

ANALYTIC FUNCTIONS ON TIME SCALES

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ABSTRACT. The paper considers analytic functions defined on time scales. It is shown that locally the sum of the sequence of almost uniformly convergent analytic functions on some subset of time scale is also an analytic function on this subset. The main result proves that the solution of the initial value problem on any time scale with right hand analytic function is also the analytic function on this time scale.

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1. INTRODUCTION

It is known that in the mathematical modeling of engineering, physical, biological, economical and other phenomena in most cases time is a crucial, continuous or discrete variable. Moreover, in many situations the constructed models are described by differential/difference equations. On the other hand, one can notice that there exist quite a big class of different results that in continuous- and discrete-time cases look very similar or even identical. The examples can be multiplied. E.T.Bell in 1937 said "A major task of mathematics today is to harmonize the continuous

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and discrete, to include them in one comprehensive mathematics and eliminate obscurity from both.” Calculus on time scales, originated in 1988 by S. Hilger [10], is a perfect language for this task. A time scale is a model of time. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, but there are many examples of time models, which may be partially continuous and partially discrete. Let us remember that the basic idea of the time scale calculus is the concept of delta derivative as a generalization of the standard differential calculus on the one hand, and the calculus of finite differences on the other hand. Similarly, the integral of real functions defined on the time scale is an extension of the Riemann integral in the continuous time and finite sum in discrete time case [6, 9]. In the consequence, differential equations as well as difference equations are naturally accommodated into this theory. Moreover, calculus on time scales can be implemented in mathematical packages as, for example, **MATHEMATICA** [13, 14].

The application of time scales theory in some problems has opened the way for unification and extension of existing results, for example the unification and extension of the classical Kalman’s rank conditions for controllability or observability of control systems [2], unification and extension of Euler-Lagrange optimality condition [11, 8], etc.

Differential equations with right hand side given by power series are of great importance. Classical Cauchy’s theorem says that *initial value problem* $\dot{x}(t) = \sum_{j,k=0}^{\infty} a_{jk}(t - t_0)^j(x - x_0)^k$, $x(t_0) = x_0$, with right hand analytic side has unique analytic solution. Surprisingly enough, the theory of analytic functions on time scales has only been touched [4, 3]. In [3] a concept of analyticity for functions defined on a product of two time scales was introduced. In [4] a globally right convergent power series on time scale was defined and used to solve an initial value problem containing a dynamic version of the transport equation.

The aim of the paper is an extension of Cauchy’s theorem to any time scale case. The paper is organized as follows: in Section 2, based on [6, 1] we give a short introduction to calculus on time scales. In Section 3, based on [4, 6, 12] we recall basic facts about polynomials and generalized polynomial power series on time scales. In Section 4 we introduce the idea of analyticity, similar as in [4], of the function defined on any time scale $\mathbb{T}$. We prove that the limit of the sequence

of almost uniform convergent analytic functions on time scales is also an analytic function on time scales. The main result of the paper (in Section 5) is the claim that the solution of the initial value problem on any time scale with an analytic right hand side is an analytic function, i.e. function f on time scale that can be locally expressed as Taylor series. On the one hand this fact extends the well known fact for ordinary differential equations to any time scales. On the other hand, it extends the result proved in [7], which gave a closed form for the solution of time-varying linear system $x^\Delta(t) = A(t)x(t)$, $x(t_0) = x_0$, $t \in \mathbb{T}$, via the use of generalized form of the Peano-Baker series.

2. INTRODUCTION TO DIFFERENTIAL CALCULUS ON TIME SCALES

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$.

For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, the *graininess function* $\mu : \mathbb{T} \rightarrow [0, \infty)$ by $\mu(t) := \sigma(t) - t$. Using these operators we can classified points in real line: if $\sigma(t) > t$, then t is called *right-scattered*, if $\rho(t) < t$ is called *left-scattered*, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*, if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*.

By $\mathbb{T}^k$ we denote the set

$$\mathbb{T}^k := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases}$$

i.e. $\mathbb{T}^k$ does not contains maximal points.

Definition 2.1. [6] Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$. The *delta derivative* of f at t , denoted by $f^\Delta(t)$ (or by $\frac{\Delta}{\Delta t} f$), is the real number (provided it exists) with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that $|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$ for all $s \in U$. We say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

Theorem 2.2. [6] Let $f : \mathbb{T} \rightarrow \mathbb{R}$, $g : \mathbb{T} \rightarrow \mathbb{R}$ be delta differentiable functions at $t \in \mathbb{T}^k$. Let $t \in \mathbb{T}$. Then

- (1) if $t \in \mathbb{T}^k$ then f has at most one Δ -derivative at t
- (2) if f^Δ exists, then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$
- (3) for any constants a, b holds $(af(t) + bg(t))^\Delta = af^\Delta(t) + bg^\Delta(t)$
- (4) $((f(t)g(t))^\Delta = f^\Delta(t)g(\sigma(t)) + f(t)g^\Delta(t) = f(\sigma(t))g^\Delta + f^\Delta(t)g(t)$
- (5) if $g(t)g(\sigma(t)) \neq 0$, then $\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}$

Remark 2.3. If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$ we have $\sigma(t) = t = \rho(t)$ and the graininess function $\mu(t) \equiv 0$. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if $f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} = f'(t)$ i.e. if and only if f is differentiable in the ordinary sense at t . If $\mathbb{T} = \mathbb{Z}$, then for every $t \in \mathbb{Z}$ we have $\sigma(t) = t + 1$, $\rho(t) = t - 1$ and the graininess function $\mu(t) \equiv 1$. A function $f : \mathbb{Z} \rightarrow \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t + 1) - f(t) = \Delta f(t)$ where Δ is the usual forward difference operator defined by the last equation above.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points at $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

A continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) D , provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$. For any regulated function f there exists a function F that is pre-differentiable with region of differentiation D such that $F^\Delta(t) = f(t)$ for all $t \in D$. Such F is called pre-antiderivative of f . Then *indefinite integral* of f is defined by $\int f(t)\Delta t := F(t) + C$ where C is an arbitrary constant. *Cauchy integral* is $\int_r^s f(t)\Delta t = F(s) - F(r)$ for all $r, s \in \mathbb{T}^k$. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. It can be shown that every rd-continuous function has an antiderivative.

Remark 2.4. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau)\Delta\tau = \int_a^b f(\tau)d\tau$ where the integral on the right is the usual Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int_a^b f(\tau)\Delta\tau = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

3. POLYNOMIAL SERIES ON TIME SCALES

Let $\mathbb{T}$ be an arbitrary time scale. Let us define a sequence of real valued functions h_k on $\mathbb{T} \times \mathbb{T}$ in the following way: let us put $h_0(t, s) \equiv 1$. Then $h_{k+1}(t, s) := \int_s^t h_k(\tau, s) \Delta\tau$ for all $t, s \in \mathbb{T}$. Sequence of functions h_k are called (*generalized*) *polynomial functions* on time scales. In general, finding the formula for generalized polynomials h_n , $n = 1, 2, 3, \dots$, can be difficult. But for a particular given time scale it might be easy.

Example 3.1. [6] If $\mathbb{T} = \mathbb{R}$ then $h_k(t, s) = \frac{(t-s)^k}{k!}$. If $\mathbb{T} = \mathbb{Z}$ then $h_k(t, s) = \binom{t-s}{k}$. For $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, for all $k \in \mathbb{N}$ we have that $h_k(t, s) = \prod_{\nu=0}^{k-1} \frac{t-q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}$ holds for all $t, s \in \mathbb{T}$.

Theorem 3.2. [5] Let $k \in \mathbb{N}_0$. Then $0 \leq h_k(t, s) \leq \frac{(t-s)^k}{k!}$ for all $t \geq s$, $t, s \in \mathbb{T}$.

For each fixed $t_0 \in \mathbb{T}$, let us denote by $h_k^\Delta(t, t_0)$ the delta derivative of polynomial $h_k(t, t_0)$ with respect to t . Then for $k \in \mathbb{N}_0$ and $t \in \mathbb{T}^\kappa$: $h_k^\Delta(t, t_0) = h_{k-1}(t, t_0)$ and $h_0^\Delta(t, t_0) = 0$.

Let $n \in \mathbb{N}$ and let a function f be n times differentiable on $\mathbb{T}^{k^n}$, $t \in \mathbb{T}$. Let $t_0 \in \mathbb{T}^{k^{n-1}}$. Then, [4]

$$(1) \quad f(t) = \sum_{k=0}^{n-1} f^{\Delta^k}(t_0) h_k(t, t_0) + \int_{t_0}^t f^{\Delta^n}(\tau) h_{n-1}(t, \sigma(\tau)) \Delta\tau.$$

Formula (1) is called *Taylor's formula*. It can be shown [4] that for any $t \geq t_0$ the remainder term $R_n(t) = \int_{t_0}^t f^{\Delta^n}(\tau) h_{n-1}(t, \sigma(\tau)) \Delta\tau$ of (1) tends to zero as $n \rightarrow \infty$.

Let us consider the *functional series*

$$(2) \quad f_1(t) + f_2(t) + f_3(t) + \dots = \sum_{n=1}^{\infty} f_n(t)$$

where all functions $f_1, f_2, \dots$ are defined on some subset $\mathbb{T}_E \subseteq \mathbb{T}$. It is known that the partial sums $s_n(t) = f_1(t) + f_2(t) + \dots + f_n(t)$, $n = 1, 2, 3, \dots$, forms *functional sequence* defined in $\mathbb{T}_E$. If the sequence $\{s_n(t)\}$ converges to the function $f(t)$ at any point $t \in \mathbb{T}_E$, then $f(t)$ is called the *limit function* and $f(t)$ is simultaneously a *sum* of the series (2). Hence, analysis of the functional series and its sum is nothing else as analysis of the functional sequence and its limit function.

We say that the function series (sequence) (respectively 2) $(\{s_n(t)\})$ *uniformly converges* to the $f(t)$ on set $\mathbb{T}_E \subseteq \mathbb{T}$ if for any positive $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}_0$ such that for all $t \in \mathbb{T}_E$ and for all $n > n_0$ holds $|f_n(t) - f(t)| < \varepsilon$ (respectively $|s_n(t) - f(t)| < \varepsilon$). The function series (sequence) *almost uniformly converges* in the set $\mathbb{T}_E \subseteq \mathbb{T}$ if it uniformly converges in any closed, bounded subset of $\mathbb{T}_E$. Function series *converges absolutely* if the power series $\sum_{k=0}^{\infty} |f_n(t)|$ converges.

Theorem 3.3. [12] *Let us assume that functions f_n are regulated on set $E \subseteq \mathbb{T}$ for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} f_n(t)$ uniformly converges on E , then*

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(t) \right) \Delta t = \sum_{n=1}^{\infty} \int_a^b f_n(t) \Delta t.$$

Theorem 3.4. [12] *Suppose that partial sums s_n of the series (2) are pre-differentiable with $\mathbb{T}_E$ for each $n \in \mathbb{N}$. Let us assume that for all $t \in \mathbb{T}^k$ there exists a compact neighborhood $U = U(t) \cap \mathbb{T}$ such that the sequence $\{s_n\}_{n \in \mathbb{N}}$ uniformly converges on the set $U \cap \mathbb{T}_E$. Then the series (2) is pre-differentiable with $\mathbb{T}_E$ and*

$$\left(\sum_{n=1}^{\infty} f_n(t) \right)^{\Delta} = \sum_{n=1}^{\infty} f_n^{\Delta}(t).$$

By a *polynomial series* originated at $t_0 \in \mathbb{T}$ we mean the expression

$$(3) \quad \sum_{k=0}^{\infty} a_k h_k(t, t_0)$$

with real coefficients a_k . For any fixed $t \in \mathbb{T}$ it is an ordinary number series. If it is convergent, then we say that the power series *converges* at t .

Let us assume that $t \in \mathbb{T}$ is not fixed. Now, the series (3) can be treated as the function series defined on $\mathbb{T}$.

Proposition 3.5. [12] *If there exists a natural number n_0 such that for all $n \geq n_0$ and for all $t \in \mathbb{T}_E \subseteq \mathbb{T}$ holds $|s_n(t)| = \left| \sum_{k=0}^n a_k h_k(t, t_0) \right| \leq b_n$ and the number series $\sum_{n=0}^{\infty} b_n$ converges, then the power series (3) converges uniformly and absolutely.*

4. ANALYTIC FUNCTIONS

Suppose that $\sup \mathbb{T} = \infty$. Let us consider a function $f : \mathbb{T} \rightarrow \mathbb{R}$. We say that f is *analytic at the point* $t_0 \in \mathbb{T}$ if there exists $\varepsilon > 0$ such that f can be written in the

form

$$(4) \quad f(t) = \sum_{k=0}^{\infty} a_k h_k(t, t_0), \quad a_k \in \mathbb{R}$$

on the set $(t_0 - \varepsilon; t_0 + \varepsilon) \cap \mathbb{T}$. It is obvious that the sum of analytic functions at the same point t_0 is again an analytic function at this point. If t_0 is isolated then every function is analytic at this point and $f(t_0) = ah_0(t_0, t_0) = a$.

Proposition 4.1. *Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ is an analytic function at t_0 and for all $t \in (t_0 - \varepsilon, +\infty) \cap \mathbb{T}$ holds that $f(t) = \sum_{k=0}^{\infty} a_k h_k(t, t_0)$. Then f is infinitely many times differentiable at t_0 and $f^{\Delta^k}(t) = a_k$.*

Proof. This follows from the fact that $h_k^{\Delta}(t, t_0) = h_{k-1}(t, t_0)$ for $k \in \mathbb{N}$, $t \in \mathbb{T}^k$. So, after k steps we obtain $f^{\Delta^k}(t) = a_k$. $\square$

Remark 4.2. Proposition 4.1 is still true if we assume that $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap \mathbb{T}$ and the subinterval $(t_0, t_0 + \varepsilon) \cap \mathbb{T}$ contains an infinitely many points.

As usual, we call $\sum_{k=0}^{\infty} f^{\Delta^k}(t_0) h_k(t, t_0)$ the *Taylor series* of the function f at point t_0 .

Example 4.3. Let $\mathbb{T} = \mathbb{Z}$ and $a \in \mathbb{R}$, $a \neq -1$. Let $f(t) = (1 + a)^t$ for $t \in \mathbb{T}$. An easy calculation shows that $f^{\Delta^k}(0) = a^k$. Thus the Taylor series of this function is $\sum_{k=0}^{\infty} a^k \binom{t}{k}$. Observe that for $t \geq 0$ the series is finite (so convergent) and its

sum is equal to $f(t)$. On the other hand, for $t = -1$ we have $\binom{-1}{k} = (-1)^k$ and the series becomes infinite. For $|a| > 1$ it is not convergent, thus in this case the Taylor expansion holds only for nonnegative integers. However, if we take $\varepsilon = 1/2$, the series converges on $(-\varepsilon, +\infty) \cap \mathbb{T}$, so f is analytic at $t_0 = 0$.

Proposition 4.4. *Let us assume that*

- (1) *there exists an interval $\mathbb{T}_E = (t_0 - \varepsilon, t_0 + \varepsilon) \cap \mathbb{T}$ such that the set $(t_0, t_0 + \varepsilon) \cap \mathbb{T} \subseteq \mathbb{T}_E$ contains infinitely many points,*
- (2) *there exists $t_0 \in \mathbb{T}$ such that functions $f_n : \mathbb{T} \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0$ are analytic on $\mathbb{T}_E$ and the series*

$$(5) \quad \sum_{n=0}^{\infty} f_n(t)$$

converges on $\mathbb{T}_E$

If the series (5) almost uniformly converges on $\mathbb{T}_E$ to the function $f(t)$, then $f(t)$ is an analytic function (on $\mathbb{T}_E$).

Proof. It is obvious that $\sum_{n=0}^{+\infty} f_n(t) = \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} a_{nk} h_k(t, t_0)$ on $\mathbb{T}_E$ and

$$\begin{aligned} f(t) &= (a_{0_0} + a_{1_0} + a_{2_0} + \dots)h_0(t, t_0) + (a_{1_0} + a_{1_1} + a_{1_2} + \dots)h_1(t, t_0) + \dots + \\ &\quad (a_{n_0} + a_{n_1} + a_{n_2} + \dots)h_n(t, t_0) + \dots = \\ &\quad A_0 h_0(t, t_0) + A_1 h_1(t, t_0) + \dots + A_n h_n(t, t_0) + \dots \end{aligned}$$

where $A_k = \sum_{n=0}^{\infty} a_{nk} = \sum_{n=0}^{\infty} f_n^{\Delta^k}(t_0) = f^{\Delta^k}(t_0)$. Hence, $f(t)$ is an analytic function on $\mathbb{T}_E$. $\square$

Corollary 4.5. *If the terms f_n of the sequence $\{f_n(t)\}$ are analytic functions on $\mathbb{T}_E$ and this sequence is almost uniformly convergent (on $\mathbb{T}_E$) to the function $f(t)$, then $f(t)$ is an analytic function (on $\mathbb{T}_E$).*

Proof. Proof immediately follows from Proposition 4.4. $\square$

5. DIFFERENTIAL EQUATIONS ON TIME SCALES

Let us recall [6] that function $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called

- (1) *rd-continuous*, if g defined by $g(t) = f(t, x(t))$ is rd-continuous for any continuous function $x : \mathbb{T} \rightarrow \mathbb{R}^n$
- (2) *bounded* on a set $S \subseteq \mathbb{T} \times \mathbb{R}^n$, if there exist constants m, M such that $m \leq f(t, x) \leq M$ for all $(t, x) \in S$.
- (3) *Lipschitz continuous* on a set $S \subseteq \mathbb{T} \times \mathbb{R}^n$ with respect to the variable x , if there exists a constant $L > 0$ such that for all $(t, x_1), (t, x_2) \in S$ holds $|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$, where $|\cdot|$ denotes Euclidian norm in $\mathbb{R}^n$.

In fact, (1) says that

- a) function f is continuous at each (t, x) with right-dense or maximal t , and
- b) the limits $f(t^-, x) := \lim_{(s, y) \rightarrow (t, x)_{s < t}} f(s, y)$ and $\lim_{y \rightarrow x} f(t, y)$ exist at each (t, x) with left-dense t . Point (t, x) that satisfies condition **a**) is called *right-dense* and point (t, x) satisfying condition **b**) is called *left-dense*.

Proposition 5.1. *Let $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.*

- (1) If function f is rd-continuous on some compact set $S \subseteq \mathbb{T} \times \mathbb{R}^n$, then it is bounded on S .
- (2) Let us assume that $S \subseteq \mathbb{T} \times \mathbb{R}^n$ is closed and bounded. If there exists a continuous partial derivative $\frac{\partial f}{\partial x}$ and is bounded on S , then f is Lipschitz continuous on S .

Proof. (1) Let us assume that function f is unbounded on the set S . It means that for any m there exists a point $(t_m, x_m) \in S$ such that

$$(6) \quad |f(t_m, x_m)| > m$$

The sequence of points $\{(t_m, x_m)\}$ is convergent (as a sequence from a compact set S), so we can take a subsequence $\{(t_{m_k}, x_{m_k})\}$ that is convergent. Let $\{(t_{m_k}, x_{m_k})\} \rightarrow (t_0, x_0)$ where $(t_0, x_0) \in S$. Function f is rd-continuous at the point (t_0, x_0) , so at right-dense points obviously $f(t_{m_k}, x_{m_k}) \rightarrow f(t_0, x_0)$. At the left-dense points we have $g(t_0) = \lim_{t \rightarrow t_0^-} g(t) = f(t_0^-, x(t_0))$. Hence, f is bounded, so contradiction with (6).

(2) Proof is standard (as in classical analysis). $\square$

Theorem 5.2. [6] Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$ and a be any positive constant. Let us assume that $\inf \mathbb{T} \leq t_0 - a$ and $\sup \mathbb{T} \geq t_0 + a$. Suppose that $f : (t_0 - a, t_0 + a) \cap \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$, $x_0 \in A$, is rd-continuous, bounded and Lipschitz continuous. Then there exists α , $\alpha \leq a$ such that the initial value problem

$$(7) \quad x^\Delta = f(t, x), \quad x(t_0) = x_0$$

has exactly one solution on the interval $[t_0 - \alpha; t_0 + \alpha] \cap \mathbb{T}$. If t_0 is right-scattered and $\alpha < \mu(t_0)$, then the unique solution exists on the interval $[t_0 - \alpha, \sigma(t_0)] \cap \mathbb{T}$.

Let $C(X, Y)$ denote Banach space of all continuous functions $f : X = (t_0 - a, t_0 + a) \cap \mathbb{T} \rightarrow Y = \mathbb{R}^n$ (with the norm $\| \cdot \| = \sup_X |f(t)|$), where a is positive constant such that $\inf \mathbb{T} \leq t_0 - a$, $\sup \mathbb{T} \geq t_0 + a$. Let $A \subseteq Y$. Let $\lambda : X \rightarrow Y$ be a continuous function. Then the function g defined by $g(t) = f(t, \lambda(t))$ is rd-continuous. Let us define operator $F_{x_0} : C(X, \mathbb{R}^n) \rightarrow C(X, \mathbb{R}^n)$ as $F_{x_0}(\lambda)(t) := x_0 + \int_{t_0}^t f(\tau, \lambda(\tau)) \Delta \tau$. It can be shown, that (for some a) operator F is a contraction map [6]. By Banach's fixed point theorem, the equation $\lambda = F(\lambda)$ has exactly one solution $\tilde{\lambda} \in C$ such that $\tilde{\lambda}^\Delta(t) = f(t, \tilde{\lambda}(t))$, $\tilde{\lambda}(t_0) = x_0$, so $\tilde{\lambda}$ is the unique solution of the equation (7).

Let us take any point $t_1 \in [t_0 - \alpha; t_0 + \alpha] \cap \mathbb{T}$. Hence there exists a nonnegative real number ε such that the length of the interval $[t_1, t_0 + \alpha] \cap \mathbb{T}$, i.e. distance between $t_0 + \alpha$ and t_1 , is equal ε . Let us put $x(t_1) = x_1$ and start to search the solution of the initial value problem:

$$(8) \quad x^\Delta = f(t, x), \quad x(t_1) = x_1$$

If the point (t_1, x_1) satisfies the assumptions of the theorem 5.2, then there exists a unique solution of (8) defined on the set $(t_1 - \alpha_1, t_1 + \alpha_1) \cap \mathbb{T}$. This solution coincides with solution of (7) on the intersection of the intervals $(t_0 - \alpha; t_0 + \alpha) \cap \mathbb{T}$ and $(t_1 - \alpha_1, t_1 + \alpha_1) \cap \mathbb{T}$. If $\alpha_1 > \varepsilon$, then the solution can be extended to the interval $(t_0 - \alpha, t_1 + \alpha_1) \cap \mathbb{T}$. If possible, the solution can be extended also to the point $t_1 + \alpha_1$. So, the solution of the equation (7), denoted by $\tilde{\lambda}(\cdot, t_0, x_0)$, can be obtained as a limit of the sequence $F_{n, x_0}(x_0)$, where n denotes a number of steps in which F_{x_0} is calculated, i.e. $\tilde{\lambda}(\cdot, t_0, x_0) = \lim_{n \rightarrow \infty} F_{n, x_0}(t_0)$. Hence $\tilde{\lambda}(\cdot, t_0, x_0)$ tends to the solution of (7).

Let us consider a local initial value problem with analytic right hand side:

$$(9) \quad x^\Delta(t) = \sum_{j,k=1}^{\infty} c_{jk} h_j(t, t_0) (x - x_0)^k =: f(t, x), \quad x(t_0) = x_0$$

where $c_{j,k}$ are real coefficients.

Theorem 5.3. *Let us assume that function f is rd-continuous and that number series $\sum_{j,k=1}^{\infty} c_{jk}$ corresponding to (9) converges. If function $f(t, x)$ defined in (9) is analytic function on the set $S \subseteq \mathbb{T} \times \mathbb{R}^n$, then the initial value problem*

$$(10) \quad x^\Delta(t) = \sum_{j,k=1}^{\infty} c_{jk} h_j(t, t_0) (x - x_0)^k, \quad x(t_0) = x_0$$

has a unique solution which is also an analytic function on the set S .

Proof. Proof is similar in the classical case. From the theorem 3.2 we can notice that the right hand side of (10) is bounded on some subset $\tilde{\mathbb{T}} \times B(x_0, \varsigma)$ where B denotes a ball with radius $\varsigma < r$, r is such number that $|x - x_0| < r$ and $\tilde{\mathbb{T}}$ is a bounded subset of $\mathbb{T}$. The partial derivative $\frac{\partial f}{\partial x}$ is also bounded on the set $\mathbb{T} \times B(x_0, \varsigma)$, so the function f is Lipschitz continuous. Then, the $n + 1$ approximation of the unique solution of equation (10) is of the form $x_{n+1}(t) = x_0 + \int_{t_0}^t \sum_{j,k=1}^{\infty} c_{jk} h_j(\tau, t_0) (x_n(\tau) - x_0)^k \Delta \tau$ for

$n = 1, 2, 3, \dots$. Let us note that $\sum_{j,k=1}^{\infty} c_{jk} h_j(t, t_0) (x - x_0)^k \leq \sum_{j,k=1}^{\infty} c_{jk} \frac{(t-t_0)^j}{j!} (x_n(t) - x_0)^k$. Hence, it uniformly converges of the series (10).

Note that if n -th approximation of (10) is analytic, then $(n+1)$ -th approximation is also right analytic. It is obvious that x^0 , as a constant function on time scales, is analytic. Hence, all approximations x_n are (locally) analytic functions. $\square$

Theorem 5.3 states that the solution of the differential equations with analytic right hand side is the analytic function. Hence, it can locally be uniquely represented as power series of the form $x(t) = \sum_{j=0}^{\infty} a_j h_j(t, t_0)$.

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