LIMSUP SHADOWING PROPERTY AND CHAOTIC BEHAVIORS

F. H. GHANE
FERDOWSI UNIVERSITY OF MASHHAD
E-MAIL: GHANE@MATH.UM.AC.IR
A. FAKHARI
DAMGHAN UNIVERSITY OF BASIC SCIENCE
E-MAIL: FAKHARI@DUBS.AC.IR
A. SARIZADEH
FERDOWSI UNIVERSITY OF MASHHAD
E-MAIL: AL_ SA45@STU-MATH.UM.AC.IR

Abstract. Here, we study the kind of shadowing, limsup shadowing, with ignoring a finite number of shadowing errors. It is shown that the limsup shadowing property (Abbrev. LSSP) is stronger than the pseudo orbit tracing property (Abbrev. POTP) in the shadowing way and discuss the relationship between the POTP and the LSSP. First, we present some of its general properties and then we characterize this notion of the shadowing in terms of chaos.

AMS Classification: 34D05, 37C50
Keywords: limsup shadowing property, Chain component, Topological mixing, Topological transitive, Lyapunov stable

1. INTRODUCTION

An important goal in the theory of dynamical systems is to discuss the chaotic dynamics. In the study of chaotic systems, direct analytical methods are not available. Moreover, chaotic systems are closely related with shadowing property and

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 6, NUMBER 2 (2008) 113-121.

topological transitivity (see [4, 5] and references therein).

The term chaos in a connection with a map was first used by Li and Yorke [9]. Today, chaotic systems have been studied extensively by many authors. Some authors try to find some kinds of shadowing which imply topological transitivity.

In a recent work [6], Gu introduced the notion of the asymptotic average shadowing property (Abbrev. AASP) and investigated the relation between the AASP and transitivity. By adding the strong condition, $\mathcal{L}$-hyperbolicity, Gu proved that the asymptotic average shadowing property is topologically transitive.

In the other note [7], he proved that a Lyapunov stable system with having the average-shadowing property is topologically ergodic.

In numerical techniques it seems that the calculation of average and asymptotic average shadowing errors is not so easy. To overcome this problem, we introduce the kind of shadowing, limsup shadowing, with ignoring a finite number of shadowing errors.

In section 3, it is shown that the limsup shadowing property is stronger than the pseudo orbit tracing property in the shadowing way and discussed the relationship between the POTP and the LSSP.

We will assert the chain mixing property is deduced from the LSSP, in section 4. In particular, we prove that on a compact metric space X, if a homeomorphism f has the LSSP then it is topologically mixing. This implies that it has sensitive dependence to initial conditions and so it is chaotic in the sense of Li-Yorke and Auslander-Yorke. Moreover, if f is expansive and has the LSSP then f satisfies in specification property.

2. Some basic terminology

Let (X, d) be a compact metric space and let $f : X \to X$ be a homeomorphism of X onto itself. If $x \in X$ then the orbit of x under f is defined by

$$\mathcal{O}(x, f) = \{f^i(x) : i \in Z\}.$$

A point $x \in X$ is called non-wandering if for any neighborhood U of x, there exists natural number n such that $f^n(U) \cap U \neq \emptyset$. The set of all non-wandering points denoted by $\Omega(f)$.

For $\delta > 0$ a sequence $\xi = \{x_i\}_{i=a}^{b}$ is called δ-pseudo orbit from x_a to x_b if $d(f(x_i), x_{i+1}) < \delta$, for any $-\infty \leq a \leq i < b \leq \infty$. A homeomorphism f has the shadowing property, if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each δ-pseudo orbit $\xi = \{x_i\}_{i=-\infty}^{\infty}$ there exists $x \in X$ such that $d(f^i(x), x_i) < \epsilon$, for any integer i.

We say that $x \sim y$ if for each $\delta > 0$, there exist finite δ-pseudo orbits from x to y and conversely, from y to x. A point $x \in X$ is a chain-recurrent point of f, if $x \sim x$. The set of all chain-recurrent points of f is denoted by $CR(f)$. Clearly $\Omega(f) \subseteq CR(f)$; proper inclusion may be occurs. However, it is well known for any homeomorphism f with the shadowing property $\Omega(f) = CR(f)$ (see [10]). Any equivalence class of relation $\sim$ is called a chain component. A closed invariant set Λ of X is chain transitive, if for each $x, y \in \Lambda$, $x \sim y$. Clearly, each chain component is chain transitive, (see [11]). A map f is said to be chain transitive if X is a chain transitive set of f.

The map f is called topologically transitive if for any two nonempty open subsets V and U in X, there is a positive integer k such that $f^k(U) \cap V \neq \emptyset$. It is well known that a map f is topologically transitive if and only if there is a point x in X such that $\overline{\mathcal{O}(x, f)} = X$, (see [1]).

A map f is called topologically mixing if for any two nonempty open subsets V and U in X, there is a positive integer N such that $f^n(U) \cap V \neq \emptyset$, for every integer $n \geq N$.

It is well known [6, 12] that topologically mixing property implies topologically transitive property.

The continuous and surjective mapping f is called chaotic in the sense of Li and Yorke if X contains an uncountable scrambled set. It means that, there is an uncountable set such that for each x, y belong to it we have

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0 \quad , \quad \limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0.$$

The map f is called Lyapunov ϵ-unstable at $x \in X$, if for every neighborhood U of x, there is $y \in U$ and $n \geq 0$ with $d(f^n(x), f^n(y)) > \epsilon$. The map f is called unstable at a point $x \in X$ if there is $\epsilon > 0$ such that f is Luapunov ϵ-unstable at x,A map f is called chaotic in the sense of Auslander and Yorke , if every point is unstable and X contains a dense orbit. Here, sensitive dependence to initial conditions means that there exists $\epsilon > 0$ such that f is Lyapunov ϵ-unstable at every point in X.

3. Limsup shadowing property and its basic properties

At first, we introduce the notion of the limsup shadowing.

Definition 3.1. A sequence $\{x_i\}_{-\infty}^{\infty}$ in X is called δ-limsup pseudo orbit of f if

$$d(f(x_i), x_{i+1}) < \delta \quad ; \quad \forall \quad |i| \geq N,$$

for some positive integer N.
We say that f has the limsup shadowing property, if for any $\epsilon > 0$, there exists $\delta > 0$ such that each δ-limsup pseudo orbit $\{x_i\}_{-\infty}^{\infty}$ is ϵ-limsup traced, that is, there is $x \in X$ with

$$\limsup\{d(f^i(x), x_i) : \quad i \in \mathbb{Z}\} < \epsilon,$$

i.e. there exists positive integer M so that,

$$d(f^i(x), x_i) < \epsilon \quad ; \quad \forall |i| > M.$$

Denoting by $\omega(x)$ (resp. $\alpha(x)$) the ω-limit (α-limit) set of x. The following theorem shows that a system (X, f) with the limsup shadowing property is chain transitive.

Theorem 3.2. *Let f be a homeomorphism of a compact metric space X. If f has the limsup shadowing property, then f has only one chain component which is the whole of X.*

Proof. First, we show that f has only one chain component. Suppose that Λ_1 and Λ_2 are two distinct chain components of f. Let $x \in \Lambda_1$, $y \in \Lambda_2$ and $\epsilon > 0$ be given. To get a contradiction, we show that there exists an ϵ-pseudo orbit from x to y and conversely, from y to x. For this, we construct a sequence $\{x_i\}_{-\infty}^{\infty}$ as follows. Let $x_0 = x$ and $x_1 = y$. Put

$$x_{-i} = f^{-i}(x) \quad , \quad x_i = f^{i-1}(y) \quad ; \forall \ i \geq 1.$$

Clearly, $\{x_i\}_{-\infty}^{\infty}$ is an δ-limsup pseudo orbit of f, for each $\delta > 0$. Since f has limsup shadowing property, $\{x_i\}_{-\infty}^{\infty}$ is ϵ-limsup-shadowed by some point $z \in X$, that is, there exists $N > 0$ such that,

$$d(f^i(z), x_i) < \epsilon \quad ; \quad \forall \ |i| \geq N.$$

This gives an ϵ-pseudo orbit from $f^{-N-1}(x)$ to $f^N(y)$ such as follows:

$$f^{-N-1}(x), f^{-N}(z), f^{-N+1}(z), ..., z, f(z), ..., f^N(z), f^N(y).$$

Now, since each chain component is invariant so there are ϵ-pseudo orbits from x to $f^{-N-1}(x)$ and also from $f^N(y)$ to y. By concatenating three ϵ-pseudo orbits, we obtain an ϵ-pseudo orbit from x to y. Similarly, there exists an ϵ-pseudo orbit from y to x and this implies that x and y are in the same chain component. Thus f has only one chain component.

Now, we proof that $CR(f) = X$. Let $x \in X \backslash CR(f), y \in \omega(x)$ and $z \in \alpha(x)$. It is obvious that there exist two δ-pseudo orbits from x to y and from z to x. Since $\alpha(f) \cup \omega(f) \subseteq CR(f)$, so y and z belong to $CR(f)$.

By the above argument, $CR(f)$ has only one chain component and this implies that, there is an δ-pseudo orbit from y to z. By concatenating them, we obtain an δ-pseudo orbit from x to x.

This complete the proof of Theorem. □

Remark 3.3. It is known that if the chain recurrent set of a C^1-diffeomorphism f on a compact manifold M, $CR(f)$, has a hyperbolic structure, then f has a finite number of chain components, (see [11]). Moreover, f has the POTP.

There are many examples with more than one hyperbolic chain recurrent set.

However, f has only one chain component when f has the LSSP, by above theorem.

This shows that, in general, the POTP can not be equal to the LSSP.

So, it is natural asking whether the LSSP implies the POTP. It is given a positive answer to it, i.e. it is proved that the POTP is strictly weaker than the limsup shadowing property.

The following proposition is cited from [2].

Proposition 3.4. *Let X be a compact metric space. If to every $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that for each $k > 0$, each δ-pseudo orbit $\{x_i : 0 \leq i \leq k\}$ for f is ϵ-traced by some point, then f has the POTP.*

Theorem 3.5. *Let f be a homeomorphism on a compact metric space X. If f has the limsup shadowing property then f has the POTP.*

Proof. Suppose f has the LSSP. Let $\epsilon > 0$ be given. Then, there is $\delta > 0$ such that every δ-limsup pseudo orbit is ϵ-limsup traced. Let k be a positive integer and

$\{x_i : 0 \leq i \leq k\}$ be a δ-pseudo orbit. By Theorem 3.2 there is $l > 0$ and δ-pseudo orbit $\{y_i\}_{i=0}^{l}$ from x_k to x_0.

We construct an δ-limsup pseudo orbit $\{w_i\}_{-\infty}^{\infty}$ as follows:

$$\begin{array}{lcll} w_i & = & f^i(x_0) & if \;\; i \leq 0 \\ w_i & = & x_{[i \mod k+l]} & if \;\; [i \mod k+l] \in \{0, 1, \ldots, k\} \\ w_i & = & y_{[i \mod k+l]} & if \;\; [i \mod k+l] \in \{k+1, \ldots, k+l\}. \end{array}$$

Clearly, $\{w_i\}_{-\infty}^{\infty}$ is an δ-limsup pseudo orbit for f. Thus, there is $w \in X$ and positive integer N such that $d(f^i(w), w_i) < \epsilon$, for all $|i| \geq N$.

This shows that δ-pseudo orbit $\{x_i : 0 \leq i \leq k\}$ for f is ϵ-traced by $f^j(w)$, for some integer $j \geq N$. By Proposition 3.4, homeomorphism f has the POTP.

This gives the result. $\square$

Remark 3.6. It is known that [1], if $f : X \longrightarrow X$ has the POTP, then $CR(f) = \Omega(f)$. Now by Theorems 3.2 and 3.5, if f has the LSSP then $\Omega(f) = X$.

Proposition 3.7. *Let f be a chain transitive homeomorphism on a compact metric space X. If f has the POTP, then f has the limsup shadowing property.*

Proof. Let $\epsilon > 0$ be given and $\delta > 0$ corresponds to ϵ as in definition of the POTP. Suppose that $\{x_i\}_{-\infty}^{\infty}$ is an δ-limsup pseudo orbit in X. Then, there exists a positive integer N such that $d(f(x_i), x_{i+1}) < \delta$, for each $|i| \geq N$. Since f is chain transitive, we can find an δ-pseudo orbit $\{y_i\}_{i=0}^{k}$ from x_{-N+1} to x_N. By concatenating these sequences, we construct an δ-pseudo orbit $\{w_i\}_{-\infty}^{\infty}$ such as follows:

$$\ldots, x_{-i}, x_{-i+1}, \ldots, x_{-N}, y_0, y_1, \ldots, y_k, x_{N+1}, \ldots \;.$$

Since f has shadowing property, then δ-pseudo orbit $\{w_i\}_{-\infty}^{\infty}$ is ϵ-shadowed by a point $w \in X$. This implies that the δ-limsup pseudo orbit $\{x_i\}_{-\infty}^{\infty}$ can be ϵ-limsup-shadowed by w. $\square$

Now, we give a mapping with the POTP but not possessing the limsup shadowing property.

Example 3.8. Let k be a fixed integer and put $X = \{0, 1, ..., k-1\}$ with discrete topology. Consider the product space $Y = \prod_{-\infty}^{\infty} X$, equipped with product topology. It is known [15] that a subshift $\sigma : Y \to Y$ has the shadowing property if and only if it is finite type. Now, we give an example of a subshift of finite type which dose not have the limsup shadowing property. Let $X = \{0, 1, 2, 3\}$ and σ be a shift homeomorphism on X_4^Z. Take

$$Y_1 = \{(x_i)_{-\infty}^{\infty} : \ x_i = 0, 1\} \quad , \quad Y_2 = \{(x_i)_{-\infty}^{\infty} : \ x_i = 2, 3\}.$$

Then we have the following assertions:

(i) $\tilde{\sigma} = \sigma|_{Y_1 \cup Y_2}$ is a subshift of finite type. Thus $\sigma|_{Y_1 \cup Y_2}$ has the shadowing property.

(ii) $\tilde{\sigma}$ dose not have the limsup shadowing property. Indeed, $\tilde{\sigma}$ has at least two chain components Y_1 and Y_2. By Theorem 3.2, $\tilde{\sigma}$ has not the LSSP.

(iii) $\sigma|_{Y_1}$ and $\sigma|_{Y_2}$ have the LSSP.

Example 3.9. In [10], sufficient conditions for the limit shadowing property in one dimensional systems are obtained. As a corollary, it is shown that if a homeomorphism of the circle has the POTP, then it has the limit shadowing property. Pilyugin in [10] presented a homeomorphism on $\mathbb{S}^1$ with the shadowing property and two distinct chain components. By Theorem 3.2, this homeomorphism has not the limsup shadowing property. However, by Plamenevskaya' result, it has the limit shadowing property. So the limit shadowing property is not equivalent to the limsup shadowing property, generally.

4. LSSP and chaotic behaviors

The main result of this section presents the relationship between the limsup shadowing property, chain mixing and topological mixing properties. The fundamental properties of the LSSP are listed in the following theorem; the proof is easy and we omit it (note that, we can apply the argument used for the POTP in [2], with any modifications).

Theorem 4.1. *(1) f has the limsup shadowing property if and only if, for any positive integer k, f^k has the limsup shadowing property.*

(2) The limsup shadowing property is invariant by conjugacy.

A map f is called chain mixing if for any $\epsilon > 0$ and any $x, y \in X$ there is a positive integer N such that for any integer $n \geq N$ there is an ϵ-chain from x to y with length n.

Theorem 4.2. *Let $f : X \to X$ be a homeomorphism on a compact metric space X. If f has the limsup shadowing property, then f is chain mixing.*

Proof. By Theorem 3.2, if f has the limsup shadowing property, then f is chain transitive. The above Remark implies that f^n is chain transitive, for each $n \in N$. Now, by a result of Yang [13] total chain transitivity is equivalent to chain mixing property. □

The following Lemma is due to Yang [13].

Lemma 4.3. *If f has the POTP, then f is chain mixing if and only if f is topologically mixing.*

Corollary 4.4. *If f has the limsup shadowing property, then f is topologically mixing. In particular, f has sensitive dependence to initial conditions.*

Proof. By Theorem 3.5, the limsup shadowing implies the POTP. So by Theorem 4.2 and Lemma 4.3 f is topologically mixing. Now by a result of Raldjane [8], sensitive dependence to initial conditions follows by topological mixing property. □

Note that, if f has sensitive dependence to initial conditions then a source of error in numerical techniques arises. Since numerical methods introduce errors, it is virtually guaranteed that a numerically computed solution diverges away from the exact solution.

A continuous mapping f from a nondegenerate compact metric space X to itself has the specification property if for any $\epsilon > 0$ there exists $M \in N$ such that for any $k \geq 2$, for any k points $x_1, x_2, ..., x_k \in X$, for any nonnegative integers $a_1 \leq b_1 < a_2 \leq b_2 < ... < a_k \leq b_k$ with $a_i - b_{i-1} \geq M$ for each $i = 2, 3, ..., k$ and for any $p \geq M + b_k - a_1$, there exists a point $y \in X$ such that $f^p(y) = y$ and $d(f^n(y), f^n(x_i)) \leq \epsilon$ for all $a_i \leq n \leq b_i$, $1 \leq i \leq k$.

Corollary 4.5. *Let f be a homeomorphism on a compact metric space X. If f is expansive and has the LSSP, then f satisfies in specification property.*

Proof. It is known that if f is a topologically mixing expansive map with the POTP, then f satisfies specification (see Theorem 9.25 of [2]). Now Corollary 4.5 and Theorem 3.5 give the result. □

Let f be a non-hyperbolic systems with the POPT. Thus there exists a chain component Ω of the system f where $f|_\Omega$ is non-hyperbolic with the POPT. Moreover, since $f|_\Omega$ have one chain component with the POPT then f is a chaotic non-hyperbolic system.

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