

JET RIEMANN-LAGRANGE GEOMETRY AND SOME APPLICATIONS IN THEORETICAL BIOLOGY

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ABSTRACT. The aim of this paper is to construct a natural Riemann-Lagrange differential geometry on 1-jet spaces, in the sense of nonlinear connections, generalized Cartan connections, d-torsions, d-curvatures, jet electromagnetic fields and jet electromagnetic Yang-Mills energies, starting from some given nonlinear evolution ODEs systems modelling biologic phenomena like the cancer cell population model or the infection by human immunodeficiency virus-type 1 (HIV-1) model.

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1. HISTORICAL GEOMETRICAL ASPECTS

The Riemann-Lagrange geometry [5] of the 1-jet space $J^1(T, M)$, where T is a smooth "multi-time" manifold of dimension p and M is a smooth "spatial" manifold of dimension n , contains many fruitful ideas for the geometrical interpretation of the solutions of a given ODEs or PDEs system [7]. In this direction, authors like P.J. Olver [11] or C. Udriște [18] agreed that many applicative problems coming from Physics [11], [17], Biology [6], [9] or Economics [18] can be modelled on 1-jet spaces.

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In such an applicative-geometrical context, a lot of authors (G.S. Asanov [1], D. Saunders [15], A. Vondra [19] and many others) studied the *contravariant differential geometry* of 1-jet spaces. Moreover, proceeding with the geometrical studies of G.S. Asanov [1], the second author of this paper has recently elaborated that so-called the *Riemann-Lagrange geometry of 1-jet spaces* [5], which is a natural extension on 1-jet spaces of the already well known *Lagrangian geometry of the tangent bundle* due to R. Miron and M. Anastasiei [4]. We emphasize that the Riemann-Lagrange geometry of the 1-jet spaces allow us to regard the solutions of a given ODEs (respectively PDEs) system as *horizontal geodesics* [17] (respectively, *generalized harmonic maps* [7]) in a convenient Riemann-Lagrange geometrical structure. In this way, it was given a final solution for an open problem suggested by H. Poincaré [13] (*find the geometrical structure which transforms the field lines of a given vector field into geodesics*) and generalized by C. Udriște [17] (*find the geometric structure which converts the solutions of a given first order PDEs system into some harmonic maps*).

In the following, let us present the main geometrical ideas used by C. Udriște in order to solve the open problem of H. Poincaré. For more details, please see the works [17] and [18].

For this purpose, let us consider a Riemannian manifold $(M^n, \varphi_{ij}(x))$ and let us fix an arbitrary vector field $X = (X^i(x))$ on M . Obviously, the vector field X produces the first order ODEs system (*dynamical system*)

$$(1.1) \quad \frac{dx^i}{dt} = X^i(x(t)), \quad \forall i = \overline{1, n}.$$

Using the Riemannian metric φ_{ij} and its Christoffel symbols γ_{jk}^i and differentiating the first order ODEs system (1.1), after a convenient rearranging of the terms involved, C. Udriște constructed a second order prolongation (*single-time geometric dynamical system*) of the ODEs system (1.1), which has the form

$$(1.2) \quad \frac{d^2 x^i}{dt^2} + \gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = F_j^i \frac{dx^j}{dt} + \varphi^{ih} \varphi_{kj} X^j \nabla_h X^k, \quad \forall i = \overline{1, n},$$

where ∇ is the Levy-Civita connection of the Riemannian manifold (M, φ) and

$$F_j^i = \nabla_j X^i - \varphi^{ih} \varphi_{kj} \nabla_h X^k$$

is a $(1, 1)$ -tensor field which represents the *helicity* of the vector field X .

It is easy to see that any solution of class C^2 of the first order ODEs system (1.1) is also a solution for the second order ODEs system (1.2). Conversely, this statement is not true.

Remark 1.1. *The importance of the second order ODEs system (1.2) comes from its equivalency with the Euler-Lagrange equations of that so-called the **least squares Lagrangian function***

$$L_{ls} : TM \rightarrow \mathbb{R}_+,$$

given by

$$(1.3) \quad L_{ls}(x, y) = \frac{1}{2} \varphi_{ij}(x) [y^i - X^i(x)] [y^j - X^j(x)].$$

Note that the field lines of class C^2 of the vector field X are the global minimum points of the **least squares energy action**

$$\mathbb{E}_{ls}(x(t)) = \int_a^b L_{ls}(x^k(t), \dot{x}^k(t)) dt.$$

As a conclusion, the field lines of class C^2 of the vector field X are solutions of the Euler-Lagrange equations produced by L_{ls} . Because the Euler-Lagrange equations of L_{ls} are exactly the equations (1.2), C. Udriște claims that the solutions of class C^2 of the first order ODEs system (1.1) are **horizontal geodesics** on the **Riemann-Lagrange manifold** [17]

$$(\mathbb{R} \times M, 1 + \varphi, N_{(1)}^i)_j = \gamma_{jk}^i y^k - F_j^i).$$

2. RIEMANN-LAGRANGE GEOMETRICAL BACKGROUND ON 1- JET SPACES

In this Section, we regard the given first order nonlinear ODEs system (1.1) as an ordinary differential system on an 1-jet space $J^1(T, \mathbb{R}^n)$, where $T \subset \mathbb{R}$. Moreover, starting from Udriște's geometrical ideas, we construct some jet Riemann-Lagrange geometrical objects (nonlinear connections, generalized Cartan connections, d-torsions, d-curvatures, jet electromagnetic fields and jet electromagnetic Yang-Mills energies) which, in our opinion, characterize from a geometrical point of view the given nonlinear ODEs system of order one.

In this direction, let $T = [a, b] \subset \mathbb{R}$ be a compact interval of the set of real numbers and let us consider the jet fibre bundle of order one

$$J^1(T, \mathbb{R}^n) \rightarrow T \times \mathbb{R}^n, \quad n \geq 2,$$

whose local coordinates (t, x^i, x_1^i) , $i = \overline{1, n}$, transform by the rules

$$(2.1) \quad \tilde{t} = \tilde{t}(t), \quad \tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{x}_1^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot x_1^j.$$

Remark 2.1. From a physical point of view, the coordinate t has the physical meaning of **relativistic time**, the coordinate $x = (x^i)_{i=\overline{1, n}}$ represents the **spatial coordinate** and the coordinate $y = (x_1^i)_{i=\overline{1, n}}$ has the physical meaning of **direction** or **relativistic velocity**. Thus, the coordinate y is intimately connected with the physical concept of **anisotropy**.

Let us consider $X = \left(X_{(1)}^{(i)}(x^k) \right)$ be an arbitrary d -tensor field on the 1-jet space $J^1(T, \mathbb{R}^n)$, whose local components transform by the rules

$$\tilde{X}_{(1)}^{(i)} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot X_{(1)}^{(j)}.$$

Clearly, the d-tensor field X produces the jet ODEs system of order one (*jet dynamical system*)

$$(2.2) \quad x_1^i = X_{(1)}^{(i)}(x^k(t)), \quad \forall i = \overline{1, n},$$

where $x(t) = (x^i(t))$ is an unknown curve on $\mathbb{R}^n$ (i. e., a *jet field line* of the d-tensor field X) and we use the notation

$$x_1^i \stackrel{\text{not}}{=} \frac{dx^i}{dt}, \quad \forall i = \overline{1, n}.$$

Remark 2.2. *The main and refined difference between the ODEs systems (1.1) and (2.2), which have the same form, consists only in their invariance transformation groups. Thus, the ODEs system (1.1) is invariant under the transformation group $\tilde{x}^i = \tilde{x}^i(x^j)$ while the ODEs system (2.2) is invariant under the transformation group $\tilde{t} = \tilde{t}(t)$, $\tilde{x}^i = \tilde{x}^i(x^j)$ which does not ignore the temporal reparametrizations.*

Now, let us consider the Euclidian structures $(T, 1)$ and $(\mathbb{R}^n, \delta_{ij})$, where δ_{ij} are the Kronecker symbols. Using as a pattern the Udrishte's geometrical ideas, we underline that the jet first order ODEs system (2.2) automatically produces the *jet least squares Lagrangian function*

$$JL_{ls} : J^1(T, \mathbb{R}^n) \rightarrow \mathbb{R}_+,$$

expressed by

$$(2.3) \quad JL_{ls}(x^k, x_1^k) = \sum_{i=1}^n \left[x_1^i - X_{(1)}^{(i)}(x) \right]^2.$$

Because the *global minimum points* of the *jet least squares energy action*

$$\mathbb{J}E_{ls}(c(t)) = \int_a^b JL_{ls} \left(x^k(t), \frac{dx^k}{dt} \right) dt$$

are exactly the solutions of class C^2 of the jet first order ODEs system (2.2), it follows that the solutions of class C^2 of the jet dynamical system (2.2) verify the second order *Euler-Lagrange equations* produced by JL_{ls} (*jet geometric dynamics*), namely

$$(2.4) \quad \frac{\partial [JL_{ls}]}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial [JL_{ls}]}{\partial x_1^i} \right) = 0, \quad \forall i = \overline{1, n}.$$

In conclusion, because of all the arguments exposed above we believe that we may regard the jet least squares Lagrangian function JL_{ls} as a natural geometrical substitut on the 1-jet space $J^1(T, \mathbb{R}^n)$ for the jet first order ODEs system (2.2).

Remark 2.3. *A Riemann-Lagrange geometry on $J^1(T, \mathbb{R}^n)$ produced by the jet least squares Lagrangian function JL_{ls} , via its second order Euler-Lagrange equations (2.4), in the sense of nonlinear connection, generalized Cartan connection, d-torsions, d-curvatures, jet electromagnetic field and jet Yang-Mills electromagnetic energy, is now completely done in the works [5], [6] and [7].*

In this geometric background, we introduce the following concept:

Definition 2.4. *Any geometrical object on the 1-jet space $J^1(T, \mathbb{R}^n)$, which is produced by the jet least squares Lagrangian function JL_{ls} , via its Euler-Lagrange equations (2.4), is called **geometrical object produced by the jet first order ODEs system (2.2)**.*

In this context, we give the following geometrical result (this is proved in the works [6] and [9] and, for the multi-time general case, in the paper [7]) which characterizes the jet first order ODEs system (2.2). For all details, the reader is invited to consult the book [5].

Theorem 2.5. (i) *The canonical nonlinear connection on $J^1(T, \mathbb{R}^n)$ produced by the jet first order ODEs system (2.2) is*

$$\Gamma = \left(0, N_{(1)j}^{(i)}\right),$$

whose local components $N_{(1)j}^{(i)}$ are the entries of the matrix

$$N_{(1)} = \left(N_{(1)j}^{(i)}\right)_{i,j=\overline{1,n}} = -\frac{1}{2} \left[J(X_{(1)}) - {}^T J(X_{(1)}) \right],$$

where

$$J(X_{(1)}) = \left(\frac{\partial X_{(1)}^{(i)}}{\partial x^j} \right)_{i,j=\overline{1,n}}$$

is the Jacobian matrix.

(ii) *All adapted components of the canonical generalized Cartan connection CT produced by the jet first order ODEs system (2.2) vanish.*

(iii) *The effective adapted components $R_{(1)jk}^{(i)}$ of the torsion d-tensor T of the canonical generalized Cartan connection CT produced by the jet first order ODEs system (2.2) are the entries of the matrices*

$$R_{(1)k} = \frac{\partial}{\partial x^k} [N_{(1)}], \quad \forall k = \overline{1,n},$$

where

$$R_{(1)k} = \left(R_{(1)jk}^{(i)} \right)_{i,j=\overline{1,n}}, \quad \forall k = \overline{1,n}.$$

(iv) *All adapted components of the curvature d-tensor R of the canonical generalized Cartan connection CT produced by the jet first order ODEs system (2.2) vanish.*

(v) *The geometric electromagnetic distinguished 2-form produced by the jet first order ODEs system (2.2) has the expression*

$$\mathbf{F} = F_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + N_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1,n},$$

and the adapted components $F_{(i)j}^{(1)}$ are the entries of the matrix

$$F^{(1)} = \left(F_{(i)j}^{(1)} \right)_{i,j=\overline{1,n}} = -N_{(1)}.$$

(vi) *The adapted components $F_{(i)j}^{(1)}$ of the geometric electromagnetic d-form $\mathbf{F}$ produced by the jet first order ODEs system (2.2) verify the generalized Maxwell*

equations

$$\sum_{\{i,j,k\}} F_{(i)j||k}^{(1)} = 0,$$

where $\sum_{\{i,j,k\}}$ represents a cyclic sum and

$$F_{(i)j||k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k}$$

means the horizontal local covariant derivative produced by the Berwald connection $B\Gamma_0$ on $J^1(T, \mathbb{R}^n)$. For more details, please consult [5].

(vii) The **geometric jet Yang-Mills energy produced by the jet first order ODEs system (2.2)** is given by the formula

$$\mathbf{EYM}(x) = \frac{1}{2} \cdot \text{Trace} \left[F^{(1)} \cdot {}^T F^{(1)} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left[F_{(i)j}^{(1)} \right]^2.$$

In the next Sections, we apply the above jet Riemann-Lagrange geometrical results to certain evolution equations from Theoretical Biology that govern two of the most actual diseases of our times, namely the spread of cancer cells in vivo and the infection by human immunodeficiency virus-type 1 (HIV-1). We sincerely hope that our geometrical approach of these evolution equations to give useful mathematical informations for biologists.

Remark 2.6. For more geometrical methods applied to mathematical models coming from Theoretical Biology, the reader is invited to consult the book [9].

3. JET RIEMANN-LAGRANGE GEOMETRY FOR A CANCER CELL POPULATION MODEL IN BIOLOGY

The mathematical model of cancer cell population, which consists of a two dimensional system of ODEs with four parameters, was introduced in 2006 by Garner et al. [2].

It is well known that cancer cell populations consist of a combination of proliferating, quiescent and dead cells that determine tumor growth or cancer spread [3]. Moreover, recent research in cancer progression and treatment indicates that many forms of cancer arise from one abnormal cell or a small subpopulation of abnormal cells [14]. These cells, which support cancer growth and spread are called *cancer stem cells* (CSCs). Targeting these CSCs is crucial because they display many of the same characteristics as healthy stem cells, and they have the capacity of initiating new tumors after long periods of remission. The understanding of cancer mechanism could have a significant impact on cancer treatment approaches as it emphasizes the importance of targeting diverse cell subpopulations at a specific stage of development.

The nondimensionalized model introduced by Garner et al. is based on a system of Solyanik et al. [16], which starts from the following assumptions:

- (1) the cancer cell population consists of proliferating and quiescent (resting) cells;

- (2) the cells can lose their ability to divide under certain conditions and then transit from the proliferating to the resting state;
- (3) resting cells can either return to the proliferating state or die.

The dynamical system has two state variables, namely P - *the number of proliferating cells* and Q - *the number of quiescent cells*, and their evolution in time is described by the following differential equations (*cancer cell population flow*)

$$(3.1) \quad \begin{cases} \frac{dP}{dt} = P - P(P + Q) + F(P, Q), \\ \frac{dQ}{dt} = -rQ + aP(P + Q) - F(P, Q), \end{cases}$$

$$F(P, Q) = \frac{hPQ}{1 + kP^2}, \quad r = \frac{d}{b}, \quad h = \frac{A}{ac}, \quad k = \frac{Bb^2}{c^2},$$

where

- a - is a dimensionless constant that measures the relative nutrient uptake by resting and proliferating cells;
- b - is the rate of cell division of the proliferating cells;
- c - depends on the intensity of consumption by proliferating cells and gives the magnitude of the rate of cell transition from the proliferating stage to the resting stage in per cell per day;
- d - is the rate of cell death of the resting cells (per day);
- A - represents the initial rate of increase in the intensity of cell transition from the quiescent to proliferating state at small P ;
- A/B - represents the rate of decrease in the intensity of cell transition from the quiescent to proliferating state when P becomes larger.

The Riemann-Lagrange geometrical behavior on the 1-jet space $J^1(T, \mathbb{R}^2)$ of the *cancer cell population flow* is described in the following result:

Theorem 3.1. (i) *The canonical nonlinear connection on $J^1(T, \mathbb{R}^2)$ produced by the cancer cell population flow (3.1) has the local components*

$$\hat{\Gamma} = \left(0, \hat{N}_{(1)j}^{(i)}\right), \quad i, j = \overline{1, 2},$$

where, if

$$F_P = \frac{hQ(1 - kP^2)}{(1 + kP^2)^2} \quad \text{and} \quad F_Q = \frac{hP}{1 + kP^2}$$

are the first partial derivatives of the function F , then we have

$$\begin{aligned} \hat{N}_{(1)1}^{(1)} &= \hat{N}_{(1)2}^{(2)} = 0, \\ \hat{N}_{(1)2}^{(1)} &= -\hat{N}_{(1)1}^{(2)} = \frac{1}{2} [(2a + 1)P + aQ - (F_P + F_Q)] = \\ &= \frac{1}{2} \left[(2a + 1)P + aQ - \frac{hQ(1 - kP^2)}{(1 + kP^2)^2} - \frac{hP}{1 + kP^2} \right]. \end{aligned}$$

(ii) *All adapted components of the canonical generalized Cartan connection $C\hat{\Gamma}$ produced by the cancer cell population flow (3.1) vanish.*

(iii) All adapted components of the **torsion** d -tensor $\hat{T}$ of the canonical generalized Cartan connection $C\hat{T}$ **produced by the cancer cell population flow (3.1)** are zero, except

$$\begin{aligned}\hat{R}_{(1)21}^{(1)} &= -\hat{R}_{(1)11}^{(2)} = a + \frac{1}{2}(1 - F_{PP} - F_{PQ}), \\ \hat{R}_{(1)22}^{(1)} &= -\hat{R}_{(1)12}^{(2)} = \frac{1}{2}(a - F_{PQ} - F_{QQ}) = \frac{1}{2}(a - F_{PQ}),\end{aligned}$$

where

$$F_{PP} = -\frac{2hkPQ(3 - kP^2)}{(1 + kP^2)^3}, \quad F_{PQ} = \frac{h(1 - kP^2)}{(1 + kP^2)^2} \text{ and } F_{QQ} = 0$$

are the second partial derivatives of the function F .

(iv) All adapted components of the **curvature** d -tensor $\hat{R}$ of the canonical generalized Cartan connection $C\hat{T}$ **produced by the cancer cell population flow (3.1)** vanish.

(v) The **geometric electromagnetic distinguished 2-form produced by the cancer cell population flow (3.1)** has the expression

$$\hat{\mathbf{F}} = \hat{F}_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \hat{N}_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1, 2},$$

and the adapted components $\hat{F}_{(i)j}^{(1)}$, $i, j = \overline{1, 2}$, are given by

$$\begin{aligned}\hat{F}_{(1)1}^{(1)} &= \hat{F}_{(2)2}^{(1)} = 0, \\ \hat{F}_{(2)1}^{(1)} &= -\hat{F}_{(1)2}^{(1)} = \frac{1}{2}[(2a + 1)P + aQ - (F_P + F_Q)] = \\ &= \frac{1}{2} \left[(2a + 1)P + aQ - \frac{hQ(1 - kP^2)}{(1 + kP^2)^2} - \frac{hP}{1 + kP^2} \right].\end{aligned}$$

(vi) The **biologic geometrical Yang-Mills energy produced by the cancer cell population flow (3.1)** is given by the formula

$$\mathbf{EYM}^{cancer}(P, Q) = \frac{1}{4} \left[(2a + 1)P + aQ - \frac{hQ(1 - kP^2)}{(1 + kP^2)^2} - \frac{hP}{1 + kP^2} \right]^2.$$

Proof. We regard the cancer cell population flow (3.1) as a particular case of the jet first order ODEs system (2.2) on the 1-jet space $J^1(T, \mathbb{R}^2)$, with

$$n = 2, \quad x^1 = P, \quad x^2 = Q$$

and

$$\begin{aligned}X_{(1)}^{(1)}(x^1, x^2) &= x^1 - x^1(x^1 + x^2) + F(x^1, x^2), \\ X_{(1)}^{(2)}(x^1, x^2) &= -rx^2 + ax^1(x^1 + x^2) - F(x^1, x^2).\end{aligned}$$

Now, using the Theorem 2.5 and taking into account that we have the Jacobian matrix

$$\begin{aligned} J(X_{(1)}) &= \begin{pmatrix} 1 - 2P - Q + F_P & -P + F_Q \\ 2aP + aQ - F_P & -r + aP - F_Q \end{pmatrix} \\ &= \begin{pmatrix} 1 - 2P - Q + \frac{hQ(1 - kP^2)}{(1 + kP^2)^2} & -P + \frac{hP}{1 + kP^2} \\ 2aP + aQ - \frac{hQ(1 - kP^2)}{(1 + kP^2)^2} & -r + aP - \frac{hP}{1 + kP^2} \end{pmatrix}, \end{aligned}$$

we obtain what we were looking for. ■

Remark 3.2 (Open problem). *The Yang-Mills biologic energetical curves of constant level produced by the cancer cell population flow (3.1), which are different by the empty set, are in the plane POQ the curves of implicit equations*

$$\mathcal{C}_C : \left[(2a + 1)P + aQ - \frac{hQ(1 - kP^2)}{(1 + kP^2)^2} - \frac{hP}{1 + kP^2} \right]^2 = 4C,$$

where $C \geq 0$. For instance, the **zero Yang-Mills biologic energetical curve produced by the cancer cell population flow (3.1)** is in the plane POQ the graph of a rational function:

$$\mathcal{C}_0 : Q = \frac{P(1 + kP^2)[h - (2a + 1)(1 + kP^2)]}{a(1 + kP^2)^2 - h(1 - kP^2)}.$$

As a possible opinion, we consider that if the cancer cell population flow does not generate any Yang-Mills biologic energies, then it is to be expected that the variables P and Q vary along the rational curve $\mathcal{C}_0$. Otherwise, if the cancer cell population flow generates an Yang-Mills biologic energy, then it is possible that the shapes of the constant Yang-Mills biologic energetical curves $\mathcal{C}_C$ to offer useful interpretations for biologists.

4. THE JET RIEMANN-LAGRANGE GEOMETRY OF THE INFECTION BY HUMAN IMMUNODEFICIENCY VIRUS (HIV-1) EVOLUTION MODEL

It is well known that the major target of HIV infection is a class of lymphocytes, or white blood cells, known as $CD4^+$ T cells. These cells secrete growth and differentiations factors that are required by other cell populations in the immune system, and hence these cells are also called "*helper T cells*". After becoming infected, the $CD4^+$ T cells can produce new HIV virus particles (or virions) so, in order to model HIV infection it was introduced a population of uninfected target cells T , and productively infected cells T^* .

Over the past decade, a number of models have been developed to describe the immune system, its interaction with HIV, and the decline in $CD4^+$ T cells. We propose for geometrical investigation a model that incorporates viral production (for more details, please see [8], [12]). This mathematical model of infection by

HIV-1 relies on the variables $T(t)$ - the population of uninfected target cells, $T^*(t)$ - the population of productively infected cells, and $V(t)$ - the HIV-1 virus, whose evolution in time is given by the HIV-1 flow [12]

$$(4.1) \quad \begin{cases} \frac{dT}{dt} = s + (p - d)T - \frac{pT^2}{m} - kVT \\ \frac{dT^*}{dt} = kTV - \delta T^* \\ \frac{dV}{dt} = n\delta T^* - cV, \end{cases}$$

where

- s represents the rate at which new T cells are created from sources within the body, such as thymus;
- p is the maximum proliferation rate of T cells;
- d is the death rate per T cells;
- δ represents the death rate for infected cells T^* ;
- m is the T cells population density at which proliferation shuts off;
- k is the infection rate;
- n represents the total number of virions produced by a cell during its lifetime;
- c is the rate of clearance of virions.

In what follows, we apply our jet Riemann-Lagrange geometrical results to the HIV-1 flow (4.1) regarded on the 1-jet space $J^1(T, \mathbb{R}^3)$. In this context, we obtain:

Theorem 4.1. (i) *The canonical nonlinear connection on $J^1(T, \mathbb{R}^3)$ produced by the HIV-1 flow (4.1) has the local components*

$$\check{\Gamma} = \left(0, \check{N}_{(1)j}^{(i)}\right), \quad i, j = \overline{1, 3},$$

where $\check{N}_{(1)j}^{(i)}$ are the entries of the matrix

$$\check{N}_{(1)} = -\frac{1}{2} \begin{pmatrix} 0 & -kV & -kT \\ kV & 0 & kT - n\delta \\ kT & -kT + n\delta & 0 \end{pmatrix}.$$

(ii) *All adapted components of the canonical generalized Cartan connection $C\check{\Gamma}$ produced by the HIV-1 flow (4.1) vanish.*

(iii) *All adapted components of the torsion d-tensor $\check{T}$ of the canonical generalized Cartan connection $C\check{\Gamma}$ produced by the HIV-1 flow (4.1) vanish, except the entries of the matrices*

$$\check{R}_{(1)1} = \begin{pmatrix} 0 & 0 & k/2 \\ 0 & 0 & -k/2 \\ -k/2 & k/2 & 0 \end{pmatrix}$$

and

$$\check{R}_{(1)3} = \begin{pmatrix} 0 & k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\check{R}_{(1)k} = \left(\check{R}_{(1)jk}^{(i)} \right)_{i,j=\overline{1,3}}, \quad \forall k \in \{1, 3\}.$$

(iv) All adapted components of the **curvature** d-tensor $\check{\mathbf{R}}$ of the canonical generalized Cartan connection $C\check{\Gamma}$ **produced by the HIV-1 flow (4.1)** vanish.

(v) The **geometric electromagnetic distinguished 2-form produced by the HIV-1 flow (4.1)** has the expression

$$\check{\mathbf{F}} = \check{F}_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \check{N}_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1, 3},$$

and the adapted components $\check{F}_{(i)j}^{(1)}$ $i, j = \overline{1, 3}$, are the entries of the matrix

$$\check{F}^{(1)} = \frac{1}{2} \begin{pmatrix} 0 & -kV & -kT \\ kV & 0 & kT - n\delta \\ kT & -kT + n\delta & 0 \end{pmatrix}.$$

(vi) The **biologic geometric Yang-Mills energy produced by the HIV-1 flow (4.1)** is given by the formula

$$\mathbf{EYM}^{HIV-1}(T, T^*, V) = \frac{1}{4} [k^2(V^2 + T^2) + (kT - n\delta)^2].$$

Proof. Consider the HIV-1 flow (4.1) as a particular case of the jet first order ODEs system (2.2) on the 1-jet space $J^1(T, \mathbb{R}^3)$, with

$$n = 3, \quad x^1 = T, \quad x^2 = T^*, \quad x^3 = V$$

and

$$X_{(1)}^{(1)}(x^1, x^2, x^3) = s + (p - d)x^1 - \frac{p}{m}(x^1)^2 - kx^3x^1,$$

$$X_{(1)}^{(2)}(x^1, x^2, x^3) = kx^1x^3 - \delta x^2,$$

$$X_{(1)}^{(3)}(x^1, x^2, x^3) = n\delta x^2 - cx^3.$$

It follows that we have the Jacobian matrix

$$J(X_{(1)}) = \begin{pmatrix} p - d - \frac{2p}{m}T - kV & 0 & -kT \\ kV & -\delta & kT \\ 0 & n\delta & -c \end{pmatrix}.$$

In conclusion, using the Theorem 2.5, we find the required result. ■

Remark 4.2 (Open problem). The **Yang-Mills biologic energetical surfaces of constant level produced by the HIV-1 flow (4.1)** have in the system of axis OTT^*V the implicit equations

$$\Sigma_C : k^2(V^2 + T^2) + (kT - n\delta)^2 = 4C,$$

where $C \geq 0$. It is obvious that the surfaces Σ_C are some real or imaginary **cylinders**. Taking into account that the family of conics

$$\Gamma_C : 2k^2T^2 + k^2V^2 - 2kn\delta T + n^2\delta^2 - 4C = 0,$$

which generate the cylinders Σ_C , have the matrices

$$A = \begin{pmatrix} 2k^2 & 0 & -kn\delta \\ 0 & k^2 & 0 \\ -kn\delta & 0 & n^2\delta^2 - 4C \end{pmatrix},$$

it follows that their invariants are $\Delta_C = k^4(n^2\delta^2 - 8C)$, $\delta = 2k^4 > 0$ and $I = 3k^2 > 0$. As a consequence, we have the following situations:

- (1) If $0 \leq C < \frac{n^2\delta^2}{8}$, then we have the **empty set** $\Sigma_{0 \leq C < \frac{n^2\delta^2}{8}} = \emptyset$;
- (2) If $C = \frac{n^2\delta^2}{8}$, then the surface $\Sigma_{C=\frac{n^2\delta^2}{8}}$ degenerates into the **straight line**

$$\Sigma_{C=\frac{n^2\delta^2}{8}} : \begin{cases} T = \frac{n\delta}{2k} \\ V = 0 \end{cases};$$

- (3) If $C > \frac{n^2\delta^2}{8}$, then the surface $\Sigma_{C > \frac{n^2\delta^2}{8}}$ is a **right elliptic cylinder** of equation

$$\Sigma_{C > \frac{n^2\delta^2}{8}} : \frac{\left(T - \frac{n\delta}{2k}\right)^2}{a^2} + \frac{V^2}{b^2} = 1, \quad T^* \in \mathbb{R},$$

where $a < b$ are given by

$$a = \frac{\sqrt{8C - n^2\delta^2}}{2k}, \quad b = \frac{\sqrt{8C - n^2\delta^2}}{k\sqrt{2}}.$$

Obviously, it has as axis of symmetry the straight line $\Sigma_{C=\frac{n^2\delta^2}{8}}$.

There exist possible valuable informations for biologists contained in the shapes of the Yang-Mills energetical constant surfaces Σ_C ?

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