

EFFECT OF ALL QUADRATIC TERMS ON THIRD ORDER OSCILLATION

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ABSTRACT. In this paper, we consider non-linear perturbation for the third order linear oscillator. Using the center manifold reduction , we reduced system into second order differential equation. Then we study the Hopf bifurcation of the system and stability of the periodic orbits. At the end we classify the effect of 2-jet of parametric perturbation.

AMS Classification: 34C23

Keywords: Center manifold, Bifurcation, Stability, Periodic Solution

1. INTRODUCTION

Oscillatory systems are one of the most intensely studied systems in nonlinear dynamics. It serves as a basic model for self-excited oscillations in physics, electronics, biology, neurology and other disciplines [11,18,3,4,...]. Parametric perturbations of oscillatory systems are one of the most studied parts of nonlinear dynamics; but few of them concern on three dimensional oscillators. The differential equation

$$(1.1) \quad \ddot{x} + \omega^2 x = 0$$

is known as the three dimensional linear oscillator. It describes an object moving on an ellipse in three dimensional space, so all the solutions are periodic. The natural question is therefore "do periodic solutions persists under (non-)linear parametric perturbations?" This question, in its general form, has been well answered in recent studies [6,7,8,9,10,30,20], as authors considered general perturbation of an oscillatory system and presented methods guaranteeing existence of periodic solutions. However, because of the characteristic of three dimensional space, many of these methods are not applicable for the three dimensional linear oscillator. For example,

normally non degeneracy conditions of the periodic manifold is not hold for this equation, or, the symmetry conditions are too trivial. On the other hand, standard methods like, *Normal Forms*, *Averaging method*,... do not reduce the system into a simpler form. One of the earlier studies on the perturbed three dimensional linear oscillator was done by Mehri and Niksirat [20]. They considered the nonlinear equation $\dot{\ddot{X}} = F(X, \dot{X}, \ddot{X})$, $X \in R^{2n+1}$ and showed that ,under some condition ,there exists $\omega_0 > 0$ such that $X(\omega_0) = X(0)$. Then, they considered the nonlinear third-order differential equation

$$(1.2) \quad \ddot{x} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}),$$

and obtained some conditions based on homotopy methods for the existence of a periodic solution [19]. After that, Rabiei and Afsharnejad in [22] considered the equation (1.2) and prove that if the map $F(x, \dot{x}, \ddot{x})$ is the sum of the partial derivatives of a map $f(x, \dot{x})$ with $\frac{\partial^2 f}{\partial x \partial \dot{x}}(0, 0) \neq 0$, then (1.2) is reduced to a two dimensional system with an extra parameter which has infinitely many periodic orbits. Also, they considered the case $f(x, \dot{x}) = ax^2 + bx\dot{x} + c\dot{x}^2$ and imposed conditions on a, b and c , such that the system has infinitely many homoclinic orbits and periodic solutions. Then by [23], they extend the concept of [22] for non-autonomous third order differential equation

$$(1.3) \quad \ddot{x} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}, t).$$

They showed that for

$$\begin{aligned} F(x, \dot{x}, \ddot{x}, t) &= H_x(x, \dot{x}, t, \epsilon)\dot{x} + H_{\dot{x}}(x, \dot{x}, t, \epsilon)\ddot{x} + H_t(x, \dot{x}, t, \epsilon), \\ H(x, \dot{x}, t, \epsilon) &= f(x, \dot{x}) + \epsilon g(x, \dot{x}, t), \end{aligned}$$

the equation (1.3) is reduced to a two dimensional equation which has infinitely many periodic solutions and nontrivial bounded invariant sets. Then they turned to the polynomial perturbations, specially the case that

$$F(x, \dot{x}, \ddot{x}) = A_1 x + A_2 \dot{x} + A_3 \ddot{x} + A_4 x \dot{x} + A_5 \dot{x} \ddot{x} + A_6 x \ddot{x} + A_7 x^2 + A_8 \dot{x}^2 + A_9 \ddot{x}^2$$

contains linear and quadratic terms (of forcing, damping, and velocity,...). behavior of polynomial vector fields has been studied in [21,5]. But chaotic behavior in autonomous systems requires a phase-space dimension to be equal or greater than three; and nonlinearities in the model equation. For polynomial vector fields the problem "What are the most elementary functional forms of three-dimensional dynamical system that do exhibit chaotic behavior?" has been attacked in several recent publications and seems to be related not only to the number and degree of nonlinear monomials but also to the total number of appearing terms. Zhang and Heidel [29,16] have investigated the possibility of chaotic dynamic in system with only quadratic nonlinearities. They found out that quadratic system with a total of less than five terms on the right-hand side of the corresponding three order differential equations can not exhibit chaotic behavior. Their work was motivated by former studies of Sprott [26]. A further simplification was achieved in a resent study [12] by transforming the Sprott models to explicit third-order ordinary differential equations, so-called jerky dynamics, $\dot{\ddot{x}} = J(x, \dot{x}, \ddot{x})$. If $F(x, \dot{x}, \ddot{x})$ in equation (1.2) is

an polynomial then we have an jerky dynamics. Bifurcations of this kind of vector fields are important. Also we know that some kind of bifurcation cause chaotic motion. In another study, Rabiei and Afsharnejad [25] considered that quadratic terms which dose not contain derivative of any $f(x, \dot{x})$, that is

$$\ddot{x} + \omega^2 \dot{x} = Ax^2 + B\dot{x}^2 + C\ddot{x}^2,$$

where in despite of the title, they investigated conditions guaranteeing lake of pe-riodic solutions. Also, they studied the role of the iterative derivatives of the displacement function on constitution of periodic solutions, the type of the stability and global bifurcation of the system. After that, they showed that for

$$F(x, \dot{x}, \ddot{x}, t) = Ax^2 + B\dot{x}^2 + C\ddot{x}^2 + \varepsilon e(t)$$

the system has at least two periodic solutions. Now, in this paper we consider the map $F(x, \dot{x}, \ddot{x})$ with all linear and quadratic terms, that is to say

$$\ddot{x} + \omega^2 \dot{x} = A_1 x + A_2 \dot{x} + A_3 \ddot{x} + A_4 x \dot{x} + A_5 \dot{x} \ddot{x} + A_6 x \ddot{x} + A_7 x^2 + A_8 \dot{x}^2 + A_9 \ddot{x}^2,$$

¹ Then we can put

$$\bar{x}(t) = \frac{1}{\omega} x\left(\frac{t}{\omega}\right), a_1 = \frac{A_1}{\omega^3}, a_2 = \frac{A_2}{\omega^2}, a_3 = \frac{A_3}{\omega}, a_4 = \frac{A_4}{\omega},$$

$$a_5 = A_5 \omega, a_6 = A_6, a_7 = \frac{A_7}{\omega^2}, a_8 = A_8, a_9 = A_9 \omega^2,$$

so after dropping the bars, we obtain

$$(1.4) \ddot{x} + \dot{x} = a_1 x + a_2 \dot{x} + a_3 \ddot{x} + a_4 x \dot{x} + a_5 \dot{x} \ddot{x} + a_6 x \ddot{x} + a_7 x^2 + a_8 \dot{x}^2 + a_9 \ddot{x}^2$$

or system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -\dot{x} + a_1 x + a_2 \dot{x} + a_3 \ddot{x} + a_4 x \dot{x} + a_5 \dot{x} \ddot{x} + a_6 x \ddot{x} + a_7 x^2 + a_8 \dot{x}^2 + a_9 \ddot{x}^2. \end{aligned}$$

The importance of the subject is not introducing new methods or concepts, *but it is another necessary step for classifying effects of the quadratic terms* on three dimensional linear oscillator which appears in many physical phenomena [3,4,24]. Furthermore, because of the complexity of the formula, computing of the formula is a very long process such that we can say manual computation and simplification of the formula are almost impossible. This makes us that throughout this manuscript, all formulas derive by using computer softwares in order to avoid the manual computation mistakes. Also for simplifying, the computation proceed has been omitted in order to write the final form of the formulas. Our main goal here is to bring the system to its standard form and provide conditions for the system to have a Hopf bifurcation. Then we use the center manifold theorem and reduce this system to the two dimension system and study the resonance situations for equation(1.2) under the perturbations.

Therefore in section 2, we give those tools which we need for after sections. section 3, is devoted to consider the center manifold and conditions for Hopf bifurcation for

¹We can assume that $A_i = \mu \bar{A}_i$, $i = 1, \dots, 9$

our system. At the end in section 4, investigated different bifurcation in reduced system.

2. PRELIMINARY

Consider the parameterized vector field $\dot{x} = f(x, \mu)$, $x \in R^n$, $\mu \in R^p$ where f is a C^r , $r \geq 1$ function. Suppose this vector field has a fixed point at $(x, \mu) = (x_0, \mu_0)$, i.e., $f(x_0, \mu_0) = 0$. If for every μ close to μ_0 (and for x close to x_0) we have new dynamical behavior, for example, fixed point are created or destroyed and time-dependent behavior such as periodic, quasiperiodic, or even chaotic dynamic are created. Then the point (x_0, μ_0) is called a bifurcation point of the vector field. since Hopf bifurcation can provide us with information about periodic solution of the system $\dot{x} = f(x, \mu)$. Therefore in this section we briefly explain this theorem. Also center manifold theorem play an important rule in our paper. Hence we will give the theorem.

Consider $\dot{x} = f(x, \mu)$, $x \in R^n$, $\mu \in R$. The Hopf bifurcation theorem describe conditions that we can have periodic solutions for $\mu \neq 0$. There are several versions of this theorem [17, 15, 27, 28] but we give the versions mentioned in (17).

Theorem 2.1. *Suppose that $\dot{x} = f_\mu(x)$, $x \in R^n$, $\mu \in R$ has an equilibrium (x_0, μ_0) at which the following properties are satisfied:*

(H1) $D_x f_{\mu_0}(x_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

then (H1) implies that there is a smooth curve of equilibria $(x(\mu), \mu)$ with $x(\mu_0) = x_0$. The eigenvalues $\lambda(\mu)$, $\bar{\lambda}(\mu)$ of $D_x f_{\mu_0}(x(\mu))$ which are imaginary at $\mu = \mu_0$ vary smoothly with μ . If, moreover,

(H2)

$$\frac{d}{d\mu}(\text{Re}\lambda(\mu))|_{\mu=\mu_0} = d \neq 0,$$

then there is a unique three-dimensional center manifold passing through (x_0, μ_0) in $R^n \times R$ and a smooth system of coordinates (preserving the planes $\mu = \text{const.}$) for which the Taylor expansion of degree 3 on the center manifold is given by :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} (d\mu + a(x^2 + y^2))x - (\omega + c\mu + b(x^2 + y^2))y \\ \omega + c\mu + b(x^2 + y^2)x + (d\mu + a(x^2 + y^2))y \end{pmatrix}$$

If $a \neq 0$, there is a surface of periodic solution in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid $\mu = -(a/d)(x^2 + y^2)$. If $a < 0$, then this periodic solutions are stable limit cycles, while if $a > 0$, the periodic solutions are repelling.

In a two-dimensional system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0, -\omega \\ \omega, 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix},$$

with $f(0) = g(0) = 0$ and $Df(0) = Dg(0) = 0$, the formula for a is

$$a = \frac{1}{16}[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \frac{1}{16\omega}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}],$$

where f_{xy} denotes $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$, etc. Condition (H2) at above theorem can be inspect by the following proposition.

Proposition 2.2. *Let $A(\alpha)$ be a parameter-dependent real $(n \times n)$ -matrix which has a simple pair of complex eigenvalues $\lambda_{1,2} = \sigma(\alpha) \pm i\omega(\alpha)$, $\sigma(0) = 0$, $\omega(0) > 0$. then, we have*

$$\frac{d\sigma}{d\alpha}(0) = \text{Re} \langle p, A'(0) \rangle$$

where $p, q \in C^n$ satisfy;

$$A(0)q = i\omega(0)q, A^T(0)p = -i\omega(0)p, \langle p, q \rangle = 1.$$

For proof see [28].

Now we discuss and state the center manifold theorem, which provides a means for systematically reducing the dimension of the state spaces which need not to be considered when analyzing bifurcations of a given type. In the case of the Hopf bifurcation, it enables a reduction to two dimensions without losing any information concerning stability.

Theorem 2.3. *Let f be a C^r vector field on R^n vanishing at the origin ($f(0) = 0$). Let $A = Df(0)$. Divide the spectrum of A into three parts $\sigma_s, \sigma_c, \sigma_u$ with*

$$\text{Re} \lambda = \begin{cases} < 0 & \text{if } \lambda \in \sigma_s, \\ = 0 & \text{if } \lambda \in \sigma_c, \\ > 0 & \text{if } \lambda \in \sigma_u. \end{cases}$$

Let the (generalized) eigenspaces of σ_s, σ_c and σ_u be E^s, E^c and E^u respectively. Then there exist C^r stable and unstable invariant manifolds W^u and W^s tangent to E^u and E^s at 0 and a C^{r-1} center manifold W^c tangent to E^c at 0. The manifolds W^u, W^s and W^c are all invariant for the flow of f . The stable and unstable manifolds are unique, but W^c need not be.

Since the center manifold is tangent to E^c we can represent $W^c(0)$ locally by

$$\{(x, y, z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u \mid y = h_1(x), z = h_2(x), h_i(0) = Dh_i(0) = 0, i = 1, 2\}$$

We can replace above relation into essential differential equation and using invariance of center manifold, then obtain reduced system. For more details refer (17).

3. CENTER MANIFOLD FOR EQUATION (1.4)

The differential equation (1.4) has two equilibrium points, origin and $(-\frac{a_1}{a_7}, 0, 0)$. First, we consider equilibrium point $(-\frac{a_1}{a_7}, 0, 0)$.

The matrix of linearized system (1.4) at $(-\frac{a_1}{a_7}, 0, 0)$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -\beta & -\alpha \end{pmatrix}$$

where $-\beta = -1 + a_2 - \frac{a_1 a_4}{a_7}$, $-\alpha = a_3 - \frac{a_1 a_5}{a_7}$. The characteristic equation of the matrix is $\lambda^3 + \alpha\lambda^2 + \beta\lambda + a_1 = 0$.

In order to have Hopf bifurcation we need to have complex eigenvalue of the characteristic equation. For this, substitute $\lambda = i\omega$, in it. This shows the characteristic equation has a pair of purely imaginary roots $\lambda_{1,2} = \pm i\omega$, $\omega > 0$, if $\frac{a_1}{\alpha} = \beta = \omega^2$ i.e

$$\frac{\frac{a_1}{a_7} - a_3}{\frac{a_1}{a_7}} = 1 - a_2 + \frac{a_1 a_4}{a_7}, 1 - a_2 + \frac{a_1 a_4}{a_7} > 0$$

In this case the third eigenvalue is $\lambda_3 = -\frac{a_1}{\beta}$. Under transversality condition and $a_1 \neq 0$, we will have two dimensional center manifold. We reduced system on center manifold and determined conditions of stability for the periodic solutions. For this, We translated $(-\frac{a_1}{a_7}, 0, 0)$ to $(0, 0, 0)$ and, in order to write the above matrix in Jordan form; we use the following transformation

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} -\frac{a_1(a_1+\omega^3)}{2(a_1^2+\omega^6)} & -\frac{1}{2\omega} & \frac{\omega(-a_1+\omega^3)}{2(a_1^2+\omega^6)} \\ \frac{a_1(-a_1+\omega^3)}{2(a_1^2+\omega^6)} & \frac{1}{2\omega} & \frac{\omega(a_1+\omega^3)}{2(a_1^2+\omega^6)} \\ -\frac{\omega^4}{a_1^2+\omega^6} & 0 & -\frac{\omega^2}{a_1^2+\omega^6} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Here α, β are bifurcation parameters. Now we applied the translation $\bar{\alpha} = \alpha - \frac{a_1}{\omega^2}$, $\bar{\beta} = \beta - \omega^2$. For avoiding the complexity, we set $\omega = 1$, $\bar{\beta} = 0$. We omitted the bars so, obtain standard form:

$$(3.1) \quad \dot{X} = A(\alpha)X + F(x, y, z, \alpha)$$

where $X = (x, y, z, \alpha)^T$ and $F(x, y, z, \alpha) = (f(x, y, z, \alpha), g(x, y, z, \alpha), h(x, y, z, \alpha), 0)^T$ where

$$\begin{aligned} f(x, y, z, \alpha) &= \frac{1}{2(1+a_1^2)}((1-a_1)((x+y+z)(x-y-za_1)a_4 \\ &\quad - (x+y+z)(x+y-za_1^2)a_5 + (-x+y+za_1)(x+y-za_1^2)a_6 \\ &\quad + (x+y+z)^2a_7 + (-x+y+za_1)^2a_8 + (x+y-za_1^2)^2a_9 \\ g(x, y, z, \alpha) &= \frac{1}{2(1+a_1^2)}((1+a_1)((x+y+z)(x-y-za_1)a_4 \\ &\quad - (x+y+z)(x+y-za_1^2)a_5 + (-x+y+za_1)(x+y-za_1^2)a_6 \\ &\quad + (x+y+z)^2a_7 + (-x+y+za_1)^2a_8 + (x+y-za_1^2)^2a_9 \\ h(x, y, z, \alpha) &= -\frac{1}{2(1+a_1^2)}((x+y+z)(x-y-za_1)a_4 \\ &\quad + (x+y+z)(x+y-za_1^2)a_5 - (-x+y+za_1)(x+y-za_1^2)a_6 \\ &\quad + (x+y+z)^2a_7 + (-x+y+za_1)^2a_8 + (x+y-za_1^2)^2a_9 \end{aligned}$$

and

$$A(\alpha) = \begin{pmatrix} \frac{\alpha(-1+a_1)}{2(1+a_1^2)} & -1 + \frac{\alpha(-1+a_1)}{2(1+a_1^2)} & -\frac{\alpha(-1+a_1)a_1^2}{2(1+a_1^2)} & 0 \\ 1 - \frac{\alpha(1+a_1)}{2(1+a_1^2)} & -\frac{\alpha(1+a_1)}{2(1+a_1^2)} & \frac{\alpha(-1+a_1)a_1^2}{2(1+a_1^2)} & 0 \\ \frac{\alpha}{1+a_1^2} & \frac{\alpha}{1+a_1^2} & -a_1 + \alpha(-1 + \frac{1}{1+a_1^2}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The linear part of the equation (3.1) At $\alpha = 0$ is

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus from center manifold theory, the stability and bifurcation of $(-\frac{a_1}{a_7}, 0, 0)$ near $\alpha = \frac{-a_1}{\omega^2}$ can be determined by studying a one-parameter family of first-order ordinary differential equation on a center manifold, which can be represented as a graph over the x, y, α variables, i.e., $W^c(0)$ is equal to $\{(x, y, z, \alpha) \in \mathbb{R}^4 \mid z = K(x, y, \alpha), K(0, 0, 0) = DK(0, 0, 0) = 0, \text{ for } x, y, \alpha \text{ sufficiently small}\}$ where

$$K(x, y, \alpha) = bx^2 + cy^2 + d\alpha^2 + exy + g\alpha + h\alpha^2 + o(3);$$

First we study transversality condition in Hopf bifurcation theorem. Consider $\lambda_{1,2} = \sigma(\alpha) \pm i\omega(\alpha)$ with $\sigma(0) = 0, \omega(0) > 0$ are eigenvalues of $A(\alpha)$ in equation (3.1). Differentiating from $A(\alpha)$ at $\alpha = 0$,

$$A'(0) = \begin{pmatrix} \frac{-1+a_1}{2(1+a_1^2)} & \frac{-1+a_1}{2(1+a_1^2)} & -\frac{(-1+a_1)a_1^2}{2(1+a_1^2)} \\ -\frac{1+a_1}{2(1+a_1^2)} & \frac{1+a_1}{2(1+a_1^2)} & \frac{(1+a_1)a_1^2}{2(1+a_1^2)} \\ \frac{1}{1+a_1^2} & \frac{1}{1+a_1^2} & -1 + \frac{1}{1+a_1^2} \end{pmatrix}$$

Then by using proposition (2.2) we have

$$\sigma'(0) = -\frac{a_1}{1+a_1^2}$$

therefore, we have transversality condition for $a_1 \neq 0$. The coefficient of center manifold are:

$$b = -(a_1^2 a_4 - 4a_5 + 2a_1 a_5 - a_1^2 a_5 - a_1^2 a_6 + 4a_7 - 2a_1 a_7 + a_1^2 a_7 + 4a_8 + 2a_1 a_8 + a_1^2 a_8 + 4a_9 - 2a_1 a_9 + a_1^2 a_9) / (a_1(1+a_1^2)(4+a_1^2));$$

$$c = (a_1^2 a_4 + 4a_5 + 2a_1 a_5 + a_1^2 a_5 - a_1^2 a_6 - 4a_7 - 2a_1 a_7 - a_1^2 a_7 - 4a_8 + 2a_1 a_8 - a_1^2 a_8 - 4a_9 - 2a_1 a_9 - a_1^2 a_9) / (a_1(1+a_1^2)(4+a_1^2))$$

$$e = 2(-2a_4 + 2a_6 + a_1(a_5 - a_7 + a_8 - a_9)) / (4 + 5a_1^2 + a_1^4)$$

$$d = 0, g = (-1 + a_1) / (1 + a_1^2)^2, h = (1 + a_1) / (1 + a_1^2)^2$$

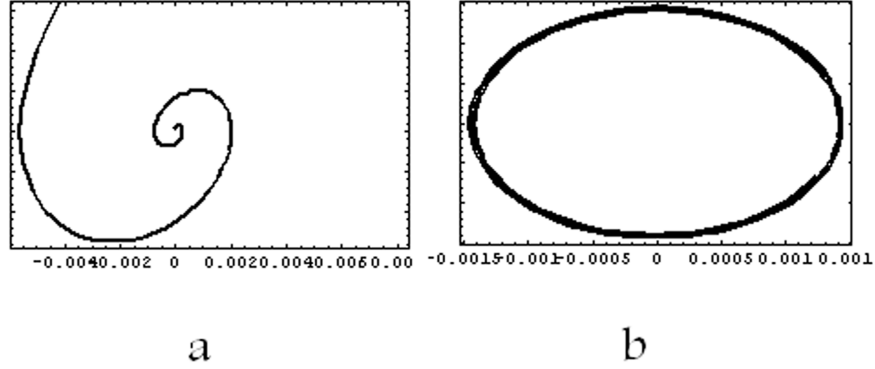


FIGURE 1. In (a) $a_1 = 1, a_2 = a_4 = a_5 = a_6 = a_8 = a_9 = 0, a_3 = -1.9$. In (b) $a_1 = 1, a_2 = a_4 = a_5 = a_6 = a_8 = a_9 = 0, a_3 = -1.01$

Finally, substituting $z = K(x, y, \alpha)$ into (3.1), we obtain the reduced vector field:

$$(3.2) \quad \begin{aligned} \dot{x} &= -y + (-1 + a_1)(Axy + Bx\alpha + By\alpha - Cx^2 + Dy^2) + O(3) \\ \dot{y} &= x + (1 + a_1)(-Axy - Bx\alpha - By\alpha + Cx^2 - Dy^2) + O(3) \end{aligned}$$

where

$$\begin{aligned} A &= \frac{(a_5 - a_7 + a_8 - a_9)}{1 + a_1^2} \\ B &= \frac{1}{2(1 + a_1^2)} \\ C &= \frac{(a_4 - a_5 - a_6 + a_7 + a_8 + a_9)}{1 + a_1^2} \\ D &= \frac{(a_4 + a_5 - a_6 - a_7 - a_8 - a_9)}{1 + a_1^2} \end{aligned}$$

Let a_p denote the Stability of the periodic solutions of the perturbed system (1.4). By the formula in the Hopf theorem, the a_p is equal to

$$\begin{aligned} &(-a_1 a_4^2 + 3a_1 a_5^2 - a_1 a_4 a_6 - a_1^2 a_5 a_6 + 2a_1 a_6^2 + a_1^2 a_4 a_7 - a_1 a_5 a_7 \\ &- 2a_1 a_7^2 + a_1^2 a_4 a_8 + 3a_1 a_5 a_8 + 2a_1^2 a_6 a_8 - 2a_1 a_7 a_8 - 4a_1 a_8^2 - a_1^2 a_4 a_9 \\ &- 11a_1 a_5 a_9 + 2a_1^2 a_6 a_9 + 6a_1 a_7 a_9 - 4a_1 a_8 a_9 + 8a_1 a_9^2)/(4(1 + a_1^2)(4 + a_1^2)) \\ &+ (-6a_4 a_5 + a_5 a_6 + 5a_4 a_7 + 5a_4 a_8 - 2a_6 a_8 + 7a_4 a_9 - 2a_6 a_9)/(2(1 + a_1^2)(4 + a_1^2)) \\ &+ (-2a_5^2 + 6a_5 a_7 - 4a_7^2 + 2a_5 a_8 - 4a_7 a_8 + 2a_5 a_9 - 4a_7 a_9)/a_1(1 + a_1^2)(4 + a_1^2). \end{aligned}$$

Figure 1 shows create (b) and damage (a) of periodic solution.

Similarly, in order the origin goes under Hopf bifurcation, we consider the linearized the equation (3.1) at origin. The matrix of the linearization at origin has

the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -\delta & -\gamma \end{pmatrix}$$

with $-\delta = -1 + a_2$, $-\gamma = a_3$ and with characteristic equation $\lambda^3 + \gamma\lambda^2 + \delta\lambda - a_1 = 0$. Similar to case $(-\frac{a_1}{a_7}, 0, 0)$ for Hopf bifurcation for the system (1.4) at the origin, we should have $\delta = -\frac{a_1}{\gamma}$, $\delta > 0$. The same way the transformation implies following standard form,

$$(3.3) \quad \dot{X} = A(\gamma)X + F(x, y, z, \gamma)$$

where $X = (x, y, z, \gamma)^T$ and $F(x, y, z, \gamma) = (f(x, y, z, \gamma), g(x, y, z, \gamma), h(x, y, z, \gamma), 0)^T$ whit

$$\begin{aligned} f(x, y, z, \gamma) &= \frac{1}{2(1+a_1^2)}((1+a_1)((x+y-z)(x-y-za_1)a_4 \\ &- (x+y-z)(x+y+za_1^2)a_5 + (-x+y+za_1)(x+y+za_1^2)a_6 \\ &+ (x+y-z)^2a_7 + (-x+y+za_1)^2a_8 + (x+y+za_1^2)^2a_9. \end{aligned}$$

$$\begin{aligned} g(x, y, z, \gamma) &= \frac{1}{2(1+a_1^2)}((1-a_1)((x+y-z)(x-y-za_1)a_4 \\ &- (x+y-z)(x+y+za_1^2)a_5 + (-x+y+za_1)(x+y+za_1^2)a_6 \\ &+ (x+y-z)^2a_7 + (-x+y+za_1)^2a_8 + (x+y+za_1^2)^2a_9. \end{aligned}$$

$$\begin{aligned} h(x, y, z, \gamma) &= \frac{1}{2(1+a_1^2)}((x+y-z)(x-y-za_1)a_4 \\ &- (x+y-z)(x+y+za_1^2)a_5 + (-x+y+za_1)(x+y+za_1^2)a_6 \\ &+ (x+y-z)^2a_7 + (-x+y+za_1)^2a_8 + (x+y+za_1^2)^2a_9. \end{aligned}$$

and

$$A(\gamma) = \begin{pmatrix} -\frac{\gamma(1+a_1)}{2(1+a_1^2)} & -1 - \frac{\gamma(1+a_1)}{2(1+a_1^2)} & -\frac{\gamma(1+a_1)a_1^2}{2(1+a_1^2)} & 0 \\ 1 + \frac{\gamma(-1+a_1)}{2(1+a_1^2)} & \frac{\gamma(-1+a_1)}{2(1+a_1^2)} & \frac{\gamma(-1+a_1)a_1^2}{2(1+a_1^2)} & 0 \\ -\frac{\gamma}{1+a_1^2} & -\frac{\gamma}{1+a_1^2} & a_1 + \gamma(-1 + \frac{1}{1+a_1^2}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, by proposition (2.2)

$$\sigma'(0) = \frac{-1}{2(1+a_1^2)} \neq 0$$

So there is a center manifold for equation (3.3) which can locally be represented at origin as $W^c(0) = \{(x, y, z, \gamma) \in \mathbb{R}^4 \mid z = H(x, y, \gamma), H(0, 0, 0) = DH(0, 0, 0) = 0, \text{ for } x, y, \gamma \text{ sufficiently small}\}$ where

$$H(x, y, \gamma) = bx^2 + cy^2 + d\gamma^2 + exy + gx\gamma + hy\gamma + o(3);$$

With

$$\begin{aligned}
b &= (-a_1^2 a_4 + 4a_5 + 2a_1 a_5 + a_1^2 a_5 + a_1^2 a_6 - 4a_7 - 2a_1 a_7 - a_1^2 a_7 \\
&\quad - 4a_8 + 2a_1 a_8 - a_1^2 a_8 - 4a_9 - 2a_1 a_9 - a_1^2 a_9)/(a_1(1 + a_1^2)(4 + a_1^2)); \\
c &= (a_1^2 a_4 + 4a_5 - 2a_1 a_5 + a_1^2 a_5 - a_1^2 a_6 - 4a_7 + 2a_1 a_7 - a_1^2 a_7 - 4a_8 \\
&\quad - 2a_1 a_8 - a_1^2 a_8 - 4a_9 + 2a_1 a_9 - a_1^2 a_9)/(a_1(1 + a_1^2)(4 + a_1^2)); \\
e &= 2(2a_4 - 2a_6 + a_1(a_5 - a_7 + a_8 - a_9))/(4 + 5a_1^2 + a_1^4); \\
d &= 0, g = (1 + a_1)/(1 + a_1^2)^2, h = (-1 + a_1)/(1 + a_1^2)^2;
\end{aligned}$$

Thus the reduced system is

$$\begin{aligned}
(3.4) \quad \dot{x} &= -y + (1 + a_1)(-Axy - Bx\gamma - By\gamma + Cx^2 - Dy^2) + O(3) \\
\dot{y} &= x + (1 - a_1)(-Axy - Bx\gamma - By\gamma + Cx^2 - Dy^2) + O(3)
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{(a_5 - a_7 + a_8 - a_9)}{1 + a_1^2} \\
B &= \frac{1}{2(1 + a_1^2)} \\
C &= \frac{(a_4 - a_5 - a_6 + a_7 + a_8 + a_9)}{1 + a_1^2} \\
D &= \frac{(a_4 + a_5 - a_6 - a_7 - a_8 - a_9)}{1 + a_1^2}
\end{aligned}$$

and stability of the periodic solution is

$$\begin{aligned}
a_o &= (a_1 a_4^2 - 3a_1 a_5^2 + a_1 a_4 a_6 - a_1^2 a_5 a_6 - 2a_1 a_6^2 + a_1^2 a_4 a_7 + a_1 a_5 a_7 \\
&\quad + 2a_1 a_7^2 + a_1^2 a_4 a_8 - 3a_1 a_5 a_8 + 2a_1^2 a_6 a_8 + 2a_1 a_7 a_8 + 4a_1 a_8^2 - a_1^2 a_4 a_9 \\
&\quad + 11a_1 a_5 a_9 + 2a_1^2 a_6 a_9 - 6a_1 a_7 a_9 + 4a_1 a_8 a_9 - 8a_1 a_9^2)/(4(1 + a_1^2)(4 + a_1^2)) \\
&\quad + (-6a_4 a_5 + a_5 a_6 + 5a_4 a_7 + 5a_4 a_8 - 2a_6 a_8 + 7a_4 a_9 - 2a_6 a_9)/(2(1 + a_1^2)(4 + a_1^2)) \\
&\quad + (+2a_5^2 - 6a_5 a_7 + 4a_7^2 - 2a_5 a_8 + 4a_7 a_8 - 2a_5 a_9 + 4a_7 a_9)/a_1(1 + a_1^2)(4 + a_1^2)
\end{aligned}$$

4. BIFURCATION OF THE LIMIT CYCLES FOR EQUATION (1.4)

In this section we used the obtained results in section 3 in order to prove the existence of the limit cycle(may be attractor). By using a_p and a_o we determined the stability condition for the periodic orbits. we arrange a_p, a_o with respect to a_7 and then solve algebraic equation $a_p = 0, a_o = 0$, we obtain a_p is zero at

$$a_7 = \frac{U + V \pm \sqrt{N + M}}{2(16 + 2a_1^2)}$$

with

$$\begin{aligned}
U &= 10a_1a_4 + a_1^3a_4 - 2a_1^2a_8, V = +24a_5 - a_1^2a_5 - 16a_8 + 6a_1^2a_9 \\
M &= 36a_1^2a_4^2 + 12a_1^4a_4^2 + a_1^6a_4^2 + 64a_5^2 + 80a_1^2a_5^2 + 25a_1^4a_5^2 \\
&\quad - 64a_1^2a_4a_6 - 8a_1^4a_4a_6 + 128a_1^2a_6^2 + 16a_1^4a_6^2 - 256a_5a_8 + 192a_1^2a_5a_8 \\
&\quad + 28a_1^4a_5a_8 + 256a_8^2 - 192a_1^2a_8^2 - 28a_1^4a_8^2 - 256a_5a_9 - 320a_1^2a_5a_9 \\
&\quad - 100a_1^4a_5a_9 + 512a_8a_9 - 384a_1^2a_8a_9 - 56a_1^4a_8a_9 + 256a_9^2 + 320a_1^2a_9^2 \\
&\quad + 100a_1^4a_9^2 \\
N &= -288a_1a_4a_5 - 68a_1^3a_4a_5 - 2a_1^5a_4a_5 + 128a_1a_5a_6 - 48a_1^3a_5a_6 \\
&\quad - 8a_1^5a_5a_6 + 320a_1a_4a_8 + 72a_1^3a_4a_8 + 4a_1^5a_4a_8 - 256a_1a_6a_8 \\
&\quad + 96a_1^3a_6a_8 + 16a_1^5a_6a_8 + 576a_1a_4a_9 + 136a_1^3a_4a_9 + 4a_1^5a_4a_9 \\
&\quad - 256a_1a_6a_9 + 96a_1^3a_6a_9 + 16a_1^5a_6a_9
\end{aligned}$$

Also, coefficient of term a_7^2 is $-\frac{8+a_1^2}{2a_1(1+a_1^2)(4+a_1^2)}$. Therefore we have the following theorem

Theorem 4.1. Consider the equation (1.4).

I) At point $(\frac{-a_1}{a_7}, 0, 0)$, For $|\alpha|$ small enough and a_7 between two values $\frac{U+V+\sqrt{N+M}}{2(16+2a_1^2)}$ and $\frac{U+V-\sqrt{N+M}}{2(16+2a_1^2)}$, a unique unstable limit cycle and for a_7 outside this two values a unique stable limit cycle (attractor) bifurcates from the equilibrium via the Hopf bifurcation. This means that at $a_7 = \frac{U+V\pm\sqrt{N+M}}{2(16+2a_1^2)}$ a bifurcation occurs for the system (1.4).

II) With the same way from $a_0 = 0$, we have

$$a_7 = \frac{-U + V \pm \sqrt{-N + M}}{2(16 + a_1^2)}$$

and the coefficient of a_7^2 is equal to $\frac{8+a_1^2}{2a_1(1+a_1^2)(4+a_1^2)}$. At the origin, for $|\gamma|$ small enough and a_7 between two values $\frac{-U+V+\sqrt{-N+M}}{2(16+2a_1^2)}$ and $\frac{-U+V-\sqrt{-N+M}}{2(16+2a_1^2)}$, a unique stable limit cycle (attractor) and for a_7 outside this two values a unique unstable limit cycle bifurcates from the equilibrium via the Hopf bifurcation. This means that for $a_7 = \frac{-U+V\pm\sqrt{-N+M}}{2(16+2a_1^2)}$ a bifurcation occurs for the system (1.4).

Proof : By theorem (2.1) in order to know about stability and instability of the limit cycles we should look at the sign of a_0 and a_p . In fact we should check in with regions we have negative or positive sign of a_0 and a_p . By using the above results for a_0 and a_p , the proof of the theorem will be completed.

At the end we consider the satiation at which we have Hopf bifurcation appear at origin and $(-\frac{a_1}{a_7}, 0, 0)$ simultonance at the value of the parameters.

Theorem 4.2. Under following condition Hopf bifurcation occurred at the same value of parameters at two equilibrium point $(\frac{-a_1}{a_7}, 0, 0)$ and origin.

$$1)a_4 = a_7(-\frac{1}{a_3} + \frac{a_7}{a_1 a_5 - a_3 a_7})$$

$$2)a_2 = \frac{-a_1 + a_3}{a_3}$$

$$3)\frac{a_1 a_7}{a_1 a_5 - a_3 a_7} > 0, \frac{a_1}{a_3} > 0$$

Proof : We solve the equation $\frac{a_1}{\frac{a_1 a_5}{a_7} - a_3} = 1 - a_2 + \frac{a_1 a_4}{a_7}, 1 - a_2 + \frac{a_1 a_4}{a_7} > 0$ and $-1 + a_2 = \frac{a_1}{-a_3}, 1 - a_2 > 0$ where these equation are condition for occurrence of Hopf bifurcation. The solutions of these equation implies the condition 1-3.

Note that a_p, a_o can be written as

$$\begin{aligned} a_p &= a_1 A_1 + a_1^2 A_2 + \frac{1}{a_1} A_3 + A_3 \\ a_o &= -a_1 A_1 + a_1^2 A_2 - \frac{1}{a_1} A_3 + A_3 \end{aligned}$$

where

$$\begin{aligned} A_1 &= (-a_4^2 + 3a_5^2 - a_4 a_6 + 2a_6^2 - a_5 a_7 \\ &\quad - 2a_7^2 + 3a_5 a_8 - 2a_7 a_8 - 4a_8^2)/(4(1 + a_1^2)(4 + a_1^2)) \\ A_2 &= (-a_5 a_6 + a_4 a_7 + a_4 a_8 + 2a_6 a_8 - a_4 a_9)/(4(1 + a_1^2)(4 + a_1^2)) \\ A_3 &= (-2a_5^2 + 6a_5 a_7 - 4a_7^2 + 2a_5 a_8 - 4a_7 a_8 \\ &\quad + 2a_5 a_9 - 4a_7 a_9)/(2(1 + a_1^2)(4 + a_1^2)) \\ A_4 &= (-6a_4 a_5 + a_5 a_6 + 5a_4 a_7 + 5a_4 a_8 - 2a_6 a_8 \\ &\quad + 7a_4 a_9 - 2a_6 a_9)/(1 + a_1^2)(4 + a_1^2) \end{aligned}$$

By these explanation we have following proposition.

Proposition 4.3. .

1) If $A_1 = A_3 = 0$ then ,stability of periodic solution at two equilibrium point are the same.

2) If $A_2 = A_4 = 0$ then ,stability of periodic solution at two equilibrium point are opposite.

3) For $|a_1| \ll 1$ stability of periodic solution at two equilibrium point are opposite .

proof: At $A_1 = A_3 = 0$ ($A_2 = A_4 = 0$), we have the same (opposite) sing for a_o and a_p . So the periodic solutions at two point have the same (opposite) stability. Also for $|a_1| \ll 1$, term $\frac{1}{a_1} A_3$ determined the sing of a_o and a_p . Therefore we have opposite stability.

Conclusion and Classification

In this paper we consider behavior of the nonlinear perturbed of third order linear oscillation

$$(4.1) \quad \ddot{x} + \omega^2 \dot{x} = 0$$

We verified the effect of terms 2-jet of $C^r, r \geq 2$ function $F(x, \dot{x}, \ddot{x})$ on the linear oscillation (4.1). We have find the center manifold for the perturbed linear oscillator. Using the center manifold as the reduced equation (1.4) which is second

order differential equation (see Eq.s(3.2),(3.4)). We have find out that under some conditions for coefficient the system (1.4) goes under Hopf bifurcation. This implies that the perturbed system have periodic solutions. Also we have considered the stability(attractor) and instability of these periodic orbits. Moreover, we have shown that under spacial values of some parameters Hopf bifurcation occurs at the equilibriums origin and point $(\frac{-a_1}{a_7}, 0, 0)$ Simultaneously. At the end we classify our results under effect of all terms of 2-jet of function $F(x, \dot{x}, \ddot{x})$. The classification of $a_i, i = 1, 2, \dots, 9$ of equation (1.4) is:

- 1)The effect of nonzero coefficients a_1, a_2 and a_3 implies thst system (1.4) goes under Hopf bifurcation at the origin.
- 2)The nonzero coefficients $a_i, i = 1, \dots, 7$ implies the Hopf bifurcation for the system (1.4) at point $(\frac{-a_1}{a_7}, 0, 0)$.
- 3)Stability and instability of periodic orbits (at origin and $(\frac{-a_1}{a_7}, 0, 0)$) depends only coefficients $a_i, i = 4, \dots, 9$.
- 4)The nonzero coefficients a_1 and a_7 (coefficients x and x^2) provide us non-origin equilibrium.
- 5)If $a_1 \equiv 0$ and $a_4 \neq 0$ the system (1.4) has many periodic solutions see [22]. But in our work we can ignore these condition and have the same result.

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