

FINER STRUCTURE OF THE MULTIFRACTAL SPECTRUM OF LOCAL ENTROPIES

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ABSTRACT. We consider a finer decomposition for the multifractal spectrum of local entropies $\mathcal{E}(\alpha) =: h_{top}(f, K_\alpha := \{x : h_\mu(f, x) = \alpha\})$. This consists in taking a partition for each multifractal set of the form $K_\alpha = \bigcup_{V \in \mathbf{R}^+} (K_{\alpha, V} \cup \mathcal{I})$. If $\mathcal{E}(\alpha, V)$ denotes $h_{top}(f, K_{\alpha, V})$, then is established that $\mathcal{E}(\alpha, V) = \mathcal{E}(\alpha)$ for a set of values of V .

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1. INTRODUCTION

The study of the nature of fractal sets by mean of invariants like for instance Hausdorff dimension or topological entropy deserved much attention in the last two decades by Physicists as well as Mathematicians. In this way emerged an interesting subarea in the field of Dimension Theory of Dynamical Systems, the *Multifractal Analysis*. It essentially consists in giving a decomposition, called a *multifractal decomposition*, of invariant sets into subsets with some scaling property. Primary

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works were done in the physical context [12], [13], [1], [2], in which was considered an attractor carrying an invariant measure μ which scales with an exponent α in a scale level r and a decomposition in sets

$$(1) \quad K_\alpha = \{x : \mu(B_r(x)) \sim r^\alpha \text{ as } r \rightarrow 0\}.$$

where $B_r(x)$ is the ball of centre x and radius r .

In [3] was presented the general concept and the a rigorous mathematical description of multifractal analysis. This was introduced by giving a set X and a map $g : X \rightarrow [-\infty, +\infty]$ and considering the level sets

$$K_\alpha = K_\alpha(g) = \{x : g(x) = \alpha\},$$

and the decomposition $X = \left(\bigcup_\alpha K_\alpha\right) \cup Y$, where Y is the set in which g is not defined, are considered. If G is a function defined on sets, and $F(\alpha) = G(K_\alpha)$, then the map F is called *the multifractal spectrum* specified by the pair (g, G) . Main interesting cases are

- The *dimension multifractal spectrum*: here $g(x) = D_\mu(x)$, the dimension of the measure μ and $F(\alpha) = \dim_H K_\alpha$ the Hausdorff dimension of the set K_α . This spectrum was previously studied for particular cases in [9], [13].
- The *multifractal spectrum of local entropies*: In this case g is taken as the local entropy of a dynamical map f and $F(\alpha)$ as the Bowen topological entropy (for non-compact sets) of the level sets.

The Hausdorff dimension and the topological entropy are special cases of “characteristic dimensions” in metric spaces. Thus there is a close relationship between the fields of multifractal analysis and dimension theory of dynamical systems. To study the complexity of the multifractal decomposition sets is necessary the knowledge of suitable characteristic dimensions. The description of multifractal spectra can be obtained by mean of family of measures $\{\mu_\alpha\}_{\alpha \in \mathbf{R}}$ such that $\mu_\alpha(K_\alpha) = 1$, for this must be found a parametrization $\alpha(q)$ of the level sets with $\mu_q(K_{\alpha(q)}) = 1$ and $\mu_q(K_\alpha) = 0$ if $\alpha \neq \alpha(q)$.

In [23], [3] were performed complete multifractal descriptions for conformal repellers and Axion-A diffeomorphisms on surfaces, while in [26] was developed a multifractal formalism for the spectrum of local entropies under weaker conditions

than imposed in [23], [3]. Other remarkable works in these subjects are [4], [27], [29], [27] among others. We have been working with the formalism by Takens and Verbitski [26] to describe multifractal spectra [17], [20] as well as to connect with other fields, like for instance the study of growth rates of Kleinian action groups on the hyperbolic plane or to establish formal thermodynamic relationships [18], [20], [21]. More recently Pesin and Zhang [24] developed a formalism for continuous maps on compact spaces which admit the so called inducing schemes. Other subject which drawn the attention was the study of the “dimensionless” of subsets of the multifractal sets, this kind of problems were treated in [5], [22], [11].

In [16] Jordan and Pollicott investigated a finer multifractal decomposition for dimension spectrum of C^2 unidimensional maps. In this article we do a similar analysis for the spectrum of local entropies and for more general dynamical systems which satisfy the properties of expansiveness and specification. Thus we shall consider the multifractal sets

$$K_\alpha = \{x : h_\mu(f, x) = \alpha\},$$

where $h_\mu(f, x)$ is the local measure-theoretic entropy. Further we take a decomposition of each level set of the form $K_\alpha = \bigcup_{V \in \mathbf{R}^+} (K_{\alpha, V} \cup \mathcal{I})$ (in the next section we shall specify these sets) and analyze the spectrum $h_{top}(f, K_{\alpha, V})$ obtaining a suitable family of measures to describe it.

2. PRELIMINARIES

Let us begin by recalling the main aspects of the multifractal formalism for the spectrum of local entropies by Takens and Verbitski [26] and the necessary background to understand it. Let X be a compact metric space, and $f : X \rightarrow X$ a homeomorphism, let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a partition of X , for any $x \in X$ can be assigned a word $\underline{j} = j_0 j_1 \dots j_{n-1}$ of length n (we denote $x \leftrightarrow \underline{j} = j_0 j_1 \dots j_{n-1}$ by $f^i(x) \in A_{j_i}$, $i = 0, 1, \dots, n-1$). The *cylinder of length n* containing the element $x \leftrightarrow \underline{j}$, denoted by $\mathcal{C}_n = \mathcal{C}_{n, \underline{j}}(x)$, is the set formed for all the points $y \leftrightarrow \underline{j}$.

The *upper and lower measure-theoretic local entropies* are

$$\overline{h}_\mu(f, x, \mathcal{A}) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup -\frac{1}{n} \log \mu(\mathcal{C}_n(x))$$

$$\underline{h}_\mu(f, x, \mathcal{A}) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu(\mathcal{C}_n(x)).$$

If μ is a f -invariant measure, then (Brin-Katok theorem [8]), the local entropy does exist, i.e. $\overline{h}_\mu(f, x, \mathcal{A}) = \underline{h}_\mu(f, x, \mathcal{A}) := h_\mu(f, x, \mathcal{A})$, for μ -a.e. $x \in X$ and if μ is ergodic then for μ -a.e. $x \in X$ holds $h_\mu(f, x, \mathcal{A}) = h_\mu(f, \mathcal{A})$ (the usual Kolmogorov measure-theoretic entropy of f with respect to the measurable partition $\mathcal{A}$).

A partition $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ is a *generator* if for any bi-infinite word $\dots j_{-1} j_0 j_1 \dots$ the intersection $\bigcap_{n=-\infty}^{\infty} f^{-n}(A_{j_n})$ is just a single point. Let us recall that a homeomorphism $f : X \rightarrow X$ is called *expansive* if there is a constant $\delta > 0$, such that $d(f^n(x), f^n(y)) < \delta$, for any integer n implies $x = y$, and that a homeomorphism expansive on a compact metric space has a generator [28]. This generator is given by a partition by cylinders with diameter $\leq \delta$. If $\mathcal{A}$ is a generating partition then $h_\mu(f, x, \mathcal{A}) = h_\mu(f, x)$, for μ -a.e.

Now the *local entropies spectrum* of f is that specified by the pair $(h_\mu(f, x), \mathcal{E}(\alpha))$ with $\mathcal{E}(\alpha) := h_{\text{top}}(f, K_\alpha)$. The set X is naturally decomposed as

$$X = \bigcup_{\alpha=-\infty}^{\infty} K_\alpha \cup (X - Y),$$

where Y is the set in which the local entropy map does not exist and is usually called the *irregular part* of the spectrum. By the Brin-Katok theorem $\mu(X - Y) = 0$, for any f -invariant measure μ . As definition of *topological pressure* [25] associated to f and a potential $\varphi : X \rightarrow \mathbf{R}$, can be taken the variational form

$$(2) \quad P(\varphi) = \sup_{\mu} \left\{ h_\mu(f) + \int \varphi d\mu \right\},$$

where the supreme is taken over all the f -invariant Borel measures μ on X .

An *equilibrium state* for the potential φ is a measure μ_φ for which:

$$(3) \quad P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi.$$

The set of equilibrium states for the potential φ will be denoted by $\mathcal{M}_\varphi(X)$.

A homeomorphism $f : X \rightarrow X$ has the *specification property* [7].if: $I_1, \dots, I_k$ is a finite disjoint collection of integer intervals $I_1, \dots, I_k$, then for $\varepsilon > 0$, there is a integer $M(\varepsilon)$ and a function $\Phi : I = \cup I_i \rightarrow X$, such that:

- i) $\text{dist}(I_i, I_j) > M(\varepsilon)$ (Euclidean distance)
- ii) $f^{n_1 - n_2}(\Phi(n_1)) = \Phi(n_2)$

iii) $d(f^n(x), \Phi(n)) < \varepsilon$, for some $x : f^m(x) = x$, with $m \geq M(\varepsilon) + \text{length}(I)$ and for every $n \in I$.

This fulfillment of this condition ensures abundance of periodic points

The *statistical sum* of a potential φ is

$$(4) \quad S_n(\varphi)(x) = \sum_{i=0}^{n-1} \varphi(f^i(x)).$$

and says that a potential φ belongs to the class $\nu_f(X)$ [15] or [26], if there is a constant $K > 0$ such that if x, y belong to the same cylinder of length n then

$$|S_n(\varphi)(x) - S_n(\varphi)(y)| < K$$

Let $\varphi \in \nu_f(X)$ and let $P_n(f) = \{x : f^n(x) = x\}$, then set

$$(5) \quad \mu_{\varphi,n}(A) = \frac{1}{\tilde{Z}(f, \varphi, n)} \sum_{x \in P_n(f)} \exp(S_n(\varphi)(x)) \delta_x(A),$$

where $\tilde{Z}(f, \varphi, n) = \sum_{x \in P_n(f)} \exp(S_n(\varphi)(x))$ and δ_x is the unit measure at x :

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

If X is compact the sequence $\{\mu_{\varphi,n}\}$ has an accumulation point and under the above conditions it has a weak limit μ_φ , i.e.:

$$(6) \quad \lim_{n \rightarrow \infty} \int \psi(x) d\mu_{\varphi,n} = \int \psi(x) d\mu_\varphi$$

for every continuous ψ ([15, 25]).

Definition: A f -invariant measure μ is a *Gibbs state* if sufficiently small $\varepsilon > 0$, there are constants $A_\varepsilon, B_\varepsilon > 0$, such that for any $x \in X$ and for any positive integer n :

$$A_\varepsilon (\exp(S_n(\varphi)(x)) - nP(\varphi)) \leq \mu(\mathcal{C}_n(x)) \leq B_\varepsilon (\exp(S_n(\varphi)(x)) - nP(\varphi)),$$

Theorem ([15, 25]): Let f be an expansive homeomorphism with the specification property and φ a potential belonging to the class $\nu_f(X)$, then μ_φ is the unique equilibrium state associated to φ . Besides μ_φ is ergodic.

Next recall the definition of the Bowen topological entropy for non-compact nor invariant sets [6]. Let $f : X \rightarrow X$ be a continuous map and (X, d) a compact metrizable space. Let $\mathcal{U} = \{U_1, \dots, U_N\}$ be finite covering of X and a string is defined as a sequence

$L = (\ell_0, \ell_1, \dots, \ell_{n-1})$ such that $\{U_{\ell_0}, \dots, U_{\ell_{n-1}}\} \subset \mathcal{U}$, $\ell_i \in \{1, 2, \dots, N\}$. The length of the string

$L = (\ell_0, \ell_1, \dots, \ell_{n-1})$ is denoted by $n(L) = n$. Let call $W_n(\mathcal{U})$ the set of the strings L with length n for the covering $\mathcal{U}$.

Let

$$X(L) = U_{\ell_0} \cap f^{-1}(U_{\ell_1}) \cap \dots \cap f^{-n+1}(U_{\ell_{n-1}}),$$

if $Z \subset X$ it says that $\Pi = \{L = (\ell_0, \ell_1, \dots, \ell_{n-1})\}$ covers Z if

$$Z \subset \bigcup_{L \in \Pi} X(L).$$

For any real number s :

$$M(Z, \mathcal{U}, s, n) = \inf_{\Pi} \sum_{L \in \Pi} \exp(-sn(L)),$$

where the infimum is taken over all the collections of strings $\Pi \subset W_n(\mathcal{U})$ which covers Z .

Let

$$M(\mathcal{U}, Z, s) = \lim_{n \rightarrow \infty} M(\mathcal{U}, Z, s, n).$$

There is an unique number $\bar{s}$ such that $M(\mathcal{U}, Z, \cdot)$ jumps from $+\infty$ to 0, now let

$$h_{top}(f, Z, \mathcal{U}) = \bar{s} = \sup \{s : M(\mathcal{U}, Z, s) = +\infty\} = \inf \{s : M(\mathcal{U}, Z, s) = 0\}.$$

Finally the number

$$(7) \quad h_{top}(f, Z) = \lim_{\Delta(\mathcal{U}) \rightarrow 0} h_{top}(f, Z, \mathcal{U}), \quad \Delta(\mathcal{U}) = \text{diameter of } \mathcal{U}$$

is the *topological entropy of f* restricted to the set Z .

The conditions of expansiveness and specification are much weaker than the existence of Markov partitions. We review now the main results of the multifractal formalism for local entropies developed by Takens and Verbitski [26], which was done for expansive homeomorphism with the specification property and potentials in the class $\nu_f(X)$.

Let $T(q) = P(q\varphi) - qP(\varphi)$, $q \in \mathbf{R}$, called the *free energy* of φ ,

i) the function $T(q)$ is convex and continuously differentiable with $T'(q) = \int \varphi d\mu_q - P(\varphi)$, where μ_q is the Gibbs state for the potential $\varphi_q := q\varphi - qP(\varphi)$.

. This map has a Legendre transform $\mathcal{E}(\alpha) = \inf_{q \in \mathbf{R}} \{q\alpha - T(q)\}$, $\mathcal{E}(\alpha)$ describes local entropies spectrum f .

ii) If $K_\alpha = \{x : h_{\mu_\varphi}(f, x) = \alpha\}$, ($\mu_\varphi \neq \mu_{\max}$, the measure maximal entropy), then $\mathcal{E}(\alpha) = h_{top}(f, K_\alpha)$. Besides

$$\mathcal{E}(\alpha(q)) = \mathcal{E}_\varphi(\alpha(q)) = q\alpha(q) + T(q); \quad \alpha(q) := -T'(q), \quad q = \mathcal{E}'(\alpha).$$

Let $\alpha_i = \lim_{q \rightarrow \infty} \alpha(q) = \inf_{q \in \mathbf{R}} \{\alpha(q)\}$, $\alpha_s = \lim_{q \rightarrow -\infty} \alpha(q) = \sup_{q \in \mathbf{R}} \{\alpha(q)\}$, then $K_\alpha = \emptyset$, if $\alpha \notin (\alpha_i, \alpha_s)$, so that the domain of definition of $\mathcal{E}(\alpha)$ is the range of $T'(q)$.

The the family of measures $\{\mu_q = \mu_{q\varphi - qP(\varphi)} = \mu_{q\varphi}\}$ describes the multifractal spectrum of local entropies, in fact $\mu_q(K_{\alpha(q)}) = 1$, with $\alpha(q) = -T'(q)$.

We shall consider the following decomposition of any level set:

$$(8) \quad K_\alpha = \bigcup_{V \in \mathbf{R}^+} (K_{\alpha, V} \cup \mathcal{I}),$$

with

$K_{\alpha, V} = \left\{ x \in K_\alpha : \limsup_{n \rightarrow \infty} \frac{\mu(\mathcal{C}_n(x)) - n\alpha}{\sqrt{n \log \log n}} = V \right\}$, where $\mu \neq \mu_\varphi$ and $\varphi \in \nu_f(X)$ being the “irregular part” $\mathcal{I}$ taken as the set in which $\limsup$ is not finite. As we mentioned in the introduction the issue is to analyze the spectrum $\mathcal{E}(\alpha, V) = h_{top}(f, K_{\alpha, V})$.

3. FINER DECOMPOSITION OF THE SPECTRUM OF LOCAL ENTROPIES

The systems that we shall work are those for the Takens-Verbistki formalism works, i.e. the dynamics are expansive homeomorphisms with the specification property and the potential are in principle in the class $\nu_f(X)$. This setting is, as we commented before, more general than systems with Markov partitions, particularly hyperbolic systems. The motivation of considering a decomposition like the proposed herein is based on the work of Jordan and Pollicott [16] in which as we mentioned is analyzed the multifractal dimension spectrum for unidimensional maps. In that article is used the dynamical version of the law of the iterated logarithm presented in [10], which in our framework may be formulated as:

Let $\varphi, \psi \in \nu_f(X)$ with $\int \psi d\mu_\varphi = 0$ and such that the variance $Var(\psi, \mu_\varphi) := \int \psi^2 d\mu_\varphi$ is strictly positive then

$$\lim_{n \rightarrow \infty} \frac{S_n(\psi)(x)}{\sqrt{n \log \log n}} = \sqrt{2Var(\psi, \mu_\varphi)} \text{ for } \mu - a.e. x \in X.$$

The condition $Var(\psi, \mu_\varphi) > 0$ can be ensured by considering potentials φ, ψ such that $\mu_\varphi \neq \mu_{\max}$, $\mu_\psi \neq \mu_{\max}$, where $\mu_{\max}$ is the measure of maximal entropy, i.e. that associated to the potential $\varphi \equiv 0$. This can be seen by mean of the following classification theorem of Gibbs states [26], recalling also that the Gibbs states are supported on periodic points.

Let $\varphi, \psi \in \nu_f(X)$ where X is a compact metric space and f an expansive homeomorphism with specification, then $\mu_\varphi = \mu_\psi$ if and only if there is a constant C such that $S_n(\varphi)(x) = S_n(\psi + C)(x)$, for any n and for every $x \in P_n(f) = \{x : f^n(x) = x\}$

Besides that condition must imposed in order that the spectrum do no exhibit a degenerate behavior, since if $\mu = \mu_{\max}$, then

$$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) = \begin{cases} h_{top} & \text{if } \alpha = h_{top} \\ 0 & \text{if } \alpha \neq h_{top} \end{cases}$$

In the next lemma is showed how to compute the variance by the derivative of the pressure function:

Lemma 1: Let $\varphi, \psi \in \nu_f(X)$, with f a homeomorphism with specification, then $\frac{d^2 P(\varphi + \lambda\psi)}{d\lambda^2} |_{\lambda=0} = Var(\psi, \mu_\varphi)$, seeing the measure as a functional by $\mu_\varphi(\psi) = \int \psi d\mu$, and μ_φ the Gibbs state associated to φ .

Proof: By the multifractal formalism described earlier

$\frac{dP(\varphi + \lambda\psi)}{d\lambda} |_{\lambda=0} = \mu_\varphi(\psi) = \int \psi d\mu_\varphi$. Let us denote $\mu_\lambda = \mu_{\varphi + \lambda\psi}$ and so we have $\frac{dP(\varphi + \lambda\psi)}{d\lambda} = \mu_\lambda(\psi)$. Let us recall (c.f. eqs 5 and 6) that the Gibbs state for a potential φ in the class $\nu_f(X)$ is defined as the weak limit μ_φ of the ‘‘Gibbs ensembles’’

$$(9) \quad \mu_{\varphi,n}(\{y\}) = \frac{\exp(S_n(\varphi)(y))}{\sum_{x \in P_n(f)} \exp(S_n(\varphi)(x))}.$$

By the compactness of X this sequence has an accumulation point which can be interpreted as its “thermodynamic limit”. Thus for obtaining the second derivative it must be differentiated $\mu_{n,\lambda}(\psi)$. Doing this we have

$$\frac{d\mu_{n,\lambda}(\psi)}{d\lambda} = \frac{\sum_{x \in P_n(f)} \psi^2 \exp(S_n(\varphi + \lambda\psi)(x))}{\sum_{x \in P_n(f)} \exp(S_n(\varphi + \lambda\psi)(x))} - \left[\frac{\sum_{x \in P_n(f)} \psi(x) \exp(S_n(\varphi + \lambda\psi)(x))}{\sum_{x \in P_n(f)} \exp(S_n(\varphi + \lambda\psi)(x))} \right]^2$$

and then

$$\frac{d^2 P(\varphi + \lambda\psi)}{d\lambda^2} \Big|_{\lambda=0} = \mu_\varphi(\psi^2) - (\mu_\varphi(\psi))^2$$

■

In the law of the iterated logarithm are considered potentials of media zero, i.e. $\mu_\varphi(\psi) = 0$, and so the expression of the variance is just $\int \psi^2 d\mu_\varphi$

The main result is

Theorem 1: If $\varphi \in \nu_f(X)$ with $\mu_\varphi \neq \mu_{\max}$ then $\mathcal{E}(\alpha(q)) = \mathcal{E}(\alpha(q), V(q))$, for any real q and for enough large $V(q)$.

A special value for which is verified the claim of the theorem is precisely $V(q) = \text{Var}(\tilde{\varphi}_q, \mu_q)$, with $\tilde{\varphi}_q := \varphi - P(\varphi) - \alpha(q) = \varphi - \int \varphi d\mu_q$. Thus $\int \tilde{\varphi}_q d\mu_q = 0$ and by the law of the iterated logarithm

$$(10) \quad \lim_{n \rightarrow \infty} \frac{S_n(\tilde{\varphi}_q)(x)}{\sqrt{n \log \log n}} = \sqrt{2\text{Var}(\tilde{\varphi}_q, \mu_q)}.$$

But

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{S_n(\tilde{\varphi}_q)(x)}{\sqrt{n \log \log n}} &= \lim_{n \rightarrow \infty} \frac{S_n(\varphi)(x) - n(P(\varphi) - \alpha(q))}{\sqrt{n \log \log n}} \\ &= \lim_{n \rightarrow \infty} \frac{\mu(C_n(x)) - n\alpha(q)}{\sqrt{n \log \log n}}. \end{aligned}$$

The point is now to obtain values greater than $Var(\tilde{\varphi}_q, \mu_q)$. To do this the measures μ_q must be approximated by measures μ'_q such that $\alpha(q) = P(\varphi) - \int \varphi d\mu'_q$ and studied Gibbs measures for $S_n(\varphi)(x)$.

By the above classification of Gibbs states if $\mu_\varphi \neq \mu_{\max}$ then $S_n(\varphi)(x) \neq 0$ for some $x \in P_n(f)$, then in particular there are points x, x' fixed points of f such that $\varphi(x) \neq 0, \varphi(x') \neq 0$. Let us denote by $\underline{j} = j_0 j_1 \dots j_{n-1}$ and $\underline{k} = k_0 k_1 \dots k_{n-1}$ the words associated to x and x' respectively and corresponding to fixed a generating partition. Each cylinder $\mathcal{C}_{n,\underline{i}}$ can be represented by the image under contractible transformations $S_{i_0 i_1 \dots i_{n-1}}(X)$, thus given the word $i_0 i_1 \dots i_{n-1}$ let $z = S_{i_0 i_1 \dots i_{n-1}}(x)$ and let us define the number $\gamma_{i_0 i_1 \dots i_{n-1}}(x)$ by the condition

$$S_n(\varphi)(z) = S_n(\varphi)(S_{i_0 i_1 \dots i_{n-1}}(x)) =$$

$$\gamma_{i_0 i_1 \dots i_{n-1}}(x) S_n(\varphi)(S_{j_0 j_1 \dots j_{n-1}}(x)) + (1 - \gamma_{i_0 i_1 \dots i_{n-1}}(x)) S_n(\varphi)(S_{k_0 k_1 \dots k_{n-1}}(x))$$

Therefore, since the words are respect to a generating partition, we have that

$$\lim_{n \rightarrow \infty} S_{j_0 j_1 \dots j_{n-1}}(x) = x \quad \text{and} \quad \lim_{n \rightarrow \infty} S_{k_0 k_1 \dots k_{n-1}}(x) = x' \quad \text{and so the sequence}$$

$$\gamma_{i_0 i_1 \dots i_{n-1}}(x) \text{ is convergent.}$$

Let $G_{n,\varepsilon} = \{\mathcal{C}_{n,\underline{i}} : \mathcal{C}_{n,\underline{i}} \cap \{y : |S_n(\psi)(y)| < \varepsilon\} \neq \emptyset\}$, then for any word $i_0 i_1 \dots i_{n-1}$ let us introduce perturbations $S_{n,\lambda}(\psi)(x)$, $\lambda \in (0, 1)$, of the statistical sum $S_n(\psi)(x)$ defined, following [16], by

$$\exp((S_{n,\lambda}(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) = (1 - \lambda) (\exp(S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x))))$$

if $\mathcal{C}_{n,\underline{i}} \in G_{n,\varepsilon}$

$$\begin{aligned} \exp((S_{n,\lambda}(\psi)(S_{j_0 j_1 \dots j_{n-1}}(x)))) &= \exp((S_n(\psi)(S_{j_0 j_1 \dots j_{n-1}}(x)))) + \\ &\quad \lambda \sum_{\mathcal{C}_{n,\underline{i}} \in G_{n,\varepsilon}} \gamma_{i_0 i_1 \dots i_{n-1}}(x) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) \end{aligned}$$

$$\begin{aligned} \exp((S_{n,\lambda}(\psi)(S_{k_0 k_1 \dots k_{n-1}}(x)))) &= \exp((S_n(\psi)(S_{k_0 k_1 \dots k_{n-1}}(x)))) + \\ &\quad \lambda \sum_{\mathcal{C}_{n,\underline{i}} \in G_{n,\varepsilon}} (1 - \gamma_{i_0 i_1 \dots i_{n-1}}(x)) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) \end{aligned}$$

The statistical sums on cylinders corresponding to another contractions are not modified.

Consider now the Ruelle operator [25]

$$(11) \quad \mathcal{L}_\psi \phi(x) = \sum_{f(y)=x} \exp(\psi(y)) \phi(y),$$

and let us recall that the Gibbs states for the potential ψ can be found as fixed points of the dual $\mathcal{L}_\psi^*$.

Lemma 2: It is valid:

- i) $\mathcal{L}_{S_n, \lambda}(\psi)(1) = \mathcal{L}_{S_n(\psi)}(1)$;
- ii) $\mathcal{L}_{S_n, \lambda}(\psi)S_n(\phi) = \mathcal{L}_{S_n(\psi)}S_n(\phi)$.

Proof:

$$\begin{aligned} \text{i) } \mathcal{L}_{S_n, \lambda}(\psi)(1) &= (1 - \lambda) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) + \\ &\quad \exp((S_n(\psi)(S_{j_0 j_1 \dots j_{n-1}}(x)))) + \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) + \\ &\quad \lambda \left[\sum_{\mathcal{C}_{n, \underline{i}} \in G_{n, \varepsilon}} \gamma_{i_0 i_1 \dots i_{n-1}}(x) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) + \right. \\ &\quad \left. \sum_{\mathcal{C}_{n, \underline{i}} \in G_{n, \varepsilon}} (1 - \gamma_{i_0 i_1 \dots i_{n-1}}(x)) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) \right] = \\ &\quad \mathcal{L}_{S_n(\psi)}(1) - \lambda \sum_{\mathcal{C}_{n, \underline{i}} \in G_{n, \varepsilon}} \left(\exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) + \right. \\ &\quad \left. \gamma_{i_0 i_1 \dots i_{n-1}}(x) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) + \right. \\ &\quad \left. (1 - \gamma_{i_0 i_1 \dots i_{n-1}}(x)) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) \right) \\ &= \mathcal{L}_{S_n(\psi)}(1). \end{aligned}$$

$$\begin{aligned} \text{ii) } \mathcal{L}_{S_n, \lambda}(\psi)S_n(\phi) &= \mathcal{L}_{S_n(\psi)}S_n(\phi) - \lambda \left[\sum_{\mathcal{C}_{n, \underline{i}} \in G_{n, \varepsilon}} \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) + \right. \\ &\quad \left. \gamma_{i_0 i_1 \dots i_{n-1}}(x) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) + \right. \\ &\quad \left. (1 - \gamma_{i_0 i_1 \dots i_{n-1}}(x)) \exp((S_n(\psi)(S_{i_0 i_1 \dots i_{n-1}}(x)))) \right] S_n(\phi) \\ &= \mathcal{L}_{S_n(\psi)}S_n(\phi). \end{aligned}$$

It can be assumed, without losing generality, that any pair of potentials φ, ψ satisfy $\mathcal{L}_\psi \psi = 0$ and $\mathcal{L}_\psi 1 = 1$. This is due that can be considered renormalizations

of the potentials in the following way: let $\tilde{\varphi} = \varphi - P(\varphi)$, since the spectral radius of $\mathcal{L}_\psi$ is $\exp(-P(\varphi))$ [25] we get $\mathcal{L}_{\tilde{\varphi}}(1) = \exp(-P(\tilde{\varphi})) = 1$. Recall that the Gibbs states are translation invariant. For the other condition, putting a suitable norm in $\nu_f(X)$ we have that $\|\mathcal{L}_\psi^n \psi\| \leq C\beta^n \|\psi\|$, for some $\beta \in (0, 1)$. Thus if $\phi = \sum_{n=1}^{\infty} \mathcal{L}_\psi^n \psi$ then this series converges in $(\nu_f(X), \|\cdot\|)$. Let us consider now for a potential ψ the renormalization $\tilde{\psi} = \psi - C_f \phi + \phi$, where C_f is the composition operator $C_f \phi = \phi \circ f$, therefore $\mathcal{L}_\psi(C_f \phi) = \phi$ and

$\mathcal{L}_\psi \tilde{\psi} = \mathcal{L}_\psi \psi - \phi + \mathcal{L}_\psi \phi = \mathcal{L}_\psi \psi - \mathcal{L}_\psi \psi = 0$. So we can assume that $\mathcal{L}_{S_n(\psi)}(1) = 1$, $\mathcal{L}_{S_{n,\lambda}(\psi)} S_n(\phi) = 0$ and consequently, by lemma 2, $\mathcal{L}_{S_{n,\lambda}(\psi)}(1) = 1$, $\mathcal{L}_{S_{n,\lambda}(\psi)} S_n(\phi) = 0$. Besides notice that $\mathcal{L}_\psi \psi = 0$ then $\int \psi d\mu_\varphi = 0$, this is because $\int \psi d\mu_\varphi = \int \psi d\mathcal{L}_\psi^*(\mu_\varphi) = \int \mathcal{L}_\psi \psi d\mu_\varphi = 0$.

Before proceeding to the proof of the theorem we point out some facts to be used in. We have

$\lim_{n \rightarrow \infty} \frac{1}{n} (S_n(\varphi)(S_{j_0 j_1 \dots j_{n-1}}(x))) = \lim_{n \rightarrow \infty} \frac{1}{n} (\varphi(S_{j_0 j_1 \dots j_{n-1}}(x)) + (n-1)\varphi(x)) = \delta := \varphi(x)$ and similarly

$\lim_{n \rightarrow \infty} \frac{1}{n} (S_n(\varphi)(S_{j_0 k_1 \dots k_{n-1}}(x))) = \delta' := \varphi(x')$. By the bounded distortion property of the potentials in the class considered

(12)

$$S_n(\varphi)(S_{j_0 j_1 \dots j_{n-1}}(x')) - K \leq S_n(\varphi)(S_{j_0 j_1 \dots j_{n-1}}(x)) \leq K + S_n(\varphi)(S_{j_0 j_1 \dots j_{n-1}}(x'))$$

(13)

$$S_n(\varphi)(S_{k_0 k_1 \dots k_{n-1}}(x')) - K \leq S_n(\varphi)(S_{k_0 k_1 \dots k_{n-1}}(x)) \leq K + S_n(\varphi)(S_{k_0 k_1 \dots k_{n-1}}(x'))$$

and so that

(14)

$$S_n(\varphi)(S_{j_0 j_1 \dots j_{n-1}}(x))^2 \geq (K + n\delta)$$

$$S_n(\varphi)(S_{k_0 k_1 \dots k_{n-1}}(x))^2 \geq (K - n\delta').$$

If μ_λ denotes the Gibbs state for the potential $S_{n,\lambda}(\psi)$, with the assumption $\mathcal{L}_{S_{n,\lambda}(\psi)} S_n(\varphi) = 0$, then by the above bounds can be got the estimation

$$(15) \quad \text{Var}(S_n(\varphi), \mu_\lambda) \geq \text{Var}(K + n\delta, \mu_\lambda) + \text{Var}(K - n\delta', \mu_\lambda),$$

here $K + n\delta, K - n\delta'$ means the constant functions with this values.

Proof of the theorem 1: Let us fix a potential $\varphi \in \nu_f(X)$ which will be that for the spectrum $\mathcal{E}(\alpha(q))$ is computed and let the already used function

$\tilde{\varphi}_q := \varphi - P(\varphi) - \alpha(q)$, for which $\int \tilde{\varphi}_q d\mu_q = 0$, where μ_q is the Gibbs state for the potential $\varphi_q := q\varphi - qP(\varphi)$. We also have that if $\mu_{\lambda,q}$ is the Gibbs state associated to the potential $S_{n,\lambda}(\varphi_q)$ then $\int \tilde{\varphi}_q d\mu_{\lambda,q} = 0$, because the possibility of assuming that $\mathcal{L}_{S_{n,\lambda}(\varphi_q)} S_n(\tilde{\varphi}_q) = 0$. Let $V(q) := \text{Var}(\tilde{\varphi}_q, \mu_q)$ and let the map

$\lambda \mapsto \text{Var}(S_n(\tilde{\varphi}_q), \mu_{\lambda,q})$, since $\mu_{\lambda,q}$ is the eigenvector corresponding to the eigenvalue $\exp(-P(S_{n,\lambda}(\varphi_q)))$ and the assignation $\lambda \mapsto P(S_{n,\lambda}(\varphi_q))$ is continuously differentiable by the conditions on the potentials the map $\lambda \mapsto \text{Var}(S_n(\tilde{\varphi}_q), \mu_{\lambda,q})$ is continuous [26]. Thus if $\bar{V}(q) \geq V(q) = \text{Var}(S_n(\tilde{\varphi}_q), \mu_{0,q})$ then, by the estimations of eq.(15), $\text{Var}(S_n(\tilde{\varphi}_q), \mu_{\lambda,q}) \geq \bar{V}(q)$ and so there is a $\bar{\lambda} \in (0, 1)$ such that $\text{Var}(S_n(\tilde{\varphi}_q), \mu_{\bar{\lambda},q}) = \bar{V}(q)$. Therefore, by the law of the iterated logarithm

$$\lim_{m \rightarrow \infty} \frac{S_{nm}(\tilde{\varphi}_q)(x)}{\sqrt{m \log \log m}} = \sqrt{2n\bar{V}(q)}.$$

Let $r = nm + k$, then

$$\frac{S_r(\tilde{\varphi}_q)(x)}{\sqrt{r \log \log r}} = \frac{S_{nm}(\tilde{\varphi}_q)(x) + S_k(\tilde{\varphi}_q)(f^{nk}(x))}{\sqrt{(nm+k) \log \log (nm+k)}},$$

and so

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{S_r(\tilde{\varphi}_q)(x)}{\sqrt{r \log \log r}} &= \lim_{m \rightarrow \infty} \frac{S_{nm}(\tilde{\varphi}_q)(x)}{\sqrt{nm \log \log (nm+k)}} = \sqrt{\frac{1}{n}} \lim_{m \rightarrow \infty} \frac{S_{nm}(\tilde{\varphi}_q)(x)}{\sqrt{m \log \log m}} \\ &= \sqrt{2\bar{V}(q)}. \end{aligned}$$

Thus $\mathcal{E}(\alpha(q)) = \mathcal{E}(\alpha(q), \bar{V}(q))$, with the spectrum $\mathcal{E}(\alpha(q), \bar{V}(q))$ described by the family of measures $\{\mu_{\bar{\lambda},q}\}$.

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