IMMERSED HYPERDYNAMICAL SYSTEMS

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Abstract. In this paper by using of immersed hypergroups and hyperdynamical systems the notion of immersed hyperdynamical system is considered. Quotient immersed hyperdynamical systems are studied.

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1. Introduction

Theory of hypergroups and immersed hypergroups have brought many results in mathematics [1, 2, 5, 6]

In this paper are going to consider hyperdynamical systems [5, 6] which are immersed, and we will prove the following results for them:

(1) If (M, D, H) and (N, E, I) are two immersed hyperdynamical systems, then $(M \times N, F, H \times I)$, where $F = \{g^{(a,b)} : M \times N \to M \times N : (a, b) \in H \times I\}$ is an immersed hyperdynamical system.

(2) If R is an equivalence relation compatible with (M, D, H), where (M, D, H) is an immersed hyperdynamical system, and if $f^a : M/R \to M/R$, then $(M/R, \{f^a\}, H)$

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is an immersed hyperdynamical system.

(3) If (M, D, H) is an immersed hyperdynamical system, and $\psi : H \to R^k$ is a smooth one to one immersion, where $\psi(H)$ has the join operation $\psi(ab) = \psi(a)\psi(b)$, then $(M, D, \psi(H))$ is an immersed hyperdynamical system.

(4) If (M, D, H) is an immersed hyperdynamical system, and U is any open subset of H with a join operation $\star_U$ on $U \times U$ defined by $a \star_U b = ab \bigcap U$, then (M, D, H) is an immersed hyperdynamical system.

(5) If $(H, \circ)$ is an immersed hypergroup and $\{A(x)\}$ is a family of pairwise disjoint nonempty open subsets of a differentiable manifold M, and (M, D, H) is an immersed hyperdynamical system, then (M, D, K_H) is an immersed hyperdynamical system, where $K_H = \bigcup_{x \in H} A(x)$.

2. Immersed Hyperdynamical Systems

Let M be a manifold, H be an immersed hypergroup [2], and

$$D = \{h^a : M \to M : h^a \text{ is a diffeomorphism, where } a \in H\}.$$

Then we have the next definition:

Definition 2.1. (M, D, H) is an immersed hyperdynamical system, if (M, D, H) is a hyperdynamical system and $h : H \times M \to M$ defined by $(a, m) \mapsto h^a(m)$ is a smooth map.

Example 2.1. Let H be the set of 2-dimensional subspaces of $(R^2, +)$ as a Grassmann manifold [3]. For $a, b \in H$, we define $ab = \{c :$ c is a 2-dimensional subspace of $a + b\}$. Then H with this join operation and the above differentiable structure is an immersed hypergroup. Let $D = \{h^{(a,b)} : \ h^{(a,b)} : R^2 \to R^2$ is defined by $(x, y) \mapsto (a + x, b + y)$, $(a, b) \in H\}$. Then (R^2, D, H) is a hyperdynamical system. The mapping $h : H \times R^2 \to R^2$, defined by $h((a, b), (x, y)) = (a + x, b + y)$, is a smooth map. Hence (R^2, D, H) is an immersed hyperdynamical system.

Theorem 2.1. Let (M, D, H) and (N, E, I) be two immersed hyperdynamical systems, where $D = \{h^a : a \in H\}$, $E = \{f^b : b \in I\}$. Moreover, let $F = \{g^{(a,b)} : M \times N \to M \times N : (a, b) \in H \times I \textit{ defined by } (m, n) \mapsto (h^a(m), f^b(n))\}$. Then $(M \times N, F, H \times I)$ is an immersed hyperdynamical system.

Proof. We know that H and I are immersed hypergroups. We show that $H \times I$ is an immersed hypergroup. Since H and I are smooth manifolds, then $H \times I$ is a smooth manifold. We only show that for given $(a_1, a_2), (b_1, b_2) \in H \times I$, $a_1b_1 \times a_2b_2$

is an immersed submanifold of $H \times I$.
Since $i_1 : a_1b_1 \to H \; and \; i_2 : a_2b_2 \to I$ are immersion, then the map $i : a_1b_1 \times a_2b_2 \to H \times I$ defined by $i(c_1, c_2) = (i_1(c_1), i_2(c_2))$ is an immersion. Because it is a one to one map and $di(c_1, c_2) = (di_1(c_1), di_2(c_2))$ is also a one to one map.
Now we show that $(M \times N, F, H \times I)$ is a hyperdynamical system.
Definition of F implies for given $(a, b) \in H \times I$, there exists $h^{(a,b)} \in F$.
If $g^{(a,b)},\ g^{(c,d)} \in F$, then $g^{(a,b)} \circ g^{(c,d)}(m, n) = ((h^a \circ h^c(m), h^b \circ h^d(n))$.
So $g^{(a,b)} \circ g^{(c,d)}(m, n) \in g^{(ac,bd)}(m, n)$.
We show that $h : H \times I \times M \times N \to M \times N$, defined by $(a, b, m, n) \mapsto h^{(a,b)}(m, n)$ is a smooth map.
Since (M, D, H) is an immersed hyperdynamical system, then there exists a smooth map $h_1 : H \times M \to M$, defined by $(a, m) \to h^a(m)$.
Since (N, E, I) is an immersed hyperdynamical system, then there exists a smooth map $h_2 : I \times N \to N$, defined by $(b, n) \mapsto f^b(n)$. Let h be a map from $H \times I \times M \times N$ to $M \times N$, defined by $(a, b, m, n) \mapsto (h^a(m), f^b(n))$, then it is a smooth map, because $h_1 \times h_2$ and the mapping from $H \times I \times M \times N$ to $H \times M \times I \times N$ defined by $(a, b, m, n) \mapsto (a, m, b, n)$ are smooth maps.

3. Quotient Immersed Hyperdynamical Systems

Let M be a differentiable manifold and $R \subseteq M \times M$ be an equivalence relation on M. Moreover, let M/R be the set of the equivalence classes of M with respect to R. Let $P : M \to M/R$ be the natural projection $m \mapsto [m]$.
We define the quotient topology on M/R, i.e., $U \subset M/R$ is open if and only if $P^{-1}(U)$ is open in M. Thus $P : M \to M/R$ is continuous map. Moreover, for any continuous map $F : M \to N$, there exists a unique continuous map $\bar{F} : M/R \to N$ such that $F = \bar{F} \circ P$.
Hence we have the following commutative diagram;

$$\begin{tikzcd}
M \arrow[r, "F"] \arrow[d, "P"'] & N \\
\frac{M}{R} \arrow[ur, "\overline{F}"'] &
\end{tikzcd}$$

In general, M/R may not be a manifold. We now assume that M/R has a differentiable structure such that $P : M \to M/R$ is a submersion. Since P is a continuous map, for all open set U in M/R, $P^{-1}(U)$ is an open set in M. P is also an open map, because it is a submersion [4].

Moreover, if the map F from M into a differentiable manifold N is differentiable, then the map $\bar{F} : M/R \to N$ is also differentiable, and the differentiable structure on M/R is unique [4].

M/R is called the quotient manifold of M with respect to R, if it is allows a differentiable structure such that $P : M \to M/R$ is a submersion [4].

The equivalence relation R on M is called an equivalence relation compatible with (M, D, H) [5] if $(m, n) \in R$ then $h^a(m) = h^a(n)$ for all $a \in H$.

Theorem 3.1. Let R be an equivalence relation compatible with (M, D, H), where (M, D, H) is an immersed hyperdynamical system. Moreover, let $f^a : M/R \to M/R$ be defined by $f^a([m]) = [h^a(m)]$, where $a \in H$. Then $(M/R, \tilde{D} = \{f^a\}, H)$ is an immersed hyperdynamical system.

Proof. The compatibility of R implies that $\tilde{D}$ contains well defined mappings. For all $a \in H$, there is $f^a \in \tilde{D}$.

If $f^a, f^b \in \tilde{D}$, then $(f^a \circ f^b)([m]) = [(h^a \circ h^b)(m)]$, where $[m] \in M/R$. So we have $(f^a \circ f^b)([m]) \in f^{ab}([m])$. Thus $(M/R, \tilde{D}, H)$ is a hyperdynamical system.

Since (M, D, H) is an immersed hyperdynamical system, there exists a smooth map $h : H \times M \to M$, defined by $(a, m) \mapsto h^a(m)$. The mapping $\tilde{h} : H \times M/R \to M/R$, defined by $(a, [m]) \mapsto h^a([m])$ is a smooth map. Because $I \times P \; and \; P \circ h$ are smooth maps and the following diagram is commutative;

$$
\begin{tikzcd}
H \times M \arrow[r, "Poh"] \arrow[d, "I \times P"'] & M/R \\
H \times M/R \arrow[ur, "\overline{h}"'] &
\end{tikzcd}
$$

Thus $(M/R, \tilde{D}, H)$ is an immersed hyperdynamical system.

Theorem 3.2. Let (M, D, H) be an immersed hyperdynamical system, and let

$\psi : H \to R^k$ be a smooth one to one immersion, where $\psi(H)$ has the join operation $\psi(a)\psi(b) = \psi(ab)$. Then $(M, D, \psi(H))$ is an immersed hyperdynamical system.

Proof. $\psi(H)$ is a hypergroup, because $\psi(a)\psi(H) = \psi(aH) = \psi(Ha) = \psi(H)\psi(a) = \psi(H)$, for all $a \in H$.

$\psi(a)\{\psi(b)\psi(c)\} = \psi(a)\psi(bc) = \psi(abc) = \psi(ab)\psi(c) = \{\psi(a)\psi(b)\}\psi(c)$, for all $a, b, c \in H$.

Moreover $\psi(H)$ with the smooth structure created by ψ as a diffeomorphism, is a smooth manifold.

Let $\psi(a)$ *and* $\psi(b) \in \psi(H)$ be given. Then there exists an immersion $i : ab \to H$. So the mapping $\psi \circ i : ab \to R^k$ is an immersion. Hence $\psi(ab) = \psi \circ i(ab)$ is an immersed sub-manifold of R^k.

Thus $\psi(H)$ is an immersed hypergroup.

Now we show that $(M, D, \psi(H))$ is a hyperdynamical system.

For all $b \in \psi(H)$, there exists a $h^b \in D$, since (M, D, H) is a hyperdynamical system, then for all $a \in H$, there is $h^a \in D$. Since $a \in H$, then $\psi(a) \in \psi(H)$. Since $b \in \psi(H)$, then $b = \psi(a)$, and $\psi^{-1}(b) = a \in H$.

Hence $h^{\psi^{-1}(b)} \in H$. Thus $h^b \in \psi(H)$.

We know that if $h^a, h^b \in D$, then $h^a \circ h^b \in h^{ab}$, for all $a, b \in H$. Now if $\psi(a)$ *and* $\psi(b) \in \psi(H)$, then $h^{\psi(a)} \circ h^{\psi(b)} \in h^{\psi(a)\psi(b)} = h^{\psi(ab)}$.

Hence $(M, D, \psi(H))$ is a hyperdynamical system.

Since (M, D, H) is a immersed hyperdynamical system, then there exists a smooth map $h : H \times M \to M$, defined by $(a, m) \mapsto h^a(m)$.

We define the map $\tilde{h} : \psi(H) \times M \to M$, by $(\psi(a), m) \mapsto h^{\psi(a)}(m)$.

Since h and ψ are smooth maps then $\tilde{h}$ is smooth. So $(M, D, \psi(H))$ is an immersed hyperdynamical system.

Theorem 3.3 Let (M, D, H) be an immersed hyperdynamical system. Moreover, let U be any open subset of H with the join operation $\star_U$ on $U \times U$ defined by $a \star_U b = ab \bigcap U$. Then (M, D, U) is an immersed hyperdynamical system.

Proof. U with the atlas $\mathring{A}_U = \{smooth\ charts\ (V, \Theta)\ of\ H :\ V \subset U\}$ is a smooth manifold. Moreover, it is hypergroup [5].

If $a, b \in H$, then $i : ab \to H$ is an immersion. So $i_U : ab \bigcap U \to U$ is also an immersion. Thus U is an immersed hypergroup.

Since (M, D, H) is a hyperdynamical system, then, for all $a, b \in U \subset H$ there exists

$h^a \in D$, and $h^a \circ h^b \in h^{ab}$.

Since (M, D, H) is an immersed hyperdynamical system, then the map $h : H \times M \to M$, defined by $(a, m) \mapsto h^a(m)$ is smooth. Thus $\tilde{h} : U \times M \to M$, defined by $(a, m) \mapsto h^a(m)$ is smooth and the proof is complete.

Suppose $\langle H, \circ \rangle$ is a hypergroupoid [1] and assume that $\{A(x)\}$ is a family of pairwise disjoint nonempty sets and $K_H = \bigcup_{x \in H} A(x)$. We define $g(a) = x$ if and only if $a \in A(x)$.

Now in K_H we have the hyperoperation $a \bigodot b = \bigcup_{z \in g(a) \circ g(b)} A(z)$, where $a, b \in K_H$. We know that $(H, \circ)$ is a hypergroup if and only if $\langle K_H, \bigodot \rangle$ is a hypergroup [1].

Theorem 3.4. Let $(H, \circ)$ be an immersed hypergroup and let $\{A(x)\}$ be a family of pairwise disjoint nonempty open subsets of a differentiable manifold M. Moreover, let (M, D, H) be an immersed hyperdynamical system. Then (M, D, K_H) is an immersed hyperdynamical system.

Proof. Let $a, b \in K_H$ be given. Then $a \odot b$ is also a union of open subsets of M. So it is an open submanifold of K_H. Thus it is an immersed submanifold of K_H. If $a, b \in K_H$, then there exists $h^a \in D$, and $h^a \circ h^b \in h^{ab}$. So (M, D, H) is hyperdynamical system.

Since (M, D, H) is an immersed hyperdynamical system, then there exists a smooth map $h : H \times M \to M$, defined by $(a, m) \mapsto h^a(m)$.

So the map $\tilde{h} : K_H \times M \to M$, defined by $\tilde{h}(a, m) = h^a(m)$ is a smooth map. Thus (M, D, K_H) is an immersed hyperdynamical system.

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