ATTRACTORS FOR MULTIFUNCTIONS CREATED BY SEMI-DYNAMICAL SYSTEMS

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Abstract. In this paper by using of semi-bornnology the notion of semi-bornological attractors for semi-dynamical systems is studied. It is proved that: this concept of attractor is invariant up to a special kind of lower semi-continuous multifunctions. Attractors for semi-dynamical systems on vectorical spaces are studied. The notion of semi-bornological stability is presented. It is proved that: the semi-bornological conjugacy preserves attractor. The notion of chaotic multifunction is considered.

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1. Introduction

We assume that X is a non-empty set, S is a semigroup, and $D = \{\varphi^s : s \in S\}$ is a family of maps from X to X. A triple (X, S, D) is called a semi-dynamical system if:
$\varphi^s o \varphi^g = \varphi^{sg}$ for all $s, g \in S$.

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We associate to a semi-dynamical system (X, S, D) a multifunction F on X defined by $F(x) = \{\varphi^s(x) : s \in S\}$. $F(x)$ is called the orbit of x, where $x \in X$.
$F^-(x)$ is the set $\{y \in X\colon\ \varphi^g(y) = x$ for some $g \in S\}$. For $n \geq 1$ $F^n(x) = F(F^{n-1}(x))$, and $F^{-n}(x) = F^-(F^{-(n-1)}(x))$.
A covering β^s of X is called a semi-bornology on X [3, 4], if β^s is closed under finite unions. The elements of β^s are called semi-bounded sets. In this case (X, β^s) is called a semi-bornological space.
Clearly each bornological space [1] is a semi-bornological space.
If (X, β^s) is a semi-bornological space and (A_n) is a sequence of subsets of X, then the lower bound $Li\ A_n$ is the set of points $x \in X$ such that for each $B \in \beta^s$, with $x \in B$ there is $n_0 \in N$ so that $B \bigcap A_n \neq \emptyset$ for all $n \geq n_0$.
$Ls\ A_n$ is the set of points $x \in X$ such that for all $B \in \beta^s$ with $x \in B$ there exists $n_0 \in N$ so that $B \cap A_n \neq \emptyset$ for infinitely $n \geq n_0$. If $Ls\ A_n = Li\ A_n$, then this common limit denoted by $Lb\ A_n$ and it is called the bounded limit of A_n [4]. The set $Ls\ A$ is called the closure of A and denoted by $\overline{A}$.
If (X, β^s) is a semi-bornological space and $F : X \to X$ is a multifunction, then F is called a lower semi-continuous (l.s.c) multifunction if $F(\overline{A}) \subseteq \overline{F(A)}$ for all $A \subseteq X$.
Theorem 1.1 Let (X, S, D) be a semi-dynamical system on a semi-bornological space (X, β^s) where (X, d) is a metric space and β^s is the semi-bornology generated by the set $\{B(x, \varepsilon)\colon\ x \in X, \varepsilon \in (0, \infty)\}$. Moreover let $F : X \to X$ be a lower semi-continuous multifunction. Then for all sequence $\{x_n\}_{n\geq 1} \subseteq X$ with $lim_{n\to\infty} x_n = x$ we have $F(x) \subseteq LiF(x_n)$.
Proof. If there is a sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $lim_{n\to\infty} x_n = x$ and $F(x)$ is not a subset of $LiF(x_n)$, then there exists $y \in F(x)$ so that $y \notin LiF(x_n)$. Hence there is subsequence $\{x_{n_k}\}_{k\geq 1} \subseteq \{x_n\}_{n\geq 1}$ and $\varepsilon > 0$ such that $F(x_{n_k}) \cap B(y, \varepsilon) = \emptyset$ for all $k \geq 1$.
Let $A = \{x_{n_k} : k \geq 1\}$. Then $x \in \overline{A}$ and since F is l.s.c then $F(x) \subseteq F(\overline{A}) \subseteq \overline{F(A)}$. Hence $y \in \overline{F(A)}$ but we have $F(x_{n_k}) \cap B(y, \varepsilon) = \emptyset$ for all $k > 1$. This is a contradiction. So $F(x) \subseteq LiF(x_n)$.$\square$
Definition 1.2 Let $(X, \{\varphi^g\}_{g\in S}, S)$ and $(Y, \{\psi^g\}_{g\in S}, S)$ be two semi-dynamical systems with semi-bornologies β^s and Γ^s respectively. Moreover let $\varphi^g : X \to X$ and $\psi^g : Y \to Y$ be one to one and onto maps, for all $g \in S$. Then we say that: $(X, \{\varphi^g\}_{g\in S}, S)$ is *bounded like* to $(Y, \{\psi^g\}_{g\in S}, S)$ if there is a continuous bijection $\alpha : X \to Y$ such that the multifunction $F : X \to Y$ defined by $F(x) = \{\psi^g o \alpha o \phi^{g^{-1}}(x) : g \in S\}$ and the multifunction $F^- : Y \to X$ are l.s.c. maps.

One can deduce the next theorem from theorem 1.1 of [4]

Theorem 1.2 Bounded like relation on the class of semi-dynamical systems with semi-bornologies which their evolution operators are bijections is an equivalence relation.

In section 2 the concept of attractor will be studied. We will prove that: it is an invariant object up to a l.s.c. corresponding multifunction. Then in the third section attractors for semi-dynamical systems on semi-bornological vectorical spaces will be considered. In section four semi-bornological stability and chaotic multifunctions are studied.

2. Attractors

Now we would like to define the notion of attractor for multifunction $F : X \to X$ where (X, S, D) is a semi-dynamical system with a semi-bornology β^s on X.

Definition 2.1 If (X, S, D) is a semi-dynamical system on a semi-bornological space (X, β^s) with the corresponding multifunction F, then a non-empty set A of X is called a semi-bornological attractor for D if for all $B \in \beta^s$, $LbF^n(B) = A$. Similarly a repeller for D is a non-empty subset R of X, such that $R = LbF^{-n}(B)$, for all $B \in \beta^s$.

Theorem 2.1 Let (X, S, D) be a semi-dynamical system on a semi-bornological space (X, β^s) where (X, d) is a metric space and let β^s be the semi-bornology generated by the set $\{B(x, \varepsilon) : x \in X, \varepsilon \in (0, \infty)\}$. If its corresponding multifunction is a l.s.c multifunction, then $F(A) \subseteq A$.

Proof. Let $y \in F(A)$ be given. Then there is $x_0 \in A$ such that $y \in F(x_0)$. Since $x_0 \in A$ then $x_0 \in LbF^n(B)$ for all $B \in \beta^s$. Thus there is $z \in B$ such that $x_0 \in LbF^n(z)$. We can select a sequence $\{x_n\}_{n\geq 1} \subseteq F^n(z)$ such that $x_n \to x_0$ when $n \to \infty$. Theorem 1.1 implies that $F(x_0) \subseteq LiF^n(z) \subseteq LbF^n(z) \subseteq A$. Thus $y \in A$.□

3. Attractor of multifunction on vectorical spaces

In this section we assume that X is a vectorical space, i.e. X is a vector space on $\mathbb{R}$ or $\mathbb{C}$ [2]. Moreover we assume that X is a semi-bornological space. Let $D_x = \{\lambda x : |\lambda| \leq 1\}$ where $x \in X$, and let X_{D_x} be the subspace generated by D_x. Then $P_{D_x} = inf\{| t | : y \in tD_x\}$ is called the Minkowski norm on X_{D_x}.

Remark 3.1 Let X be a vectorical space and let β^s be the semi-bornology generated by the set $\{D_x : x \in X\}$. Moreover let $(S, .)$ be a semigroup such that there is $g \in S$ with $\varphi^g(x) = 0$ for all $x \in X$. Then $0 \in A$.

Theorem 3.1 [4] Let X be a vectorical space and let β^s be the semi-bornology genrated by the set $\{D_x : x \in X\}$. Moreover let (X, S, D) be a semi-dynamical system with the corresponding multifunction F such that for all $B \in \beta^s$ and $b \in LiF^n(B)$ there is $n_0 > 0$ so that $F^n(B) \subseteq D_b$ for all $n \geq n_0$. Then $LiF^n(C) = LiF^n(B)$ and $LsF^n(B) = LsF^n(C)$ for all $B, C \in \beta^s$.

Corollary 3.1 With the assumptions of Theorem 3.1 if there is $B \in \beta^s$ with $LbF^n(B) = LsF^n(B) = LiF^n(B)$, then $LbF^n(B)$ is the attractor of F.

Now suppose that (X, d) is a metric space and β^s is the semi-bornology generated by the set

$$\{B(x, \varepsilon) : \ x \in X, \varepsilon \in (0, \infty)\}.$$

Definition 3.1 A multifunction F corresponding to a semi-dynamical system (X, S, D) is called uniformly contractive if for every bounded non-empty subset A of X, and each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $diam(F^n(A)) < \varepsilon$ for all $n \geq n_0$.

Theorem 3.2 [4] Let F be a uniformly contractive multifunction corresponding to (X, S, D). Then $LiF^n(A) = LiF^n(B)$ and $LsF^n(A) = LsF^n(B)$, where A and B are non-empty bounded subsets of X.

Corollary 3.2 With the assumption of Theorem 3.2 if there is $A \in \beta^s$ with $LiF^n(A) = LsF^n(A)$. Then $LiF^n(A)$ is the attractor of F.

Definition 3.2 Let (X, β^s) be a semi-bornological space and let $A \subseteq X$. Then we define $diam_a A = sup\{P_{D_a}(x - y) : x, y \in A \cap X_{D_a}\}$ and $diam A = sup_{a \in A} diam_a A$.

Theorem 3.3 [4] Let (X, D, S) be a semi-dynamical system and $a \in X$ be given where X is a semi-bornological space with the semi-bornology $\beta^s = \{B \subseteq X : B \subseteq X_{D_a}\} \cup \{X\}$. Moreover let for all $A \in \beta^s$, $F^n(A) \subseteq X_{D_a}$ for all arbitrary large n and let for all $\varepsilon > 0$ there is $n_0 > 0$ such that $diam F^n(A) < \varepsilon$ when $n \geq n_0$. Then $LiF^n(A) = LiF^n(B)$ and $LsF^n(A) = LsF^n(B)$ for all $A, B \in \beta^s$.

4. Semi-bornological Stability

Now in this section we are going to define conjugacy between semi-dynamical systems, and introduce the notion of semi-bornological stability.

Definition 4.1 Let (X, S, D) and (X, S, D') be two semi-dynamical systems with a common semi-bornology β^s on X. Also let $F : X \to X$ and $G : X \to X$ be the multifunctions corresponding to (X, S, D) and (X, S, D') respectively. Then we say

these two semi-dynamical systems are conjugate if F and G are semi-bornological conjugate [4] i.e. there exists a bijection $r : X \to X$ with the following properties:
(a) $roF = Gor$;
(b) For all $B \in \beta^s$, $r(B) \in \beta^s$, and $r^-(B) \in \beta^s$.
In the case of conjugacy we use of the notation: $(X, S, D) \sim (X, S, D')$.
Theorem 4.1 [4] " $\sim$ " is an equivalence relation on semi-dynamical systems on X.
Theorem 4.2 The semi-bornological conjugacy preserves attractors.
Proof. Let $r : X \to X$ be a semi-bornological conjogacy between multifunctions $F : X \to X$ and $G : X \to X$ corresponding to semi-dynamical systems (X, S, D) and (X, S, D') respectively. At first we show that $rLbF^n(B) = LbrF^n(B)$. Let $x \in rLbF^n(B)$ be given. Then $x = r(y)$ for a unique $y \in LbF^n(B)$. Let $C = r(C') \in \beta^s$ be given such that $x \in C$. Then $y \in C'$ and there is n_0 such that $C' \cap F^n(B) \neq \emptyset$ for all $n \geq n_0$. Thus $C \cap rF^n(B) \neq \emptyset$ for all $n \geq n_0$. So $x \in LbrF^n(B)$. Thus $rLbF^n(B) \subseteq LbrF^n(B)$.
Similarly we have $r^{-1}LbrF^n(B) \subseteq Lbr^{-1}rF^n(B)$. Thus $LbrF^n(B) \subseteq rLbF^n(B)$. Hence $LbrF^n(B) = rLbF^n(B)$.
Now let A be an attractor for F. Then $A = LbF^n(B)$ for all $\emptyset \neq B \in \beta^s$. So $r(A) = rLbF^n(B) = LbrF^n(B) = LbG^n r(B)$ for all $\emptyset \neq B \in \beta^s$. Thus $r(A)$ is an attractor for G.□
Definition 4.2 Let (X, β^s) be a semi-bornological space and let $(S, .)$ be a semigroup. Also let $Y = \{F : X \to X : F$ is a multifunction corresponding to a semi-dynamical system on X with the semigroup S $\}$ and let γ^s be a semi-bornology on Y. Then (X, S, D) with $D = \{\varphi^g : X \to X : g \in S$ and φ^g is map $\}$ is called a stable semi-dynamical system with respect to γ^s, if there is $\lambda \in \gamma^s$ such that $F \in \lambda$, and each $G : X \to X$ with $G \in \lambda$, corresponding to (X, S, D') is semi-bornological conjugate to F.
Remark 4.1 If (X, S, D) is semi-dynamical system with multifunction F and $\{F\} \in \gamma^s$ where γ^s is a semi-bornology on multifunctions. Then (X, S, D) is an stable semi-dynamical system.
Remark 4.2 If (X, β^s) is a semi-bornological space, $(S, .)$ is a semigroup, and $Y = \{F : X \to X : F$ is multifunction $\}$ where $F(x) = \{\varphi^g(x) : g \in S\}$, then we can define a semi-bornology on Y such that each (X, S, D) is a stable semi-dynamical system. For this purpose let $[D] = \{(X, S, D') : (X, S, D') \sim (X, S, D)\}$, and γ^s be the semi-mornology generated by the set $\{[D] : D$ corresponding $F \in Y\}$.

Then (X, S, D) is a stable semi-dynamical system.
If $x \in X$ the the orbit of x in semi-dynamical system (X, S, D) is the set $O_D(x) = \{\varphi^{g^n}(x) : g \in S, n \in \mathbb{Z}\}$, and the semi-bornological conjugacy preserves orbits.
Now we define a chaotic multifunction.

Definition 4.3 Let (X, m, μ) be a measurable space where m is a σ-algebra and μ is an outermeasure on X. Then we define $\rho : X \times X \to \mathbb{R}^+$ by
$\rho(x, y) = inf\{\mu(B) : x, y \in B, B \in m\}$.
If $\{x, y\}$ is not a subset of B for all $B \in m$ then we define $\rho(x, y) = 0$.
(X, ρ) is semi-metric space, i.e. $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Definition 4.4 Suppose that (X, m, μ) be a measurable space and μ be an outermeasure. Also let $(S, .)$ be a semigroup such that (X, S, D) is a semi-dynamical system and β^s is a semi-bornology on X with $\beta^s \subseteq m$. The multifunction $F : X \to X$ corresponding to (X, S, D) is called a chaotic multifunction if
i) For all $B, B' \in \beta^s$ there is $k > 0$ such that $F^k(B) \cap B' \neq \emptyset$;
ii) Periodic points of F are dense in X;
iii) There is $\delta > 0$ such that for all $B \in \beta^s$ and $x \in B$ there is $y \in B$, $k > 0$ such that $sup_{z \in F^k(x),\ t \in F^k(y)} \rho(t, z) > \delta$.

Example 4.1 Let $(\mathbb{R}, m, \mu)$ be a measurable space where $m = \{(-a, a) : a \in \mathbb{R}\}$ and $\mu : m \to \mathbb{R}^+$ be defined by $\mu(-a, a) = a$. Moreover let $\beta^s = \{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\} \cup \{(-n, n) : n \in \mathbb{N}\} \subseteq m$ be the semi-bornology on $\mathbb{R}$ and $(S, .)$ $(S \subseteq \mathbb{R})$ be the semigroup generated by the set $\{\frac{1}{2}, 3\}$. Then $F(x) = \{(\frac{1}{2})^n x, 3^n x : n \in \mathbb{N}\}$ is a chaotic multifunction.

Remark 4.3 If the miultifunction $F : X \to X$, corresponding to a semi-dynamical system (X, S, D), is a chaotic multifunction and there is $G : X \to X$ corresponding to semi-dynamical system (X, S, D') such that $(X, S, D) \sim (X, S, D')$. Then G is a chaotic multifunction.

Suppose that (X, m) is a measurable space and $\beta^s \subseteq m$ is a semi-bornology on X and $\mu : m \to \mathbb{R}^+$ is an outermeasure, then (X, β^s) is called a precompact space if for all $\varepsilon > 0$ there is a finite set $S \subseteq X$ such that $X = \cup_{x \in S} B(x, \varepsilon)$ where $B(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}$.

Remark 4.4 In the definition of chaotic multifunction axiom (iii) is independent of the other two axioms. For example: let $(\mathbb{R}, .)$ be a semigroup generated by the set $\{1, \frac{1}{2}\}$ and $X = S^1$ be the semi-bornology $\beta^s = \{(-\theta, \theta) : \theta \in \mathbb{R}^+\}$. Then $\rho(\alpha, \beta) = max\{\alpha, \beta\}$. Let $F : S^1 \to S^1$ be the multifunction corresponding to

$(S^1, \mathbb{R}, D)$ where $D = \{\varphi^g : S^1 \to S^1 : \varphi^g(x) = gx\}$. Then F is a l.s.c multifunction and satisfies the conditions (i) and (ii). But for all $\delta > 0$ and $y \in (-\delta, \delta)$ and $k > 0$ with $x = \frac{\delta}{2}$, $sup_{z\in F^k(x),\ t\in F^k(y)}\rho(t, z) < \delta$. So F does not satisfy the condition (iii).

Theorem 4.3 Let (X, m, μ) be a non precompact measurable space and $\beta^s \subseteq m$ be a semi-bornology on X. Also let $(S, .)$ be a semigroup and $F : X \to X$ be a multifunction corresponding to a semi-dynamical system (X, S, D) satisfies the conditions (i) and (ii). Moreover let for given $x \in X$, $\varepsilon > 0$ there is $B \in \beta^s$ such that $\mu(B) < \varepsilon$. Then F satisfies the axiom (iii).

Proof. Suppose $F : X \to X$ satisfies two conditions (i) and (ii) but does not satisfy the condition (iii).

We prove that: there is a periodic point p such that $\cup_{i=1}^{n-1} B(F^i(p), \delta) = X$ where n denotes the period of p. Let $\delta > 0$ be given and let $\delta' = \frac{\delta}{4}$. Since F does not satisfy (iii), then there is $B \in \beta^s$ and $x \in B$ such that for all $y \in B$

$$d(F^k(x), F^k(y)) = sup_{z\in F^k(x) t\in F^k(y)}\rho(z, t) \leq \delta.$$

Since F satisfies (ii) then there is a periodic point $p \in B$ such that $d(F^k(x), F^k(p)) \leq \delta$, for all $k > 0$.

So

$d(F^k(p), F^k(y)) \leq d(F^k(p), F^k(x)) + d(F^k(x), F^k(y)) \leq 2\delta' < 3\delta'$ for all $k > 0$.

Thus $F^k(B) \subset B(F^k(p), 3\delta')$ for all $k > 0$ where $B(F^k(p), 3\delta') = \{x \in X : d(F^k(p), x) < 3\delta'\}$.

Let n be the period of p then $\cup_{k=0}^{\infty} F^k(B) \subseteq \cup_{i=0}^{n-1} B(F^i(p), 3\delta')$. Now since F satisfies (i) and for all $z \in X$, there is $B' \in \beta^s$ such that $z \in B'$, $\mu(B') < \delta'$, and $F^k(B) \cap B' \neq \emptyset$, then $(\cup_{k=0}^{\infty} F^k(B)) \cap B' \neq \emptyset$.

So $(\cup_{k=0}^{n-1} B(F^k(p), 3\delta')) \cap B' \neq \emptyset$. Therefore there is $z' \in B'$ such that $z' \in B(F^i(p), 3\delta')$, $0 \leq i \leq n-1$ and

$d(z, z') = \rho(z, z') < \delta'$, $d(z', F^i(p)) < 3\delta'$.

Hence $d(z, F^i(p)) < 4\delta' = \delta$. Thus $\cup_{i=0}^{n-1} B(F^i(p), \delta) = X$ for all $z \in X$.$\square$

References

[1] Almeida J., Barreira L., Hausdorff Dimension in Convex Bornological Spaces, *J. Math. Anal. 268* (2002) 590-601.

[2] Lasota Andrzej, Myjak Jozef, Semifractals on Polish Spaces, *Bull. Pol. Ac.: Math., 46* (1998) 179-196.

[3] Molaei M.R., Mathematical Structures Based on Completely Simple Semigroups, Hadronic Press, 2005.

[4] Molaei M.R., Waezizadeh T., Repellers for Multifunctions of Semi-bornological Spaces, *Acta Mathematica Scientia, Vol. 28, Number 3* (2008) 545-550.