

# NUMERICAL TREATMENT OF DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH INDEX 2 BY REDUCING THE INDEX

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**ABSTRACT.** We will consider the solution of Hessenberg index-2 differential algebraic equations(DAE) numerically. We first introduce a method to reduce the index from 2 to index 1 resulting in an easier problem to solve. For the numerical treatment, we will use power series solutions coupled with pade' approximation for better convergence results. Numerical examples will be presented also.

**AMS Classification:** 65L80

**Keywords:** Index-2 Differential Algebraic systems, Index reduction, Power series

## 1. INTRODUCTION

Recently many authors showed interest in solving differential-algebraic systems which are common in applied sciences and engineering. Traditionally, the methods used to solve these systems differentiate the algebraic equations to transform the problem into a system of ordinary differential equations, then solving the resulting system of ODE's. Such approach proved to be inefficient since chances are that the numerical solution of the resulting system is significantly different than the original problem, see Brenan et. al.[5].

A system of DAE's is characterized by its index, which is simply the number of differentiations required to convert it into a system of ordinary differential equations. We will define and consider the index-2 DAE's. Note that generally, the higher the index of the problem, the harder the method of solution. To simplify we will consider a special method to reduce the index by one which will result in an index-1 DAE, see Babolian and Hosseini[4]. This makes it easier to solve numerically. More on

such index-1 DAE can be found in Attili[1, 2] where the systems considered have some types of singularities.

Many researchers were also concerned with the numerical solution of DAEs of index-1. Arclength parameterization was considered by Ren and Spence[12], Differential transforms for DAE of index-1 was considered by Ayaz[3], Series and Pade' approximations were considered by Celik and Bayram[7, 8]. The methods of reducing and modified reducing index have been generalized and numerically implemented by Hosseini[11], the parameterization method for DAE arising in optimal control problems was done by Gorbunov and Lutoshkin[10] while fractional step methods were considered by Vijalapura et al.[15]. For index 2 DAEs we mention for example Celik[9] where Pade' approximation was used to solve the DAE. Runge-Kutta methods were suggested and used by Schropp[13] and Sun and Jiang[14]. In this study, we will consider differential-algebraic equations that has the form

$$\begin{aligned} x' &= f(t, x(t), y(t)) \\ 0 &= g(t, x(t), y(t)) \end{aligned}$$

which is semi-explicit DAE. Here  $g_x f_z$  is nonsingular for all  $t$ . This condition makes the system above of index-2 DAE. We will present how to reduce the index 2 problem to index 1 DAE which becomes easier to solve numerically. For the numerical treatment and as mentioned before, we will first transform the problem to index-1 DAE and then use power series solutions coupled with pade' approximation for better convergence results. This is done by first expanding the solution in power series format, up to an arbitrary order using Mathematica, then we transform the resulting solution into Pade' series form. This approach of coupling power series with Pade' approximation proved to be simple and effective and the rate of convergence is usually fast.

## 2. INDEX – 2 DIFFERENTIAL ALGEBRAIC EQUATIONS

The problem under consideration is of the form

$$(2.1) \quad \begin{aligned} x'(t) &= f(x(t), y(t)) \\ 0 &= g(x(t), y(t)) \end{aligned}$$

which can be written in the form  $F(t, x, x') = 0$  with some given initial condition  $x(0) = x_0$ . This system is referred to Differential Algebraic Equation (DAE). Here  $f : D_1 \times D_2 \rightarrow R^n$ ;  $g : D_1 \times D_2 \rightarrow R^m$ ; with  $D_1$  and  $D_2$  are open nonempty subsets of  $R^n$  and  $R^m$  respectively. Also,  $g$  is continuously differentiable,  $f$  and  $g'$  are Lipschitz-continuous and

$$\text{rank}(g'(x(t), y(t))) = m.$$

Any solution  $(x, y)$  must satisfy the constraint manifold

$$M = \{(x, y) \in D_1 \times D_2; g(x(t), y(t)) = 0\}.$$

Notice that if  $g_y(x(t), y(t))$  is nonsingular then the problem is called index 1 - DAE. This condition if satisfied means that the constraint equation  $g(x(t), y(t)) = 0$  is uniquely solvable for  $y = y(x)$  locally. This is guaranteed by the Implicit Function

Theorem. We will be concerned here with index-2 DAE; that is , where  $g_x f_y$  in nonsingular. Consider the linear model problem where

$$(2.2) \quad \begin{aligned} f(x(t), y(t)) &= Mx + Ny + P \\ g(x(t), y(t)) &= Rx + Q \end{aligned}$$

where as above

$$f : D_1 \times D_2 \rightarrow R^n; g : D_1 \times D_2 \rightarrow R^m$$

with  $D_1$  and  $D_2$  are as before open none empty sets of  $R^n$  and  $R^m$  respectively and  $g$  is continuously differentiable. Also  $M$ ,  $N$ ,  $R$  and  $P$  are respectively  $n \times n$ ,  $n \times 1$ ,  $1 \times n$ ,  $n \times 1$  matrices with  $n \geq 2$ . The index 2 condition is equivalent to saying that the scalar  $RN(t) \neq 0$  for all  $t \in D_1$ . Eliminating  $y$  from the first part of (2), notice that

$$(2.3) \quad y = \frac{1}{RN} R[x' - Mx - P]$$

and hence

$$x' = Mx + N \frac{1}{RN} R[x - Mx - P] + P \text{ or } [I - N(RN)^{-1}R][x - Mx - P] = 0$$

The system (2) becomes

$$(2.4) \quad \begin{aligned} [I - N(RN)^{-1}R][x - Mx - P] &= 0 \\ Rx + Q &= 0 \end{aligned}$$

an overdetermined system. Now expanding (4) and if we look at  $[I - N(RN)^{-1}R]$  for  $n = 2$  with  $N = [n_1 \ n_2]^T$ ,  $R = [r_1 \ r_2]$ , then with  $RN = r_1 n_1 + r_2 n_2 \neq 0$ , by assumption, we will have

$$(2.5) \quad \begin{aligned} r_2[n_2 x_1 - n_1 x_2 + (n_1 m_{21} - n_2 m_{11})x_1 + (n_1 m_{22} - n_2 m_{12})x_2 - n_2 p_1 + n_1 p_2] &= 0 \\ -r_1[n_2 x_1 - n_1 x_2 + (n_1 m_{21} - n_2 m_{11})x_1 + (n_1 m_{22} - n_2 m_{12})x_2 - n_2 p_1 + n_1 p_2] &= 0 \end{aligned}$$

Now since  $r_1(t) \neq 0$  or  $r_2(t) \neq 0$  (recall  $RN(t) \neq 0$ ), then we can arbitrarily eliminate one of the equations given in (4). This leads to the system

$$\begin{aligned} n_2 x_1' - n_1 x_2' + (n_1 m_{21} - n_2 m_{11})x_1 + (n_1 m_{22} - n_2 m_{12})x_2 - n_2 p_1 + n_1 p_2 &= 0 \\ r_1 x_1 + r_2 x_2 + Q &= 0 \end{aligned}$$

Or in matrix format

$$(2.6) \quad Ax' + Bx + C = 0,$$

where

$$\begin{aligned} A &= \begin{bmatrix} n_2 & -n_1 \\ 0 & 0 \end{bmatrix}; C = \begin{bmatrix} n_2 p_1 - n_1 p_2 \\ -Q \end{bmatrix}; \\ B &= \begin{bmatrix} n_1 m_{21} - n_2 m_{11} & n_1 m_{22} - n_2 m_{12} \\ r_1 & r_2 \end{bmatrix}. \end{aligned}$$

This system is of index 1 and has full rank. Note that once (6) is solved, we can retrieve  $y$  from (3) before. This argument will lead to the following theorem;

**Theorem 2.1.** *The Hessenberg index- 2 system given by (2) with  $n=2$  is equivalent to index- 1 DAE system of the form given in (6).*

The proof follows directly from the details given above prior to the theorem.

One can generalize the Theorem by considering a linear model problem of the form

$$\begin{aligned} x^{(k)} &= \sum_{j=1}^k M_j x^{(j-1)} + Ny + P \\ 0 &= Rx + Q. \end{aligned}$$

which has index  $k + 1$ . Again with

$$y = (RN)^{-1} R \left[ x^{(k)} - \sum_{j=1}^k M_j x^{(j-1)} - P \right]$$

one can obtain an implicit DAE with index  $k$  of the form

$$\sum_{j=0}^k H_j x^{(j)} = \hat{Q}$$

with one equation less and  $H_j(t)$ ,  $j = 1, 2, \dots, k$  are  $n \times n$  matrices. With  $n > 2$ , the overdetermined system given by (2) will be transformed to a full rank system of index  $k$  by adding another condition. For that reason, define

$$E = \left[ \sum_{i=1}^k r_i n_i \left( I - N (RN)^{-1} R \right) \right]$$

where  $E$  is an  $n \times n$  matrix. With this we can prove the following theorem.

**Theorem 2.2.** *Consider problem (2) with  $n > 2$ , if there exists an  $m$  with  $1 \leq m \leq n$ ,  $r_m \neq 0$  for all  $t \in [0, a]$ , then the  $m$ -th row of the matrix*

$$\left( I - N (RN)^{-1} R \right)$$

*is linearly dependent with respect to other rows.*

*Proof.* Consider the matrix  $\left( I - N (RN)^{-1} R \right) = \sum_{i=1}^k r_i n_i E$  where

$R = [r_1, r_2, \dots, r_n]^T$  and  $N = [n_1, n_2, \dots, n_n]$ . Let  $t_1 \in [0, a]$  be arbitrary and define

$$I_1 = \{i : 1 \leq i \leq n, \text{ and } r_i(t_1) \neq 0\}$$

Here  $E_{is} = 0$  (the  $(is)$ -element of  $E$ ) for  $s = 1, 2, \dots, i-1, i+1, \dots, n$  and  $E_{ii} \neq 0$  since  $RN(t) = \sum_{i=1}^k (r_i n_i)(t) \neq 0$  for all  $t \in [0, a]$ . This means for all  $i \in I_1$ , the  $i$ -th row of  $E$  is linearly independent with respect to other rows of  $E$ .

Note that  $m$  with  $r_m \neq 0$  does not belong to  $I_1$ . Now define

$$I_2 = \{i : 1 \leq i \leq n, \text{ and } i \notin I_1 \text{ and } i \neq m\}.$$

We need to show that

$$(2.7) \quad (r_m E_m)(t_1) = \sum_{l \in I_2} (r_l E_l)(t_1)$$

which means  $E_m$  (the  $m$ -th row of  $E$ ) is linearly dependent with respect to  $E_i$  (the  $i$ -th row of  $E$ ). Notice that

$$E_{ls} = \begin{cases} -n_l r_s & ; \quad l \neq s \\ \sum_{j \in I_2} n_j r_j + n_m r_m & ; \quad l = s \end{cases}.$$

For  $1 \leq s \leq n$ , we have the left side and right side of (2.7) are equal since if  $s \neq m$ , we have  $r_m E_m = -r_m n_m r_s$  while

$$\sum_{l \in I_2} r_l E_l = \sum_{\substack{l \in I_2 \\ l \neq s}} r_l (-n_l r_s) + r_s \left( \sum_{\substack{l \in I_2 \\ j \neq s}} n_j r_j + n_m r_m \right) = r_m n_m r_s$$

just a minus sign difference.

Similarly it can be easily seen that the two sides are equal when  $s = m$ , completing the proof.  $\square$

As a result,  $\text{Rank}(E) = n - 1$  and the DAE is of index 1 and of full rank.

### 3. NUMERICAL DETAILS AND EXAMPLES

Recall that the DAE we are considering has the form

$$F(t, x, x') = 0, \quad x(t_0) = x_0.$$

We will consider power series solution to the given DAE with the initial condition. For that reason, assume that the solution has the form

$$x = x_0 + e_1 t + e_2 t^2 + e_3 t^3 + \dots + e_k t^k + \dots$$

where  $e_i$ ;  $i = 1, 2, 3, \dots$  are vector functions which are the same size as  $x_0$  and are to be determined. Using this approximation, we will first consider the linear approximation and drop higher order terms and solve the resulting linear system for  $e_1$ . Once  $e_1$  is computed we consider the next higher term to compute the next  $e$ . We repeat the same with higher order terms until satisfied. Once a polynomial of certain degree is obtained, we use rational approximation; Pade' approximation, for such polynomial of the form  $r(t)$  of degree  $N$  where

$$r(t) = \frac{p(t)}{q(t)}$$

here  $p(t)$  and  $q(t)$  are polynomials whose degree add up to  $N$  ( $N = m + n$ ). So we can write  $r(t)$  as

$$r(t) = \frac{p(t)}{q(t)} = \frac{p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n}{q_0 + q_1 t + q_2 t^2 + \dots + q_n t^n}$$

which approximates  $f(t)$  on a closed interval containing zero. Note that for  $r(t)$  to be defined at zero  $q_0 \neq 0$  or we can assume that  $q_0 = 1$ . The Pade approximation technique which is used the extension of Taylor polynomial approximation to rational functions with  $N + 1$  parameters, so that

$$f^{(k)}(0) = r^{(k)}(0)$$

where  $k = 0, 1, 2, \dots, N$ , more can be found in any standard numerical analysis book, see for example Burden and Faires[6].

For the numerical examples, we will start by solving an index-1 DAE using series method then with an example on how to transform an index-2 DAE to an index-1 problem according to the procedure explained earlier then solve the resulting system.

**Example 3.1.** Consider the following DAE, see[8, 9]:

$$\begin{bmatrix} 1 & -x & x^2 \\ 0 & 1 & -x \\ 0 & 0 & 0 \end{bmatrix} v' + \begin{bmatrix} 1 & -(x+1) & x^2 + 2x \\ 0 & -1 & x - 1 \\ 0 & 0 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ \sin[x] \end{bmatrix}$$

subject to  $v(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For which the exact solution is known, see[7], it is given as:

$$\begin{aligned} v_1(x) &= e^{-x} + x e^x \\ v_2(x) &= e^x + x \sin x \\ v_3(x) &= \sin x \end{aligned}$$

To apply the above procedure and using the initial condition, the solutions of DAE is assumed to be

$$\begin{aligned} v_1(x) &= y_{0,1} + e_1 x = 1 + e_1 x \\ v_2(x) &= y_{0,2} + e_2 x = 1 + e_2 x \\ v_3(x) &= y_{0,3} + e_3 x = e_3 x \end{aligned}$$

Substituting and solving, we obtain  $e_1 = 0, e_2 = 1, e_3 = 1$ . Repeating the process we obtain,

$$\begin{aligned} v_1(x) &= 1 + 1.5x^2 + .3333333333x^3 + .2083333333x^4 \\ &\quad .0333333333x^5 + .0097222222x^6 + .0001904761904x^7 \\ &\quad + .0002232142857x^8 + .00002204585537x^9 \\ v_2(x) &= 1 + x + 1.5x^2 + .1666666667x^3 - .125x^4 + .008333333333x^5 \\ &\quad + .009722222222x^6 + .0001984126984x^7 - .0001736111111x^8 \\ &\quad + .2755731922 * 10^{-5}x^9 \\ v_3(x) &= x - .1666666667x^3 + .008333333333x^5 - .0001984126984x^7 \\ &\quad + .2755731922 * 10^{-5}x^9 \end{aligned}$$

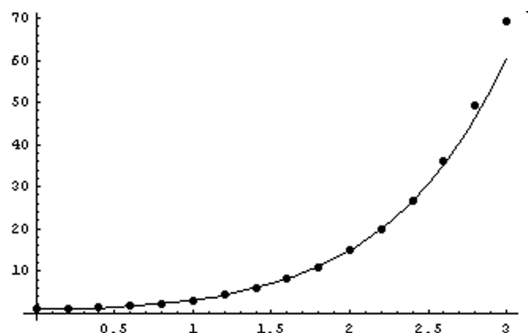


FIGURE 1. The approximate solution against the exact

which compares very well with the exact solution. Note that  $v_3(x)$  is just the first few terms of the normal Maclaurin's series of  $\sin x$ . Better results are obtained when Pade' approximation is used.

$$v_1(x) = \frac{1 - 0.357979x + 1.50937x^2 - 0.181885x^3 + 0.0984047x^4 + 0.00549595x^5}{1 - 0.357979x + 0.00937296x^2 + 0.0217502x^3 - 0.00466158x^4}$$

$$v_2(x) = \frac{1 + 0.663304x + 1.25767x^2 - 0.256179x^3 - 0.0471294x^4 - 0.0524998x^5}{1 - 0.336696x + 0.0943669x^2 - 0.0121689x^3 + 0.00460506x^4}$$

$$v_3(x) = \frac{x - 12.9416x^3 + 1.50008x^5}{1 - 12.7751x^2 - 0.637384x^4}$$

Some graphical comparisons are presented in Figures 1 which represents the exact  $v_1(x)$  with the Pade' approximation and Figure 2 the exact with the computed. The same is also done for  $v_2(x)$  in figures 3 and 4.

**Example 3.2.** Consider the index - 2 problem of the form, see[4]:

$$x_1' = \left( \lambda - \frac{1}{2-t} \right) x_1 + (2-t) \lambda y + q_1(t)$$

$$x_2' = \frac{\lambda - 1}{2-t} x_1 - x_2 + (\lambda - 1) y + q_2(t)$$

$$0 = (t+2) x_1 + (t^2 - 4) x_2 + r(t),$$

with  $x_1(0) = 1, x_2(0) = 1, r(t) = -(t^2 + t - 2)e^t$  and  $q = \begin{pmatrix} \left( \frac{3-t}{2-t} \right) e^t \\ 2e^t \end{pmatrix}$

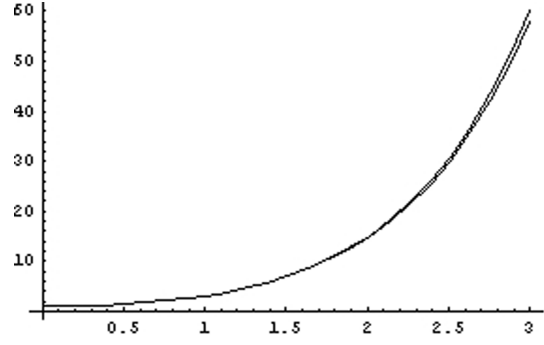


FIGURE 2. The approximate solution against the exact

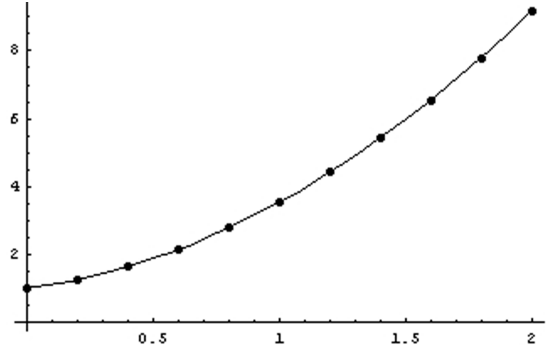


FIGURE 3. The approximate solution against the exact

Using Theorem 1, it can be transformed into index  $-1$  DAE that has the form

$$\begin{bmatrix} (\lambda - 1) & \lambda(t - 2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} + \begin{bmatrix} \frac{\lambda - 1}{2 - t} & \lambda(t - 2) \\ t + 2 & t^2 - 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ = \begin{bmatrix} \frac{-2\lambda t^2 + (7\lambda + 1)t - 5\lambda - 3}{(t^2 + t - 2)} e^t \end{bmatrix}$$

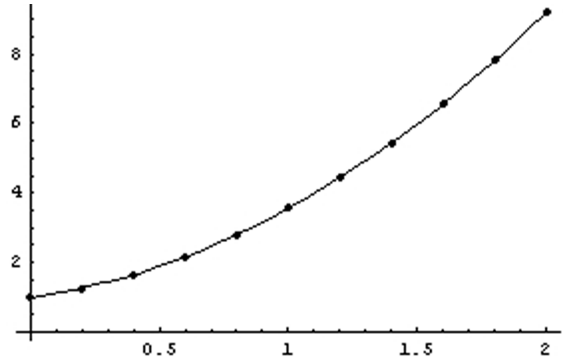


FIGURE 4. The approximate solution against the exact

To obtain a series solution, we assume

$$\begin{aligned}x_1(t) &= x_{01} + e_1 t = 1 + e_1 t \\x_2(t) &= x_{02} + e_2 t = 1 + e_2 t\end{aligned}$$

substituting into the DAE, dropping higher order terms, the resulting linear system is

$$A \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 18 & -40 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -22 \\ -1 \end{pmatrix}$$

which when solved leads to  $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , which means

$$\begin{aligned}x_1(t) &= 1 + t \\x_2(t) &= 1 + t\end{aligned}$$

To obtain the corresponding  $y(t)$  we use the formula given earlier in (3); that is,  $y = \frac{1}{RN} R[x' - Mx - P]$  which is equivalent to

$$y(t) = \frac{t-1}{(2-t)^2} x_1 + \frac{x_1'}{2-t} - x_2 - x_2' + \frac{2t^2 - 7t + 5}{(2-t)^2} e^t$$

We start by assuming  $y(t) = 0 + e_3 t$  and equating the two sides leads to the linear equation

$$4e_3 t = -8t$$

which means  $e_3 = \frac{-1}{2}$ . To obtain another term, we again assume

$$\begin{aligned}x_1(t) &= 1 + t + e_1 t^2 \\x_2(t) &= 1 + t + e_2 t^2\end{aligned}$$

and so on. Repeating the procedure until satisfied, we get up to degree 9 the following results

$$\begin{aligned} x_1(t) = x_2(t) = & 1 + t + 0.5t^2 + 0.1666666667t^3 \\ & + 0.4166666667 \times 10^{-1}t^4 + 0.8333333333 \times 10^{-2}t^5 \\ & + 0.1388888888 \times 10^{-2}t^6 + 0.1984126984 \times 10^{-3}t^7 \\ & + 0.2480158730 \times 10^{-4}t^8 + 0.2755731922 \times 10^{-5}t^9 \end{aligned}$$

and

$$\begin{aligned} y(t) = & -0.5t - 0.375t^2 - 0.2083333333t^3 \\ & - 0.9895833333t^4 - 0.4375t^5 - 0.1892361112 \times 10^{-1}t^6 \\ & - 0.8209325397 \times 10^{-2}t^7 - 0.3603980655 \times 10^{-2}t^8 \\ & - 0.1603147046 \times 10^{-2}t^9 \end{aligned}$$

Note that the exact solution for the system is

$$x_1(t) = x_2(t) = e^t, \quad y(t) = -\frac{e^t}{2-t}.$$

The Pade'(4,5) approximation of  $x_1(t)$  is given by

$$x_1(t) \approx r(t) = \frac{p(t)}{q(t)}$$

where

$$\begin{aligned} p(t) = & 1 + 0.5555556435t + 0.1388889408t^2 + 0.01984128350t^3 \\ & + 0.001653441103t^4 + 0.6613770877 \times 10^{-4}t^5 \end{aligned}$$

$$q(t) = 1 + 0.4444443565t + 0.08333329738t^2 - 0.007936502303t^3 + 0.0003306874854t^4.$$

While the Pade'(5,4) approximation for  $y(t)$  is given as

$$y(t) \approx r_1(t) = \frac{p_1(t)}{q_1(t)}$$

where

$$\begin{aligned} p_1(t) = & -0.5 - 1.213880610t - 1.607942264t^2 - 0.7596199589t^3 \\ & - 0.1536408249t^4 - 0.003665478880t^5 \end{aligned}$$

$$q_1(t) = 1 + 0.9277612200t + 0.5742426978t^2 - 1.293492321t^3 + 0.3577391259t^4$$

A comparison between the results obtained and the exact values for  $x_1(t)$  is given in Table 1. It is clear that the Pade' approximation gives much better results than without the rational approximation. For  $y(t)$  a graphical comparison between the

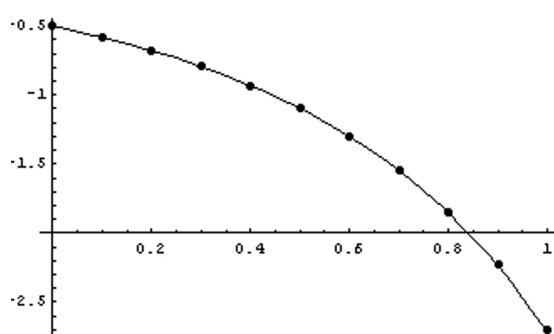


FIGURE 5. The approximate solution against the exact

exact solution(solid line) and the Pade' approximation(dots) is given in Figure 5.

| $x$ | $x_1(t)$    | <i>Exact</i> | <i>Pade</i> (5,4) |
|-----|-------------|--------------|-------------------|
| 0.0 | 1.0         | 1.0          | 1.0               |
| 0.1 | 1.105170918 | 1.105170918  | 1.105170918       |
| 0.2 | 1.221402758 | 1.221402758  | 1.221402758       |
| 0.3 | 1.349858808 | 1.349858808  | 1.349858808       |
| 0.4 | 1.491824698 | 1.491824698  | 1.491824698       |
| 0.5 | 1.648721270 | 1.648721271  | 1.648721271       |
| 0.6 | 1.822118799 | 1.822118800  | 1.822118800       |
| 0.7 | 2.013752699 | 2.013752707  | 2.013752707       |
| 0.8 | 2.225540897 | 2.225540928  | 2.225540928       |
| 0.9 | 2.459603007 | 2.459603111  | 2.459603109       |
| 1.0 | 2.718281526 | 2.718281828  | 2.718281823       |

Table 1

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