EQUIVALENCE OF DISCRETE TIME-VARIANT CONTROL SYSTEMS ON TIME SCALES

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ABSTRACT. The concept of dynamic equivalence of nonlinear time-variant systems defined on discrete-time scales is introduced. There are studied relations between classical static state feedback and external dynamic feedback. There is proved that if two time-variant systems defined on any discrete time scale are locally equivalent with respect to the regular static state feedback transformations then they are locally equivalent with respect to the external dynamic feedback transformations.

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1. INTRODUCTION

Equivalence of nonlinear time-invariant control systems as well in continuous-time as discrete-time case has been studied by many authors, for example see [7, 11, 13, 14]. In some applications deserve noticed is paid on the case of dynamic feedback equivalence. This type of equivalence first was introduced and investigated by B.Jakubczyk [11, 12]. He used a dynamical state feedback to transfer trajectories of one control system onto trajectories of the other. This concept of dynamic feedback linearization was close to the property of flatness of system [7, 16]. Jakubczyk's ideas and results were carried on discrete-time nonlinear time-invariant systems by Bartosiewicz et al. [1]. In that new setting dynamic transformations depended on future states and controls and the difference operator associated with the discrete-time system replaced the differential operator.

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Problem of the dynamic equivalence is one of the many examples where the results obtained for the continuous- and discrete-time cases look similar. The language of time scales introduced by S.Hilger [10] is a good tool for unifying the both cases. The basic concept of time scale theory is the delta derivative. It is extension of the concept of classical derivative in continuous-time case and forward difference in discrete-time case. The most important achievements in this area can be found in [6].

Time scales calculus was introduced to the control theory a few years ago (see for example [8, 2, 3]). In [5] was unified characterization of dynamic equivalence of continuous-time nonlinear control systems and nonlinear control systems. In [4] there was considered the problem of weak dynamic feedback equivalence of nonlinear time-variant control systems on homogeneous time scales. Besides the state and control transformations, also time transformation is allowed, so equivalent systems may be defined on different time scales.

The goal of this paper is to extend the notation of general dynamic transformations of nonlinear control systems with outputs (given in [14, 15]) to the time-variant systems defined on any discrete time scale. There are also studied relations between classical static state feedback and external dynamic feedback. The main result extends known result for time-invariant systems and says that if two such systems are locally equivalent with respect to the regular static state feedback transformations then they are locally equivalent with respect to the external dynamic feedback transformations.

The paper is organized as follows. In Section 2 we provide the reader with the necessary background on the calculus on time scales. Sections 3 and 4 contains the notations and some facts about dynamic transformations of discrete time-variant systems. In Section 5 there are studied relations between static state feedback transformations and external dynamic feedback transformations.

2. Calculus on time scales

We recall here basic concepts and facts of the calculus on time scales. For more information the reader is referred to [6].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. The standard examples of time scales are $\mathbb{R}$, $h\mathbb{Z}$, $h > 0$, $\mathbb{N}$, $\mathbb{N}_0$, $2^{\mathbb{N}_0}$ or $\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$. The time scales $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$.

The following operators on $\mathbb{T}$ are often used:
- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} \in \mathbb{T}$,
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$, defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} \in \mathbb{T}$,
- the *graininess functions* $\mu, \nu : \mathbb{T} \to [0, \infty[$ defined by $\mu(t) := \sigma(t) - t$ and respectively by $\nu(t) := t - \rho(t)$.

Points from the time scale can be classified as follows: a point $t \in \mathbb{T}$ is called *right-scattered* if $\sigma(t) > t$ and *right-dense* if $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *left-dense* if $\rho(t) = t$, *isolated* if it is both left-scattered and right-scattered and *dense* if it is both left-dense and right-dense. If $\mathbb{T}$ consists of only isolated points then it is called a *discrete time scale*.

We define also the sets

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases},$$

$$\mathbb{T}_{\kappa} := \begin{cases} \mathbb{T} \setminus \{m\}, & \text{if } m \text{ is right scattered minimum} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Example 2.1. If $\mathbb{T} = \mathbb{R}$ then $\rho(t) = t = \sigma(t)$ and $\mu(t) = 0$, for all $t \in \mathbb{R}$. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\rho(t) = t - h$, $\sigma(t) = t + h$ and $\mu(t) \equiv h$, for all $t \in h\mathbb{Z}$. If$\mathbb{T} = \overline{q^{\mathbb{Z}}} := \left\{q^k : k \in \mathbb{Z}\right\} \cup \{0\}$, where $q > 1$, then $\rho(t) = \frac{t}{q}$ $\sigma(t) = qt$ and $\mu(t) = (q-1)t$, for all $t \in \mathbb{T}$.

Let $\zeta : \mathbb{T} \to \mathbb{R}$ be a strictly increasing function such that $\tilde{\mathbb{T}} = \zeta(\mathbb{T})$ is also a time scale.

Corollary 2.2. [6] *If $\tilde{\sigma}$ is the forward jump operator on time scale $\tilde{\mathbb{T}}$, then*

$$\zeta \circ \sigma = \tilde{\sigma} \circ \zeta \tag{1}$$

where σ denotes the forward jump operator on time scale $\mathbb{T}$.

Definition 2.3. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then the number $f^{\Delta}(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \ \forall s \in U,$$

is called the delta derivative of f at t. *The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the* delta derivative *of f on $\mathbb{T}^{\kappa}$. We say that f is* delta differentiable *on $\mathbb{T}^{\kappa}$, if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.*

Definition 2.4. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$. Then the number $f^{\nabla}(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$|[f(\rho(t)) - f(s)] - f^{\nabla}(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|, \ \forall s \in U,$$

is called the nabla derivative of f at t. *The function $f^{\nabla} : \mathbb{T}_k \to \mathbb{R}$ is called the* nabla derivative *of f on $\mathbb{T}_{\kappa}$. We say that f is* nabla differentiable *on $\mathbb{T}_{\kappa}$, if $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_{\kappa}$.*

Remark 2.5. If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is both delta differentiable and nabla differentiable at $t \in \mathbb{R}$ iff

$$f^{\Delta}(t) = f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t),$$

i.e. f is differentiable in the ordinary sense at t.

Remark 2.6. If $\mathbb{T}=\mathbb{Z}$, then $f:\mathbb{Z}\to\mathbb{R}$ is always delta differentiable and nabla differentiable on $\mathbb{Z}$ and

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}=f(t+1)-f(t),$$

$$f^{\nabla}(t)=\frac{f(t))-f(\rho(t))}{\mu(t)}=f(t)-f(t-1),$$

for all $t\in\mathbb{Z}$.

Proposition 2.7. [9]

i) *Assume that $f:\mathbb{T}\to\mathbb{R}$ is delta differentiable on $\mathbb{T}^k$. Then f is nabla differentiable at t and*

$$f^{\nabla}(t)=f^{\Delta}(\rho(t)) \tag{2}$$

for $t\in\mathbb{T}_k$ such that $\sigma(\rho(t))=t$. If, in addition, f^{Δ} is continuous on $\mathbb{T}^k$, then f is nabla differentiable at t and (2) holds for any $t\in\mathbb{T}_k$.

ii) *Assume that $f:\mathbb{T}\to\mathbb{R}$ is nabla differentiable on $\mathbb{T}_k$. Then f is delta differentiable at t and*

$$f^{\Delta}(t)=f^{\nabla}(\sigma(t)) \tag{3}$$

for $t\in\mathbb{T}^k$ such that $\rho(\sigma(t))=t$. If, in addition, f^{∇} is continuous on $\mathbb{T}_k$, then f is delta differentiable at t and (3) holds for any $t\in\mathbb{T}^k$.

Let us notice that if $t\in\mathbb{T}^{\kappa}$ satisfies $\rho(t)=t<\sigma(t)$, then the forward jump operator σ is not delta differentiable at t.

A function $f:\mathbb{T}\to\mathbb{R}$ is called *regulated* if its right–sided limits exist (finite) at all right–dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function f is called *rd–continuous* if it is continuous at right–dense points in $\mathbb{T}$ and its left–sided limits exist at all left–dense points in $\mathbb{T}$. The function f which is infinitely many times Δ-differentiable on some interval is called Δ*-smooth function*.

Similarly, a function $f:\mathbb{T}\to\mathbb{R}$ is called *ld–continuous* provided it is continuous at left–dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The function f which is infinitely many times ∇-differentiable on some interval is called ∇*-smooth function*.

A time scale $\mathbb{T}$ is called *homogeneous* if μ and ν are constant on respectively $\mathbb{T}^{\kappa}$ and $\mathbb{T}_{\kappa}$. The time scales $\mathbb{R}$, $h\mathbb{Z}$, $[0,1]$ are homogeneous, whereas $\overline{q^{\mathbb{Z}}}$ or $\mathbb{N}$ are not.

A time scale $\mathbb{T}$ is called *regular* if the following two conditions are satisfied simultaneously

i) $\sigma(\rho(t))=t$, for all $t\in\mathbb{T}$
ii) $\rho(\sigma(t))=t$, for all $t\in\mathbb{T}$

From (i) it follows that the operator $\sigma:\mathbb{T}\to\mathbb{T}$ is "onto" while (ii) implies that σ is "one-to-one". Therefore, if $\mathbb{T}$ is regular, then σ is invertible and $\rho:\mathbb{T}\to\mathbb{T}$ is also invertible. Moreover, $\sigma^{-1}=\rho$ and $\rho^{-1}=\sigma$. Time scales $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=h\mathbb{Z}$, $h>0$,

$\mathbb{T} = \overline{q^{\mathbb{Z}}}$ and $\mathbb{T} = (-\infty, 0] \cup \{\frac{1}{k} |\ k \in \mathbb{N}\} \cup \{\frac{k}{k+1} |\ k \in \mathbb{N}\} \cup [1, 2]$ are regular (for more details see [9]).

3. Trajectories and behaviors

Let $\mathbb{T}$ be a time scale and A be any nonempty set. Let us consider the set $J_i(A)$ of infinite sequences $S_i = (s(i))_{i \geq 0}$, $s(i) \in A$. Similarly $J_i(A_1, \ldots, A_k)$ will denote the set of sequences $(S_i^1, \ldots, S_i^k)$ where $S_i^j \in J_i(A_j)$ and A_j is any set, $j = 1, \ldots, k$. Let $J(A) = \dot{\bigcup}_{i \in \mathbb{N}_0} J_i(A_1, \ldots, A_k)$, where $\dot{\bigcup}$ denotes a disjoint sum. We write $J_i(s_1, \ldots, s_r)$ for $J_i(\mathbb{R}^{s_1}, \ldots, \mathbb{R}^{s_r})$ and similarly for $J(\mathbb{R}^{s_1}, \ldots, \mathbb{R}^{s_r})$.

Let B be any set. If $\gamma : J(A_1, \ldots, A_k) \to B$, then by the *extension of* γ we will mean the map $\Gamma : J(A_1, \ldots, A_k) \to J(B)$ defined by $\Gamma(S_i^1, \ldots, S_i^k) = \tilde{S}_r$ where $\tilde{S}_r = \gamma(S_i^1|_r, \ldots, S_i^k|_r)$. The definition of extension naturally generalize to the case where $\gamma : J(A_1, \ldots, A_k) \to B_1 \times \ldots B_p$. Then the extension of γ is the map $\Gamma : J(A_1, \ldots, A_k) \to J(B_1, \ldots, B_p)$.

Remark 3.1. It can be noticed that in the case when $i \in \mathbb{T}$ and $\mathbb{T}$ is any discrete time scale, then $\rho^k \Gamma = \Gamma \rho^k$ and $\sigma^k \Gamma = \Gamma \sigma^k$.

Let a function $z : \mathbb{T} \to \mathbb{R}^r$ has infinitely many delta-derivatives. Then the map $Z : \mathbb{T} \to J(r)$ defined by $Z(t) = (z(t), z^{\Delta}(t), \ldots)$ is called the (infinite) jet of z. Let A be a set and let A_∞ denote the set of sequences $(z(t), z^{\Delta}(t), z^{\Delta^2}(t), \ldots)$, $z^{z^{\Delta^i}}(t) \in A$. We have a natural projection $\Pi_s : A_\infty \to A_s$, $\Pi_s(z(t), z^{\Delta}(t), \ldots) = (z(t), z^{\Delta}(t), \ldots, z^{\Delta^i})$ and $i + j = s - 1$.

Now, let A be a topological space. The topology on A_∞ is defined as a weakest topology for which all maps Π_s are continuous. The basis $\mathcal{B}$ of this topology consists of sets defined in the following way: a set V is an element of $\mathcal{B}$ if and only if it is the inverse image of an open set in A_s under the map Π_s for some $s \in \mathbb{N}$. For products of finitely many spaces of type A_∞ we choose the product topology.

Lemma 3.2. *If A is Hausdorff then also A_∞ is Hausdorff.*

Proof: The proof is the same as one given for the case $\mathbb{T} = \mathbb{Z}$ in [14].

If A, B are sets and $V \subseteq A^\infty$, then we say that function $\varphi : V \to B$ depends on finite number of variables if there exists $k \geq 1$ and function $\tilde{\varphi} : A^k \to B$ such that $\varphi = \tilde{\varphi} \circ \Pi_k|_V$. If the map $\varphi : J(A) \to B$ depends on a finite number of variables, we say that it is of the *finite order*.

Let $\mathbb{T}$ be any time scale. Let us consider a time-variant system Σ of the following form:

$$\begin{aligned} x^{\Delta}(t) &= f(t, x(t), u(t)) \\ y(t) &= h(t, x(t), u(t)) \end{aligned} \tag{4}$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$. We shall assume that f and h are smooth, and that control u may be infinitely many times (delta and nabla) differentiated as a function on $\mathbb{T}$.

Let us consider the following sequences

$$X = \{x(t), x^{\Delta}(t), x^{\Delta^2}(t), \ldots\} \in J(n)$$
$$U = \{u(t), u^{\Delta}(t), u^{\Delta^2}(t), \ldots\} \in J(m)$$
$$Y = \{y(t), y^{\Delta}(t), y^{\Delta^2}(t), \ldots\} \in J(p)$$

By a *trajectory* of the system Σ we will mean any triple of functions $x(\cdot), (y(\cdot), u(\cdot))$ defined on a some interval $[a, b)$ that satisfies the equations of Σ. We assume that $a \in \mathbb{T}$, $b \in \mathbb{T}$ or $b = \infty$ and $[a, b)$ contains an infinitely many points. The pair $(x(\cdot), u(\cdot))$ is called an *inner trajectory* and the pair $(y(\cdot), u(\cdot))$ is an *external trajectory* of Σ. The set of all inner (external) trajectories of the system Σ forms the *inner (external) behavior* of the system. It will be denoted by $B^i(\Sigma)$ (respectively by $B^e(\Sigma)$). Then $B^e(\Sigma) = \bigcup_{r\in\mathbb{T}} B^e_r(\Sigma)$, i.e. external behavior of the given system is a sum of the set all trajectories starting at some moment r. Similar properties hold for behavior and inner behavior.

4. External dynamic equivalence

Let us consider a nonlinear time-varying control system defined on the discrete time scale $\mathbb{T}$:

$$\Sigma : \begin{array}{r} x^{\Delta}(t) = f(t, x(t), u(t)) \\ y(t) = h(t, x(t), u(t)) \end{array}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ and control system:

$$\tilde{\Sigma} : \begin{array}{r} \tilde{x}^{\Delta}(\tilde{t}) = \tilde{f}(\tilde{t}, \tilde{x}(\tilde{t}), \tilde{u}(\tilde{t}) \\ \tilde{y}(\tilde{t}) = \tilde{h}(\tilde{t}, \tilde{x}(\tilde{t}), \tilde{u}(\tilde{t})) \end{array}$$

defined on $\tilde{\mathbb{T}}$ with $\tilde{x}(\tilde{t}) \in \mathbb{R}^{\tilde{n}}$, $\tilde{y}(\tilde{t}) \in \mathbb{R}^{\tilde{p}}$, $\tilde{u}(\tilde{t}) \in \mathbb{R}^m$. Let us consider maps

$$\phi^e : \tilde{\mathbb{T}} \times J(\tilde{p}, m) \to \mathbb{R}^p, \quad \psi^e : \tilde{\mathbb{T}} \times J(\tilde{p}, m) \to \mathbb{R}^m \tag{5}$$

$$\tilde{\phi}^e : \mathbb{T} \times J(p, m) \to \mathbb{R}^{\tilde{p}}, \quad \tilde{\psi}^e : \mathbb{T} \times J(p, m) \to \mathbb{R}^m \tag{6}$$

The extensions $\Phi^e, \Psi^e, \tilde{\Phi}^e, \tilde{\Psi}^e$ of maps (5)-(6) give

$$\chi^e = \zeta \times (\Phi^e, \Psi^e) : (\tilde{\zeta}, \tilde{Y}, \tilde{U}) \mapsto (t, \Phi^e(\tilde{Y}, \tilde{U}), \Psi^e(\tilde{Y}, \tilde{U}))$$
$$\tilde{\chi}^e = \tilde{\zeta} \times (\tilde{\Phi}^e, \tilde{\Psi}^e) : (t, Y, U) \mapsto (\tilde{t}, \tilde{\Phi}^e(Y, U), \tilde{\Psi}^e(Y, U))$$

Consider the following dynamic transformations:

$$y = \phi^e(\tilde{t}, \tilde{Y}, \tilde{U}), \quad u = \psi^e(t, \tilde{Y}, \tilde{U}) \tag{7}$$

$$\tilde{y} = \tilde{\phi}^e(t, Y, U), \quad \tilde{u} = \tilde{\psi}^e(t, Y, U) \tag{8}$$

where $Y \in J(p)$, $\tilde{Y} \in J(\tilde{p})$, $U, \tilde{U} \in J(m)$, and $\phi^e, \tilde{\phi}^e, \psi^e, \tilde{\psi}^e$ are finitely presented maps.

Let A, B denote topological spaces and let $q \in \mathbb{N}_0$. If $\phi : \mathbb{T} \times J(A) \to \mathbb{T} \times B$ is a map of locally finite order then the *dynamic forward transformation of type* q, $q = q_1 + q_2$ for some natural q_1, q_2, associated with ϕ is the map $D^q_\phi : \mathbb{T} \times J(A) \to \mathbb{T} \times J(B)$

$$D^q_\phi(t, Z) = (\Phi^{\nabla^{q_1}}, \Phi^{\Delta^{q_2}})(t, Z) \tag{9}$$

This definition can be extended to the case where $\varphi : \mathbb{T} \times J(A_1, \ldots, A_k) \rightarrow \mathbb{T} \times B_1 \times \ldots \times B_l$. In particular, we are interesting in the transformation of form

$$D^q_{(\phi^e,\psi^e)} : \tilde{\mathbb{T}} \times J(\tilde{p}, m) \rightarrow \mathbb{T} \times J(p, m)$$

associated with (ϕ^e, ψ^e). If $D^q_{(\phi^e,\psi^e)}(B^e_r(\tilde{\Sigma})) \subseteq B^e(\Sigma)$, then $D^q_{(\phi^e,\psi^e)}$ will be called an *external dynamic transformation of type* q of system $\tilde{\Sigma}$ into system Σ. The external trajectories locally in the neighborhoods, respectively of the points $(t, Y, U) \in \mathbb{T} \times J(p, m)$ and $(\tilde{t}, \tilde{Y}, \tilde{U}) \in \tilde{T} \times J(\tilde{p}, \tilde{m})$ are related in the following way

$$\begin{aligned} y(t) = \phi^e(&\tilde{t}, \tilde{y}^{\nabla^{q_1}}(\tilde{t}), \ldots, \tilde{y}^{\nabla}(\tilde{t}), \tilde{y}(\tilde{t}), \tilde{y}^{\Delta}(\tilde{t}), \ldots, \tilde{y}^{\Delta^q_2}(\tilde{t}), \\ &\tilde{u}^{\nabla^{q_1}}(\tilde{t}), \ldots, \tilde{u}^{\nabla}(\tilde{t}), \tilde{u}(\tilde{t}), \tilde{u}^{\Delta}(\tilde{t}), \ldots, \tilde{u}^{\Delta^q_2}(\tilde{t})) \end{aligned} \tag{10}$$

$$\begin{aligned} u(t) = \psi^e(&\tilde{t}, \tilde{y}^{\nabla^{q_1}}(\tilde{t}), \ldots, \tilde{y}^{\nabla}(\tilde{t}), \tilde{y}(\tilde{t}), \tilde{y}^{\Delta}(\tilde{t}), \ldots, \tilde{y}^{\Delta^q_2}(\tilde{t}), \\ &\tilde{u}^{\nabla^{q_1}}(\tilde{t}), \ldots, \tilde{y}^{\nabla}, (\tilde{t})\tilde{u}(\tilde{t}), \tilde{u}^{\Delta}(\tilde{t}), \ldots, \tilde{u}^{\Delta^q_2}(\tilde{t})) \end{aligned} \tag{11}$$

where (ϕ^e, ψ^e) is of order less or equal $q = q_1 + q_2$ in the neighborhood of $(\tilde{t}, \tilde{Y}, \tilde{U})$.

Remark 4.1. *In the case when* $\mathbb{T}$ *is a homogenous discrete-time time scale similarly as in* [14] *we can show that if there exists an external dynamic feedback transformation of type* q *associated with* (ϕ^e, ψ^e) *that transformed system* $\tilde{\Sigma}$ *into system* Σ*, then for every* $p \in \mathbb{N}_0$ *there also exists an external dynamic feedback transformation of type* p*, also associated with* (ϕ^e, ψ^e)*, which transforms* $\tilde{\Sigma}$ *into system* Σ*.*

Example 4.2. [4] *Let us consider the following control systems' defined on some discrete time scale* $\mathbb{T} = \tilde{\mathbb{T}}$

$$\Sigma : \begin{cases} z_1^{\Delta}(t) = \frac{z_1^2(t) - z_2(t)}{z_1(t)} \\ z_2^{\Delta}(t) = -z_2(t) + \frac{z_2^2(t)}{z_1^2(t)} + u(t) \\ y(t) = z_1(t) + \frac{z_1(t)u(t)}{z_1^2(t) + \mu(t)(z_1^2(t) - z_2(t))} \end{cases}$$

and

$$\tilde{\Sigma} : \begin{cases} x_1^{\Delta}(t) = -x_1(t) + x_2(t) \\ x_2^{\Delta}(t) = v(t) \\ \gamma(t) = x_1(t) + v(t) \end{cases}$$

The external dynamic transformations of the system Σ *onto system* $\tilde{\Sigma}$ *are of the form* $v(t) := \frac{u(t)}{z_1(\sigma(t))}$*,* $\gamma(t) := y(t)$*. The inner dynamic transformations of* Σ *into* $\tilde{\Sigma}$ *are of the form* $x_1(t) := z_1(t)$*,* $x_2(t) := \frac{z_2(t)}{z_1(t)}$*,* $v(t) := \frac{u(t)}{z_1(\sigma(t))}$*. It is easy to check that these both type transformations are (locally) reversible. Let us notice also that the system* $\tilde{\Sigma}$ *is time invariant, but dynamic transformations depend on time and, in the efekt, system* Σ *is time-variant system.*

If $q_1 = 0$, then the dynamic transformations we will called *external anticipating feedback transformations*. In this case current values of outputs and controls of the one system depend on the future and current outputs and controls of the other one in the following way:

$$
\begin{aligned}
y(t) &= \varphi^e(\tilde{t}, \tilde{y}(\tilde{t}), \tilde{y}^{\Delta}(\tilde{t}), \ldots, \tilde{y}^{\Delta^{q_2}}(\tilde{t}), \tilde{u}(\tilde{t}), \tilde{u}^{\Delta}(\tilde{t}), \ldots, \tilde{u}^{\Delta^{q_2}}(\tilde{t})) \\
u(t) &= \psi^e(\tilde{t}, \tilde{y}(\tilde{t}), \tilde{y}^{\Delta}(\tilde{t}), \ldots, \tilde{y}^{\Delta^{q_2}}(\tilde{t}), \tilde{u}(\tilde{t}), \tilde{u}^{\Delta}(\tilde{t}), \ldots, \tilde{u}^{\Delta^{q_2}}(\tilde{t}))
\end{aligned} \tag{12}
$$

for some natural $q_2 \geq 0$ and

$$
\begin{aligned}
\tilde{y}(t) &= \tilde{\varphi}^e(\tilde{t}, y(\tilde{t}), y^{\Delta}(\tilde{t}), \ldots, y^{\Delta^{\tilde{q}_2}}(\tilde{t}), u(\tilde{t}), u^{\Delta}(\tilde{t}), \ldots, u^{\Delta^{\tilde{q}_2}}(\tilde{t})) \\
\tilde{u}(t) &= \tilde{\psi}^e(\tilde{t}, y(\tilde{t}), y^{\Delta}(\tilde{t}), \ldots, y^{\Delta^{\tilde{q}_2}}(\tilde{t}), u(\tilde{t}), u^{\Delta}(\tilde{t}), \ldots, u^{\Delta^{\tilde{q}_2}}(\tilde{t}))
\end{aligned} \tag{13}
$$

for some natural $\tilde{q}_2 \geq 0$.

If $q_2 = 0$, then the dynamic transformations we will called *external dynamic delay feedback transformations*. In this case current values of outputs and controls of the one system depend on the past and current outputs and controls of the other one in the following way:

$$
\begin{aligned}
y(t) &= \gamma^e(\tilde{t}, \tilde{y}(\tilde{t}), \tilde{y}^{\nabla}(\tilde{t}), \ldots, \tilde{y}^{\nabla^{q_1}}(\tilde{t}), \tilde{u}(\tilde{t}), \tilde{u}^{\nabla}(\tilde{t}), \ldots, \tilde{u}^{\nabla^{q_1}}(\tilde{t})) \\
u(t) &= \eta^e(\tilde{t}, \tilde{y}(\tilde{t}), \tilde{y}^{\nabla}(\tilde{t}), \ldots, \tilde{y}^{\nabla^{q_1}}(\tilde{t}), \tilde{u}(\tilde{t}), \tilde{u}^{\nabla}(\tilde{t}), \ldots, \tilde{u}^{\nabla^{q_1}}(\tilde{t}))
\end{aligned} \tag{14}
$$

for some natural $q_1 \geq 0$ and

$$
\begin{aligned}
\tilde{y}(t) &= \tilde{\gamma}^e(\tilde{t}, y(\tilde{t}), y^{\nabla}(\tilde{t}), \ldots, y^{\nabla^{\tilde{q}_1}}(\tilde{t}), u(\tilde{t}), u^{\nabla}(\tilde{t}), \ldots, u^{\nabla^{\tilde{q}_1}}(\tilde{t})) \\
\tilde{u}(t) &= \tilde{\eta}^e(\tilde{t}, y(\tilde{t}), y^{\nabla}(\tilde{t}), \ldots, y^{\nabla^{\tilde{q}_1}}(\tilde{t}), u(\tilde{t}), u^{\nabla}(\tilde{t}), \ldots, u^{\nabla^{\tilde{q}_1}}(\tilde{t}))
\end{aligned} \tag{15}
$$

for some $\tilde{q}_1 \geq 0$.

Theorem 4.3. *Systems Σ and $\tilde{\Sigma}$ defined on regular time scales $\mathbb{T}$ and $\tilde{T}$ respectively, are external anticipating feedback equivalent if and only if they are external dynamic delay feedback equivalent.*

Proof: First we should proved (by induction principle) that $z^{\Delta^r}(t) = z^{\nabla^r}(\sigma^r(t))$ and $z^{\nabla^r}(t) = z^{\Delta^r}(\rho^r(t))$. Indeed, for $r = 1$ we have

$$
z^{\nabla}(\sigma(t)) = \frac{z(\sigma(t)) - z(\rho(\sigma(t)))}{\nu(\sigma(t))} = \frac{z(\sigma(t)) - z(t)}{\sigma(t) - \rho(\sigma(t))} = \frac{z(\sigma(t)) - z(t)}{\sigma(t) - t}
$$

Now, let us assume that $z^{\Delta^r}(t) = z^{\nabla^r}(\sigma^r(t))$. Then

$$
z^{\Delta^{r+1}}(t) = (z^{\Delta^r})^{\Delta}(t) = (z^{\nabla^r})^{\Delta}(\sigma^r(t)) \tag{16}
$$

Let us denote $s(\bar{t}) := z^{\nabla^r}(\bar{t})$ and $\bar{t} := \sigma^r(t)$. Then (16) implies that

$$
z^{\Delta^{r+1}}(t) = \frac{s(\sigma(\bar{t})) - s(\bar{t})}{\mu(\bar{t})} = \frac{s(\sigma(\bar{t})) - s(\rho(\sigma(\bar{t})))}{\sigma(\bar{t}) - \rho(\sigma(\bar{t}))} =
$$

$$
s^{\nabla}(\sigma(\bar{t})) = (z^{\nabla^r})^{\nabla}(\sigma(\sigma^r(t))) = z^{\nabla^{r+1}}(\sigma^{r+1}(t))
$$

In the similar way we can show that $z^{\nabla^r}(t) = z^{\Delta^r}(\rho^r(t))$. Thus one can replace delta derivatives with nabla derivatives and vice versa, but some forward or backward shifts are involved in this operation. Moreover, the backward shift may be expressed with the aid of the nabla derivative. $\square$

5. Static state and dynamic transformations

Let us take a nonlinear control system

$$\Sigma: \begin{array}{r} x^{\Delta}(t) = f(t, x(t), u(t)) \\ y(t) = h(t, x(t), u(t)) \end{array}$$

with the initial condition $x(t_0) = x_0$ for fixed $t_0 \in \mathbb{T}$, $\mathbb{T}$ - discrete time scale, and for $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$. Let $\varphi : \mathbb{T} \times \mathbb{R}^n \times J(m) \rightarrow \mathbb{R}$ depend only on a finite number of elements in $U \in J(m)$. Define the operator δ_Σ associated with Σ in the following way

$$(\delta_\Sigma \varphi)(t, x, U) := \varphi_t^{\Delta}(t, x, U) + \int_0^1 \frac{\partial \varphi}{\partial x}(\sigma, x + h\mu f(x, u_0), U) dh \cdot f(x, u_0) + \sum_{i=0}^{\infty} \int_0^1 \frac{\partial \varphi}{\partial u_i}(\sigma, x, U + h\mu U_1) dh \cdot u_{i+1}. \tag{17}$$

where function φ depends smoothly on t for fixed (x, U).

Let $T : \mathbb{T} \times \mathbb{R}^n \times J(m) \rightarrow J(p, m)$ defined by

$$T(t, x, U) := (h, \delta_\Sigma h, \ldots, u, \delta_\Sigma u, \ldots)^*(t, x, U)$$

where $(\ldots)^*$ means transposition of the vector $(\ldots)$. Let

$$T_{|_k}(t, x, U) = (h, \delta_\Sigma h, \ldots, \delta_\Sigma^{k-1} h, u, \delta_\Sigma u, \ldots, \delta_\Sigma^{k-1} u)^*(t, x, U)$$

For a fixed U let the map $T_U : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{T} \times U$ be defined by $T_U(t, x, U) := (h, \delta_\Sigma h, \delta_\Sigma^2 h \ldots)^*(t, x, U)$. Then $T_{U_{|_k}}(t, x, U) = (h, \delta_\Sigma h, \ldots, \delta_\Sigma^{k-1} h)^*(t, x, U)$.

We say that, for the fixed $t \in \mathbb{T}$, the map T_U is an *immersion at the point* $x \in \mathbb{R}^n$ if there exists $k \in \mathbb{N}_0$ such that $T_{U_{|_k}}$ is an immersion at this point. In particular, if

$$x = x(t, x_0, U) : [t_0, a) \cap \mathbb{T} \rightarrow \mathbb{R}^n \tag{18}$$

$a \in \mathbb{T}$ - finite or infinite, is a unique forward solution of the initial value problem $x^{\Delta}(t) = f(t, x(t), u(t), x_0 = x(t_0)$, then T_U is immersion at x if there is $k \in \mathbb{N}_0$ such that $T_{U_{|_k}}$ is an immersion at x for all $t \in [t_0, a) \cap \mathbb{T}$.

We say that two points $x_1, x_2 \in \mathbb{R}^n$ are *indistinguishable* if for every control U holds

$$h \circ \sigma^l(t, x_1, U) = h \circ \sigma^l(t, x_2, U), \quad l \in \mathbb{N}_0 \tag{19}$$

for any $t \in [t_0, a) \cap \mathbb{T}$ and $\sigma^l(t, x, U) := (\sigma^k(t), x \circ \sigma^k), U \circ \sigma^k)$. If x_1, x_2 are not indistinguishable, then they are *distinguishable*. It can be noticed that (19) implies that

$$\delta_\Sigma^l h(t, x_1, U) = \delta_\Sigma^l(t, x_2, U), \quad l \in \mathbb{N}_0 \tag{20}$$

The system Σ is *observable* if any two distinct points $x_1, x_2 \in \mathbb{R}^n$ are indistinguishable. The system Σ is *locally observable at point* $x \in \mathbb{R}^n$ if there exists a neighborhood V of x such that any point $z \in V$, $x \neq z$, is distinguishable from x. The given system is *locally observable* if it is locally observable at each point from the state space $\mathbb{R}^n$.

Lemma 5.1. *If for some $U \in J(m)$ and fixed $t_0 \in \mathbb{T}$ the map T_U is an immersion at the $x \in \mathbb{R}^n$ given by (18), then the system Σ is locally observable at this point x for all $t \geq t_0$,*

Proof: The fact that the map T_U is injective follows from the immersivity of this an the point x satisfying (18) and from the Inverse Function Theorem.

Now, let us suppose that the system Σ isn't observable at x. So, in any neighborhood V_x there exists z, $z \neq x$, that is indistinguishable from x. Then for every U we have $\delta^l_\Sigma h(t,x,U) = \delta^l_\Sigma(t,x,U)$, $l \in \mathbb{N}_0$ for any $t \in [t_0, a) \cap \mathbb{T}$. Hence $T_U(t,x) = T_U(t,z)$ and map T_U isn't injective, so we have contradiction. □

From now let us assume that:

A1. for any $x \in \mathbb{R}^n$, $U \in J(m)$ and $t \in [t_0, a) \cap \mathbb{T}$ there exists $k \in \mathbb{N}_0$ such that the map $T|_k$ is proper on some neighborhood of (t, x, U).

A2. for any $x \in \mathbb{R}^n$ satisfying (18) and $U \in J(m)$ the map T_U is an embedding

Lemma 5.2. *Let us assume that conditions (A1)-(A2) are fulfilled. Then map T is left invertible and its left inverse $S : \mathrm{im}T \to \mathbb{T} \times \mathbb{R}^n \times J(m)$ is of the form $S = (S_1, id_{\mathbb{T}\times J(m)})$ where $S_1 : \mathrm{im}T \to \mathbb{R}^n$ is of the class C^∞, $id_{\mathbb{T}\times J(m)}$ is a projection on $\mathbb{T} \times J(m)$.*

Proof is similar to the one given for time $\mathbb{T} = \mathbb{Z}$ in [14], namely the left invertibility of the map T follows from the condition A2.

Let us fix a point $Q = (\bar{t}, \overline{x}_0, \overline{U})$ and consider a map $S = T^{-1}$ defined on the set $\mathrm{im}T$. Take a natural k such that condition A2 is fulfilled at the point Q. Let $T_1 : \mathbb{T}\times\mathbb{R}^n\times J(m) \to J(p)$ and $T_U|_k(Q) = Y|_k$. $Y|_k$ depends only on k elements of U, then $T_1|_k(Q) : \mathbb{T} \times \mathbb{R}^n \times (\mathbb{R}^p)^k$ and $T|_k(Q) : \mathbb{T} \times \mathbb{R}^n \times (\mathbb{R}^m)^k \to (\mathbb{R}^p)^k$ and $\mathrm{rank}T|_k$ is full on some neighborhood V_Q of the point Q. So, Inverse Function Theorem implies that $x_0 = S_1|_k(Y|_k, U|_k)$, $S_1|_k : \mathrm{rank}T|_k \to \mathbb{R}^n$ is smooth. Maps $S_1|_k$ and $T|_k$ are mutually inverse, so $(S_1|_k \circ T|_k)(t, x_0, U) = x_0$ for some $(t, x_0, U|_k) \in V_Q$. For any U we have also $(S_1 \circ T)(t, x_0, U) = x_0$. So, for any (t, x_0, U) such that $(t, x_0, U|_k) \in V_Q$ holds $(S_1|_k \circ T|_k)(t, x_0, U) = (S_1 \circ T)(t, x_0, U)$ and hence $S_1|_k(Y|_k, U|_k) = S_1(Y, U)$. □

Let us consider a static state feedback transformations

$$u(t) = \gamma(t, x(t), v(t)) \tag{21}$$

where $t \in \mathbb{T}$ and v is an external control. Recall us that control law (21) is regular if and only if matrix $\frac{\partial \gamma}{\partial v}$ is nonsingular. Let us assume

A3. there exists (t, x, U) such that matrix $\frac{\partial \gamma}{\partial v}$ has a full rank.

Theorem 5.3. *Let us assume that conditions (A1)-(A3) are fulfilled. If time-variant systems Σ and $\tilde{\Sigma}$ are locally equivalent with respect to the regular static state feedback transformations given by (21) then they are locally equivalent with respect to the external dynamic anticipating feedback transformations.*

Proof: Applying a regular static state feedback law $u(t) = \gamma(t, x(t), v(t))$ to the system Σ we obtain the system

$$\tilde{\Sigma}: \begin{array}{c} \tilde{x}^{\Delta}(t) = f(t, \tilde{x}(t), \gamma(t, x(t), v(t))) \\ \tilde{y}(t) = h(t, \tilde{x}(t), \gamma(t, x(t), v(t))) \end{array}$$

where $\tilde{x}(\cdot) := x(\cdot)$. The corresponding external feedback transformations of Σ into $\tilde{\Sigma}$ are of the form:

$$\tilde{\phi}^e(Y, U) := \tilde{Y}$$

and

$$\tilde{\psi}^e(Y, U) = \tilde{\psi}^e(Y, \Gamma(t, x, v)) = \tilde{\psi}^e(t, \mathcal{S}_1(Y, U), V)$$

where Γ is the extension of the map γ. Invertibility of these transformations follow from the assumption A3. $\square$

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