

NEWMAN-PENROSE FORMALISM AND ELECTRIC AND MAGNETIC PARTS OF WEYL TENSOR

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ABSTRACT. In Newman-Penrose formalism, the ten independent components of the Weyl tensor can be expressed in terms of five complex scalars Ψ_r ($r = 0, 1, 2, 3, 4$). By a suitable choice of the tetrad, we have expressed these scalars in terms of the electric part E_{ij} and magnetic part H_{ij} of the Weyl tensor. The results thus obtained are then applied to Gödel spacetime.

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1. INTRODUCTION

It is known that Riemann tensor can be separated into a 'trace-free' and a 'Ricci part' and this separation is established by the Weyl tensor (in n dimension) as

$$\begin{aligned} C_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}) \\ + \frac{1}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk})R \end{aligned} \quad (1)$$

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so that in 4-dimension, the Weyl tensor is related to the Riemann and Ricci tensor through the equation

$$C_{ijkl} = R_{ijkl} - \frac{1}{2}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}) + \frac{1}{6}(g_{ik}g_{jl} - g_{il}g_{jk})R \quad (2)$$

where $R = g^{ij}R_{ij}$. This tensor has all the symmetries of the Riemann tensor and also has the property that $g^{jl}C_{ijkl} = 0$ in contrast to $g^{jl}R_{ijkl} = R_{ik}$. The Riemann tensor has 20 independent components, while the Weyl and Ricci tensors have 10 independent components each. For a unit time-like vector field u_i such that $u_i u^i = 1$ (physically, the time-like vector field u_i is often taken to be the 4-velocity of the fluid), the electric and magnetic parts of the Weyl tensor are defined respectively, as

$$E_{ik} = C_{ijkl} u^j u^l \quad (3)$$

$$H_{ik} = {}^*C_{ijkl} u^j u^l = \frac{1}{2}\eta_{ij}{}^{mn}C_{mnkl} u^j u^l \quad (4)$$

The Newman-Penrose formalism[7](also known as spin-coefficient formalism) is a tetrad formalism with specific choice of the basis vectors. Accordingly a complex null tetrad $Z_\mu^i = \{ l^i, n^i, m^i, \bar{m}^i \}$ is chosen in Newman-Penrose formalism, where l^i, n^i are real null vectors and $m^i, \bar{m}^i$ are the complex null vectors. All the inner products between the tetrad vectors vanish except $l_i n^i = 1 = -m_i \bar{m}^i$. The metric tensor g_{ij} , in terms of the tetrad vectors, can be written as

$$g_{ij} = l_i n_j + n_i l_j - m_i \bar{m}_j - \bar{m}_i m_j \quad (5)$$

In the NP-formalism, the ten independent components of the Weyl tensor are projected on the null tetrad and thus represented by five (complex) scalars

$$\begin{aligned} \Psi_0 &= -C_{1313} = -C_{pqrs} l^p m^q l^r m^s \\ \Psi_1 &= -C_{1213} = -C_{pqrs} l^p n^q l^r m^s \\ \Psi_2 &= -C_{1342} = -C_{pqrs} l^p m^q \bar{m}^r n^s \\ \Psi_3 &= -C_{1242} = -C_{pqrs} l^p n^q \bar{m}^r n^s \\ \Psi_4 &= -C_{2424} = -C_{pqrs} n^p \bar{m}^q n^r \bar{m}^s \end{aligned} \quad (6)$$

The Weyl tensor can be classified on the basis of its eigenvalues is known as Petrov classification, the same classification can be obtained on the basis of complex scalars defined in (6)(various methods are discussed in Kramer et al [6]).

In this paper we have expressed complex scalars $\Psi_0, \Psi_1, \dots, \Psi_4$, in terms of E_{ij} and H_{ij} . The results are then applied to Gödel spacetime.

It is hoped that the present form is useful in the classification of Weyl tensor in terms of electric and magnetic parts.

2. NEWMAN-PENROSE FORMALISM AND ELECTROMAGNETIC PARTS OF WEYL TENSOR

We choose a null complex tetrad $Z_\mu^i = \{ l^i, n^i, m^i, \bar{m}^i \}$. In this formalism the 4-velocity vector u_i is given by

$$u^i = \frac{1}{\sqrt{2}} (l^i + n^i) \quad (7)$$

The 4-velocity vector defined above can be used to get electric part E_{ik} and magnetic part H_{ik} of Weyl tensor for a given space-time.

It is known that both E_{ik} and H_{ik} are space-like, symmetric and traceless, i.e.,

$$E_{ik} = E_{ki}, \quad E_{ik} u^k = 0, \quad E_{ik} g^{ik} = E^t{}_t = 0 \quad (8a)$$

$$H_{ik} = H_{ki}, \quad H_{ik} u^k = 0, \quad H_{ik} g^{ik} = H^t{}_t = 0 \quad (8b)$$

The Weyl tensor is said to be purely electric if $H_{ik} = 0$ and purely magnetic if $E_{ik} = 0$ and in terms of E_{ik} and H_{ik} , the Weyl tensor can be decomposed as ([6])

$$\begin{aligned} C_{hijk} = & 2u_h u_j E_{ik} + 2u_i u_k E_{hj} - 2u_h u_k E_{ij} - 2u_i u_j E_{hk} \\ & + g_{hk} E_{ij} + g_{ij} E_{hk} - g_{hj} E_{ik} - g_{ik} E_{hj} \\ & + \eta_{hi}{}^{pq} u_k u_p H_{qj} - \eta_{hi}{}^{pq} u_p u_j H_{qk} + \eta_{jk}{}^{pq} u_i u_p H_{hq} - \eta_{jk}{}^{pq} u_h u_p H_{iq} \end{aligned} \quad (9)$$

Using (6) and (9) along with (7) it is possible to express complex scalars $\Psi_0, \Psi_1, \dots, \Psi_4$ in terms of electric and magnetic parts of Weyl tensor. Thus we get,

$$\Psi_0 = -m^i m^k E_{ik} + \frac{l_p + n_p}{2} l^h m^i m^k \eta_{hi}{}^{pq} H_{qk} + \frac{l_p + n_p}{2} m^i l^j m^k \eta_{jk}{}^{pq} H_{iq} \quad (10)$$

$$\begin{aligned} \Psi_1 = & -n^i m^k E_{ik} + \frac{l_p + n_p}{2} l^h n^i m^k \eta_{hi}{}^{pq} H_{qk} - \frac{l_p + n_p}{2} l^h l^j m^k \eta_{jk}{}^{pq} H_{hq} + \\ & \frac{l_p + n_p}{2} n^i l^j m^k \eta_{jk}{}^{pq} H_{iq} \end{aligned} \quad (11)$$

$$\Psi_2 = l^i n^k E_{ik} - \frac{l_p + n_p}{2} l^h m^i \bar{m}^j \eta_{hi}{}^{pq} H_{qj} + \frac{l_p + n_p}{2} m^i \bar{m}^j n^k \eta_{jk}{}^{pq} H_{iq} \quad (12)$$

$$\begin{aligned} \Psi_3 = & -l^i \bar{m}^k E_{ik} - \frac{l_p + n_p}{2} l^h n^i \bar{m}^j \eta_{hi}{}^{pq} H_{qj} - \frac{l_p + n_p}{2} l^h \bar{m}^j n^k \eta_{jk}{}^{pq} H_{hq} + \\ & \frac{l_p + n_p}{2} n^i \bar{m}^j n^k \eta_{jk}{}^{pq} H_{iq} \end{aligned} \quad (13)$$

$$\Psi_4 = -\bar{m}^i \bar{m}^k E_{ik} + \frac{l_p + n_p}{2} n^h \bar{m}^i \bar{m}^k \eta_{hi}{}^{pq} H_{qk} + \frac{l_p + n_p}{2} \bar{m}^i n^j \bar{m}^k \eta_{jk}{}^{pq} H_{iq} \quad (14)$$

3. GÖDEL SPACETIME

It is known that the Gödel universe is a solution of Einstein's field equations (with non-zero cosmological constant, the material content is assumed to be dust). This solution is not the realistic model of the universe but due to some interesting features it has attracted many researchers. In this model the matter in universe does not expand but rotate. The geometrical and physical properties of Gödel solution, were discussed in detail by Hawking and Ellis [5]. The Gödel solution is characterized by the conditions that (see [7] and [2])

$$a_i = \theta = \sigma_{ij} = \omega_{ij} = 0 \quad (15a)$$

$$\omega = \frac{1}{a\sqrt{2}} = \text{constant} \quad (15b)$$

For Gödel spacetime, a null tetrad can be chosen as

$$l^i = \frac{1}{\sqrt{2}} (1, 0, 0, 1) \quad (16)$$

$$n^i = \frac{1}{\sqrt{2}} (-1, 0, 0, 1) \quad (17)$$

$$m^i = \frac{1}{\sqrt{2}} (0, 1, -i\sqrt{2} e^{-qy}, i\sqrt{2}) \quad (18)$$

$$\bar{m}^i = \frac{1}{\sqrt{2}} (0, 1, i\sqrt{2} e^{-qy}, -i\sqrt{2}) \quad (19)$$

Further, the Weyl tensor for Gödel spacetime is purely electric, i.e. $H_{ik} = 0$ ([1]). Also, Hasmani and Katkar [4] have obtained tetrad components of electric part of Weyl tensor for Gödel spacetime, for the null tetrad given above. The non-vanishing components are given by,

$$E_{11} = E_{22} = -E_{12} = -E_{34} = -\frac{q^2}{6} \quad (20)$$

Using these in the expressions of $\Psi_0, \Psi_1, \dots, \Psi_4$ given above in equations (10) – (14), we get

$$\Psi_0 = -E_{33} = 0$$

$$\Psi_1 = -E_{23} = 0$$

$$\Psi_2 = E_{12} = \frac{q^2}{6} \quad (21)$$

$$\Psi_3 = E_{14} = 0$$

$$\Psi_4 = E_{44} = 0$$

It can be seen from above expressions that only non-vanishing complex scalar is Ψ_2 , this means that the metric Gödel spacetime is of Petrov type-D (see also [2]).

4. CONCLUSION

We have expressed complex scalars $\Psi_0, \Psi_1, \dots, \Psi_4$ of Newman-Penrose formalism in terms of electric and magnetic parts of Weyl tensor for a general case. It is hoped that these results will be useful to analyze gravitational fields in Petrov Classification. We have applied the results to Gödel metric and we have shown that the metric is of Petrov type-D. Further it is shown that the parameter q characterizing the rotation enters in the expression of Ψ_2 , this can be regarded as the radiation is due to rotation.

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REFERENCES

- [1] Ahsan, Z.: *Indian J. Pure Appl. Maths.* 30 (1999) 863.
- [2] Ahsan, Z., Ahsan, N. and Ali, S.: *Bulletin Cal. Math. Soc.* 93 No. 5 (2001) 407.
- [3] Ellis, G.F.R.: Relativistic Cosmology. Article in *genral Relativity and Cosmology. Proc. Inter, School of physics 'Enrico Fermi', Ed. R.K. Sachs.* Academic Press, New York (1971) 104.
- [4] Hasmani, A.H. and Katkar, L.N.: *The Aligarh Bulletin of Mathematics*, 27 No.2 (2008) (To appear) .
- [5] Hawking, S.W. and Ellis, G.F.R.: *The Large Scale Structure of the Spacetime.* Cambridge University Press (1973) 168.
- [6] Kramer D., Stephani H., Herlt E.: *Exact Solutions of Einstein's Field Equations.* Cambridge University Press, Cambridge(1980).
- [7] Newman, E.T., Penrose, R.: *An approach to gravitational radiation by a method of spin-coefficients.* J. Math. Phys. 3 (1962) 566-578.