

SPECIAL BISHOP MOTION AND BISHOP DARBOUX ROTATION AXIS OF THE SPACE CURVE

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ABSTRACT. A special Frenet motion with a one parameter has been given by Bottema [5] in E^3 . In this study, we have given a generalization of [5] to Bishop motion in Euclidean 3-space. Firstly, Bishop motion is defined for space curve α and then Darboux vector of this motion is calculated for fixed and moving spaces in E^3 . The Bishop darbox rotation for space curves in Euclidean space E^3 is decomposed into two simultaneous rotations. The axes of these simultaneous rotations are joined by a simple mechanism. One of these axes is a parallel of the tangent vector of the curve, the direction of the other is the direction of the Bishop darbox vector of the curve. This decomposition of the Bishop darbox rotation yields a necessary condition for the curve to be closed.

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1. PRELIMINARIES

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. we can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(N_1(s), N_2(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point. If the

derivatives of $(N_1(s), N_2(s))$ depend only on $T(s)$ and not each other we can make $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations

$$(1.1) \quad \begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

where

$$\begin{aligned} \kappa(s) &= \sqrt{k_1^2 + k_2^2}, \\ \theta(s) &= \arctan\left(\frac{k_2}{k_1}\right), \\ \tau(s) &= -\frac{d\theta(s)}{ds} \end{aligned}$$

[2,3,4], so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0 , which disappears from τ (and hence from the Frenet frame) due to the differentiation.

2. INTRODUCTION

A one-parameter motion of a body in Euclidean 3-space is generated by the transformation

$$(2.1) \quad \begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where A is a orthogonal matrix, $A \in SO(3)$, and C is the displacement vector of the origin. A and C are C^∞ functions of a real parameter s . X, Y are $n \times 1$ matrices and

$$SO(3) = \{A = [a_{ij}] \in R_3^3 : A^{-1} = A^T, a_{ij} \in R\}$$

[5]. X and Y respectively correspond to the position vector of the same point, with respect to the orthogonal coordinate systems of the moving space H and the fixed space H' . At the same time $s = s_0$ we consider the coordinate systems H and H' are coincident.

We define a special motion in term of space curve α given in the fixed space H' . The motion is such that the moving frame O_{xyz} , moves with O along α while rotating so that the x and y axes always coincide with, respectively, the tangent and principal normal of α . This means that as O coincides with a point Q of α the O_{xyz} frame coincides with the Bishop frame at Q : $O_{\xi\eta\rho}$. This trihedron consists of the tangent Q_ξ , the principal normal Q_η , and the binormal Q_ρ , which are three mutually orthogonal axes. Obviously, the geometry of this motion is completely defined by α .

For our special motion, because O moves along α , $Y = C$ represents space curve α . We will use the arc length, s , of α as the motion parameter, and use primes to denote derivatives with respect to s .

Denote by $\{T, N_1, N_2\}$ the moving Bishop frame along the curve $\alpha = \alpha(s)$ parameterized by arc-length parameter s , i.e., $\langle \alpha'(s), \alpha'(s) \rangle = 1$. Let $T = (t_1, t_2, t_3)$, $N_1 = (n_1, n_2, n_3)$, $N_2 = (b_1, b_2, b_3)$ be the unit vectors along Q_ξ , Q_η and Q_ρ . Then we have

$$(2.2) \quad A = \begin{bmatrix} t_1 & n_1 & b_1 \\ t_2 & n_2 & b_2 \\ t_3 & n_3 & b_3 \end{bmatrix}.$$

It can be defined that an one parameter special motion of a body in Euclidean 3-space is generated by the transformation by

$$(2.3) \quad Y = AX + C$$

where $A \in SO(3)$ and X, Y, C are 3×1 real matrices [5]. Moreover, if column vectors of A are T , N_1 and N_2 , then A holds as follows since A is a positive orthogonal matrix,

$$\begin{aligned} \langle T, T \rangle &= \langle N_1, N_1 \rangle = \langle N_2, N_2 \rangle = 1, \\ \langle T, N_1 \rangle &= \langle T, N_2 \rangle = \langle N_1, N_2 \rangle = 0 \\ \langle T', T \rangle &= \langle N_1', N_1 \rangle = \langle N_2', N_2 \rangle = 0 \end{aligned}$$

Now, we will define Bishop darbox vector. The Vectors $T(s), N_1(s), N_2(s)$ change while a point on the curve drawing the curve. Hence these vectors constitute of spherical images of the curve. Assume that Bishop frame $\{T(s), N_1(s), N_2(s)\}$ of the curve makes an instantaneous helix motion about an axis at each s time. This axis is called Bishop darbox axis corresponding s parameter at point $\alpha(s)$. The vector giving oriented and direction of this axis is called Bishop darbox vector at point $\alpha(s)$ of the curve.

3. BISHOP DARBOUX VECTOR OF BISHOP MOTION

Theorem 3.1. *Let the motion H/H' be represented by the equation (2.1). Then the component of darbox vectors of the motion H/H' , respectively, are*

$$\Omega = (k_1 b_1 - k_2 n_1, k_1 b_2 - k_2 n_2, k_1 b_3 - k_2 n_3)$$

and

$$\vec{\omega} = (0, -k_2, k_1).$$

Proof. From (2.3) we obtain

$$(3.1) \quad X = A^{-1}(Y - C)$$

and as $A^{-1} = A^T$ and $\det A = +1$, we have (2.3) and (3.1), eliminating Y

$$(3.2) \quad Y'(s) = \Omega(Y - C) + C'(s)$$

with $\Omega = A' A^{-1}$, or explicitly, by means of (2.2),

$$(3.3) \quad \Omega = A' A^{-1} = A' A^T = \begin{bmatrix} t_1' & n_1' & b_1' \\ t_2' & n_2' & b_2' \\ t_3' & n_3' & b_3' \end{bmatrix} \begin{bmatrix} t_1 & t_2 & t_3 \\ n_1 & n_2 & n_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

One can find the following equalities

$$(3.4) \quad \left. \begin{aligned} T &= N_1 \times N_2 = (n_2 b_3 - n_3 b_2, n_3 b_1 - n_1 b_3, n_1 b_2 - n_2 b_1) \\ N_1 &= N_2 \times T = (b_2 t_3 - b_3 t_2, b_3 t_1 - b_1 t_3, b_1 t_2 - b_2 t_1) \\ N_2 &= T \times N_1 = (t_2 n_3 - t_3 n_2, t_3 n_1 - t_1 n_3, t_1 n_2 - t_2 n_1) \end{aligned} \right\}$$

and

$$(3.5) \quad \left. \begin{aligned} T' &= (t'_1, t'_2, t'_3) = (k_1 n_1 + k_2 b_1, k_1 n_2 + k_2 b_2, k_1 n_3 + k_2 b_3) \\ N'_1 &= (n'_1, n'_2, n'_3) = (-k_1 t_1, -k_1 t_2, -k_1 t_3) \\ N'_2 &= (b'_1, b'_2, b'_3) = (-k_2 t_1, -k_2 t_2, -k_2 t_3) \end{aligned} \right\}$$

By using (3.3) and (3.5), we get $\Omega =$

$$\begin{aligned} & \begin{bmatrix} t'_1 t_1 + n'_1 n_1 + b'_1 b_1 & t'_1 t_2 + n'_1 n_2 + b'_1 b_2 & t'_1 t_3 + n'_1 n_3 + b'_1 b_3 \\ t'_2 t_1 + n'_2 n_1 + b'_2 b_1 & t'_2 t_2 + n'_2 n_2 + b'_2 b_2 & t'_2 t_3 + n'_2 n_3 + b'_2 b_3 \\ t'_3 t_1 + n'_3 n_1 + b'_3 b_1 & t'_3 t_2 + n'_3 n_2 + b'_3 b_2 & t'_3 t_3 + n'_3 n_3 + b'_3 b_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (t'_1 t_2 + n'_1 n_2 + b'_1 b_2) & [t'_1 t_3 + n'_1 n_3 + b'_1 b_3] \\ -(*) & 0 & \{t'_2 t_3 + n'_2 n_3 + b'_2 b_3\} \\ -[\Delta] & -\{\clubsuit\} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \left(\begin{array}{c} (k_1 n_1 + k_2 b_1) t_2 \\ + (-k_1 t_1) n_2 + (-k_2 t_1) b_2 \end{array} \right) & \left(\begin{array}{c} (k_1 n_1 + k_2 b_1) t_3 \\ + (-k_1 t_1) n_3 + (-k_2 t_1) b_3 \end{array} \right) \\ -* & 0 & \left(\begin{array}{c} (k_1 n_2 + k_2 b_2) t_3 \\ + (-k_1 t_2) n_3 + (-k_2 t_2) b_3 \end{array} \right) \\ -\Delta & -\clubsuit & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & k_1 (-b_3) + k_2 (n_3) & k_1 (b_2) + k_2 (-n_2) \\ -* & 0 & k_1 (-b_1) + k_2 (n_1) \\ -\Delta & -\clubsuit & 0 \end{bmatrix} \text{ So, we obtain} \end{aligned}$$

$$(3.6) \quad \Omega = \begin{bmatrix} 0 & -(k_1 b_3 - k_2 n_3) & (k_1 b_2 - k_2 n_2) \\ (k_1 b_3 - k_2 n_3) & 0 & -(k_1 b_1 - k_2 n_1) \\ -(k_1 b_2 - k_2 n_2) & (k_1 b_1 - k_2 n_1) & 0 \end{bmatrix}$$

Since Ω is skew symmetric matrix, from equality $\vec{\Omega}\vec{\Omega} = 0$, we have

$$\vec{\Omega} = (k_1 b_1 - k_2 n_1, k_1 b_2 - k_2 n_2, k_1 b_3 - k_2 n_3).$$

Its component with respect to the moving frame follow from $\vec{\Omega} = A\vec{\omega}$ [5], and we obtain for the vector

$$\begin{aligned}
 \vec{\omega} &= A^{-1}\vec{\Omega} \\
 &= A^T\vec{\Omega} \\
 &= \begin{bmatrix} t_1 & t_2 & t_3 \\ n_1 & n_2 & n_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} k_1 b_1 - k_2 n_1 \\ k_1 b_2 - k_2 n_2 \\ k_1 b_3 - k_2 n_3 \end{bmatrix} \\
 &= \begin{bmatrix} t_1 k_1 b_1 - t_1 k_2 n_1 + t_2 k_1 b_2 - t_2 k_2 n_2 + t_3 k_1 b_3 - t_3 k_2 n_3 \\ n_1 k_1 b_1 - k_2 n_1^2 + n_2 k_1 b_2 - k_2 n_2^2 + n_3 k_1 b_3 - k_2 n_3^2 \\ k_1 b_1^2 - b_1 k_2 n_1 + k_1 b_2^2 - b_2 k_2 n_2 + k_1 b_3^2 - b_3 k_2 n_3 \end{bmatrix} \\
 &= \begin{bmatrix} k_1 (t_1 b_1 + t_2 b_2 + t_3 b_3) - k_2 (t_1 n_1 + t_2 n_2 + t_3 n_3) \\ k_1 (n_1 b_1 + n_2 b_2 + n_3 b_3) - k_2 (n_1^2 + n_2^2 + n_3^2) \\ k_1 (b_1^2 + b_2^2 + b_3^2) - k_2 (b_1 n_1 + b_2 n_2 + b_3 n_3) \end{bmatrix} \\
 &= \begin{bmatrix} k_1 \langle T, N_2 \rangle - k_2 \langle T, N_1 \rangle \\ k_1 \langle N_1, N_2 \rangle - k_2 \langle N_1, N_1 \rangle \\ k_1 \langle N_2, N_2 \rangle - k_2 \langle N_2, N_1 \rangle \end{bmatrix} \\
 &= \begin{bmatrix} k_1 0 - k_2 0 \\ k_1 0 - k_2 1 \\ k_1 1 - k_2 0 \end{bmatrix} \\
 &= [0 \quad -k_2 \quad k_1]^T.
 \end{aligned}$$

and so Thus it holds.

$$(3.7) \quad \vec{\omega} = (0, -k_2, k_1).$$

□

So much for the direction of the screw axis. To determine its position in the O_{xyz} -frame, suppose that it intersects the plane O_{xy} at $S = (x_0, y_0, 0)$ and that the translation parallel to screw axis is

$$\sigma_0(0, -k_2, k_1).$$

The velocity vector of the origin would be

$$(-k_1 x_0 - k_2 \sigma_0, -k_1 x_0, -k_2 y_0 + k_1 \sigma_0)$$

and as (by virtue of $P'_0 = C' = T$) it must be $(1, 0, 0)$ we have

$$\begin{aligned}
 x_0 &= 0, \\
 y_0 &= \frac{k_1}{k_1^2 + k_2^2} = \frac{k_1}{\kappa^2}, \\
 \sigma_0 &= \frac{k_2}{k_1^2 + k_2^2} = \frac{k_2}{\kappa^2}.
 \end{aligned}$$

The conclusion is the instantaneous screw axis intersects O_x , the principal normal, at the points S , such that

$$OS = \frac{k_1}{k_1^2 + k_2^2};$$

it is parallel to O_{yz} ; the components of $\vec{\omega}$ are $(0, -k_2, k_1)$ and that of the translation vector

$$\left(0, \frac{-k_2^2}{\kappa^2}, \frac{k_1 k_2}{\kappa^2}\right).$$

Note that, as developed here, the angular and the linear velocities are geometrical concepts with the arc length of α acting as time.

4. THE BISHOP DARBOUX ROTATION AXIS OF THE SPACE CURVE

These equations form a rotation motion with Bishop darbox vector,

$$\varpi = -k_2 N_1 + k_1 N_2.$$

[2,3,8]. Also momentum rotation vector is expressed as follows

$$\begin{aligned} T' &= \varpi \wedge T \\ N_1' &= \varpi \wedge N_1 \\ N_2' &= \varpi \wedge N_2. \end{aligned}$$

Bishop darbox rotation of Bishop frame can be separated into two rotation motions. The vector N_1 rotates with a k_1 angular speed round the vector N_2 , that is

$$N_1' = (k_1 N_2) \wedge N_1 = k_1 (N_2 \wedge N_1) = -k_1 T$$

and the vector N_2 rotates with a $-k_2$ angular speed round the vector N_1 , that is

$$N_2' = (-k_2 N_1) \wedge N_2 = -k_2 (N_1 \wedge N_2) = -k_2 T.$$

The separation of the rotation motion of the momentum. Bishop darbox axis into two rotation motions can be indicated as such: The vector $\frac{\varpi}{\|\varpi\|}$ rotates with a

$$W = \frac{k_1' k_2 - k_1 k_2'}{k_1^2 + k_2^2}$$

speed round the tangent vector T , also

$$\left(\frac{\varpi}{\|\varpi\|}\right)' = (W.T) \wedge \frac{\varpi}{\|\varpi\|}$$

and the tangent vector T rotates with a $\|\varpi\|$ angular speed round $\frac{\varpi}{\|\varpi\|}$. Bishop darbox axis, also

$$T' = \varpi \wedge T.$$

From now on we shall do a further study of momentum Bishop darbox axis. We obtain the unit vector E

$$E = \frac{\varpi}{\|\varpi\|} = \frac{-k_2 N_1 + k_1 N_2}{\sqrt{k_1^2 + k_2^2}}.$$

from Bishop darbox vector,

$$\begin{aligned}\varpi' &= -k_2' N_1 - k_2 N_1' + k_1' N_2 + k_1 N_2' \\ &= -k_2' N_1 - k_2 (-k_1 T) + k_1' N_2 + k_1 (-k_2 T) \\ &= -k_2' N_1 + k_1' N_2.\end{aligned}$$

Differentiation of E ,

$$E' = \left(\frac{\varpi}{\|\varpi\|} \right)' = \frac{\varpi' \|\varpi\| - \varpi \|\varpi\|'}{(\|\varpi\|)^2} = \frac{(k_1' k_2 - k_1 k_2')}{k_1^2 + k_2^2} \frac{(k_1 N_1 + k_2 N_2)}{\sqrt{k_1^2 + k_2^2}}$$

is found. From this,

$$E' = W (E \wedge T)$$

written. According to the Bishop frame,

$$T' = \|\varpi\| (E \wedge T)$$

and

$$\begin{aligned}(E \wedge T)' &= E' \wedge T + E \wedge T' \\ &= W.T \wedge (E \wedge T) + \|\varpi\|.E (E \wedge T) \\ &= W [\langle T, T \rangle E - \langle T, E \rangle T] + \|\varpi\| [\langle E, T \rangle E - \langle E, E \rangle T] \\ &= W.E - \|\varpi\| T\end{aligned}$$

are obtained. These three equations are in the form of the Bishop frames that is

$$\begin{bmatrix} T' \\ (E \wedge T)' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \|\varpi\| & 0 \\ -\|\varpi\| & 0 & W \\ 0 & W & 0 \end{bmatrix} \begin{bmatrix} T \\ (E \wedge T) \\ E \end{bmatrix}$$

where the first coefficient $\|\varpi\| = \sqrt{k_1^2 + k_2^2}$ is > 0 and second coefficient

$$W = \frac{(k_1' k_2 - k_1 k_2')}{k_1^2 + k_2^2} = \frac{\left(\frac{k_1}{k_2} \right)'}{1 + \left(\frac{k_1}{k_2} \right)^2}$$

related only to natural harmonic curvature $\frac{k_1}{k_2}$. Thus, the vectors $\{T, (E \wedge T), E\}$ define a rotation motion together the rotation vector,

$$\varpi_1 = -W.T + \|\varpi\|.E = -W.T + \varpi$$

Also momentum rotation vector is expressed as follows:

$$\begin{aligned}T' &= \varpi_1 \wedge T \\ (E \wedge T)' &= \varpi_1 \wedge (E \wedge T) \\ E' &= \varpi_1 \wedge E.\end{aligned}$$

Corollary 4.1. *This rotation motion of Bishop darbox axis can be separated into two rotation motions again. Here ϖ_1 rotation vector is the addition of the rotation vectors of the rotation motions. When continued in the similar way, the rotation*

motion of Bishop darboux axis is done in a consecutive manner. In this way the series of Bishop darboux vectors are obtained. That is

$$\varpi_0 = \varpi, \varpi_1 \dots$$

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