{n \in F} (f \times f)^{-n}(V_{\frac{1}{k}}).$$

Since $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is $\mathcal{F}$ -mixing, then we have $\mu \times \mu(\bigcup_{i \in F} (f \times f)^{-i}(V_{\frac{1}{k}})) = 1$ for any $F \in k\mathcal{F}$. So $\mu \times \mu(P_{\mathcal{F}}) = 1$. By Proposition 3.9 there is $\varepsilon > 0$ such that $\mu \times \mu(\mathcal{F}(X \times X - \overline{V_\varepsilon})) = 1$. Hence the proposition holds.

Proposition 4.5 Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\text{supp}(\mu) = X$. Suppose that $\mathcal{F}$ is a Furstenberg family and $k\mathcal{F}$ is a countably generated proper family. Let μ^* be an outer measure induced by μ . If $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is $\mathcal{F}$ -mixing, then

(X, f) has an $(\mathcal{F}, \mathcal{F})$ -scrambled set $K \subset X$ with $\mu^*(K) = 1$.

Proof. As X is a compact metric space, then $(X, \mathcal{B}, \mu)$ has a regular base $\mathcal{D}$ whose cardinality is not greater than the cardinality of the continuum. Let $\mathcal{R}_2 = \{(x, y) : (x, y) \text{ is an } (\mathcal{F}, \mathcal{F})\text{-scrambled pair}\}$, $\mathcal{R}_n = X^n$ when $n \geq 3$. By Proposition 4.4, $\mu^2(\mathcal{R}_2) = 1$. Clearly, $\mu^n(\mathcal{R}_n) = 1$ for any $n \geq 3$. By Lemma 4.2, there exists an $(\mathcal{F}, \mathcal{F})$ -scrambled set $K \subset X$ such that $\mu^*(K) = 1$.

By Proposition 3.10, Proposition 4.3 and Lemma 4.2 we have also:

Theorem 4.6 Let (X, f) be a TDS, $\mu \in M(X, f)$ and $\text{supp}(\mu) = X$. Suppose that μ^* is an outer measure induced by μ . If $f : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is weakly mixing, then there exists a positive c such that (X, f) has an $(M_*(c), M_*(0))$ -scrambled set $K \subset X$ with $\mu^*(K) = 1$. $M_*(c)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose lower density is not less than c , and $M_*(0)$ denotes the family of all subsets of $\mathbb{Z}_+$ whose lower density is non-zero.

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