

DYNAMICALLY DEFINED TOPOLOGICAL PRESSURE

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ABSTRACT. Dynamically defined topological pressure as an extension of the notion of topological pressure for a non compact metric space with a semi-dynamics is presented. The semi-dynamics of the state space is deduced by a continuous map with a base. Dynamically defined topological pressure from two viewpoints is considered. It is proved that the dynamically defined topological pressure is invariant under special kind of conjugate relations. The properties of pressure for dynamically defined topological pressure are proved.

AMS Classification: 37A35, 37B99

Keywords: Dynamically Defined Topological Pressure, Spanning Set, Separating Set, Topological Entropy

1. INTRODUCTION

The notion of topological pressure was introduced by Ruelle [4] and extended by Walters [5]. We would like to extend this notion for non-compact metric spaces by using of a semi-dynamics created by a continuous map with a base. More precisely we assume that (X, d) is a metric space and $f : X \rightarrow X$ is a continuous map with a base K [2], i.e. K is a compact subset of X and for all compact set $C \subset X$ there exists a natural number $m(C)$ such that $f^n(K) \supseteq C$ for all $n \geq m(C)$.

We also assume that the semi-dynamics of the space created by a continuous map

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES

VOL. 6, NUMBER 1 (2008) 75-81.

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$g : X \rightarrow X$. In the next sections we will extend the notion of topological pressure by using of f and g .

The reader may ask why we need such an extension? In fact when we consider a discrete dynamical system created by a diffeomorphism h with a hyperbolic saddle point p , then the unstable manifold of p denoted by $W^u(p)$, in many cases is a non-compact metric space, and $f = h^{-1}|_{W^u(p)}$ is a continuous map with a base. Dynamically defined topological pressure of f and $g = h|_{W^u(p)}$ is a new object associated to h which we will prove that: it is invariant under special kind of conjugate relations.

2. BASIC NOTIONS

Let $m, n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then a subset $R_{f,\varepsilon}^{m,n}$ of $f^m(K)$ is called an (m, n, f, ε) spanning set if for given $x \in f^m(K)$ there exists $y \in R_{f,\varepsilon}^{m,n}$ so that $d(g^i(x), g^i(y)) < \varepsilon$ for all $0 \leq i \leq n-1$. The minimal cardinality of (m, n, f, ε) spanning sets is a natural number and denoted by $r(m, n, f, \varepsilon)$.

Let $C(f^m(K), R)$ be the Banach algebra of real valued continuous functions on $f^m(K)$ with the supremum norm. For $u \in C(f^m(K), R)$ and $n \in \mathbb{N}$, the summation $\sum_{i=0}^{n-1} \delta_{f^m(K)}^{g^i(x)} u(g^i(x))$ denoted by $(S_{n,m}u)(x)$, where

$$\delta_{f^m(K)}^{g^i(x)} = \begin{cases} 1 & \text{if } g^i(x) \in f^m(K) \\ 0, & \text{if } g^i(x) \notin f^m(K). \end{cases}$$

We denote the infimum of the set

$$\left\{ \sum_{x \in D} e^{(S_{n,m}u)(x)} : D \text{ is an } (m, n, f, \varepsilon) \text{ spanning set} \right\}$$

by $Q_{m,n,f}(g, u, \varepsilon)$.

The $\limsup_{n \rightarrow \infty} \frac{1}{n} \log Q_{m,n,f}(g, u, \varepsilon)$ denoted by $Q(m, f, g, u, \varepsilon)$.

We denote the $\lim_{\varepsilon \rightarrow 0} Q(m, f, g, u, \varepsilon)$ by $P(m, f, g, K, u)$.

Let $C(X, R)$ be the set of real valued continuous functions on X .

Definition 2.1 The mapping $P(f, g, K, \cdot) : C(X, R) \rightarrow R \cup \{\infty\}$ defined by

$P(f, g, K, u) = \limsup_{m \rightarrow \infty} P(m, f, g, K, u)$ is the dynamically defined topological pressure of u .

One can deduce that if f is the identity map and X is a compact set then $P(f, g, X, u)$ is the topological pressure of u . So Definition 2.1 is a natural generalization of the

notion of topological pressure [6].

Now we present a separating approach.

Theorem 2.1 If $u \in C(f^m(K), R)$ and $\varepsilon < \varepsilon'$ are two positive real numbers then:

- (a) $Q_{m,n,f}(g, u, \varepsilon) \geq Q_{m,n,f}(g, u, \varepsilon')$.
- (b) $Q_{m,n,f}(g, u, \varepsilon) \leq \|e^{S_{n,m}u}\| r(m, n, f, \varepsilon)$.
- (c) $Q_{m,n,f}(g, 0, \varepsilon) = r(m, n, f, \varepsilon)$.

Proof. (a) Since

$$\left\{ \sum_{x \in D} e^{(S_{n,m}u)(x)} : D \text{ is an } (m, n, f, \varepsilon) \text{ spanning set} \right\}$$

is a subset of

$$\left\{ \sum_{x \in D} e^{(S_{n,m}u)(x)} : D \text{ is an } (m, n, f, \varepsilon') \text{ spanning set} \right\}$$

then

$$Q_{m,n,f}(g, u, \varepsilon) \geq Q_{m,n,f}(g, u, \varepsilon').$$

(b) Let D_0 be an (m, n, f, ε) spanning set such that $|D_0| = r(m, n, f, \varepsilon)$. Then

$$Q_{m,n,f}(g, u, \varepsilon) \leq \sum_{x \in D_0} e^{(S_{n,m}u)(x)} \leq \|e^{S_{n,m}u}\| |D_0|.$$

(C) $Q_{m,n,f}(g, 0, \varepsilon) = \inf\{|D| : D \text{ is an } (m, n, f, \varepsilon) \text{ spanning set}\}$
 $= r(m, n, f, \varepsilon). \square$

The straightforward calculations show that if X is compact set and f is the identity map, then $P(id, g, X, 0)$ is the topological entropy of g [1, 3]. We now present another approach to the notion of dynamically defined topological pressure.

Let $S_{f,\varepsilon}^{m,n} \subseteq f^m(K)$ be a set such that for all $x, y \in S_{f,\varepsilon}^{m,n}$ there exists $0 \leq i \leq n-1$ so that $d(g^i(x), g^i(y)) \geq \varepsilon$. Then $S_{f,\varepsilon}^{m,n}$ is called an (m, n, f, ε) separating set. The maximal cardinal number of (m, n, f, ε) separating sets is a natural number and denoted by $s(m, n, f, \varepsilon)$. If $u \in C(f^m(K), R)$ and $\varepsilon > 0$ then we have the following notations:

$$E_{m,n,f}(g, u, \varepsilon) = \sup\left\{ \sum_{x \in E} e^{(S_{n,m}u)(x)} : E \text{ is an } (m, n, f, \varepsilon) \text{ separating set} \right\}$$

and

$$E(m, f, g, u, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log E_{m,n,f}(g, u, \varepsilon).$$

Theorem 2.2 $E(m, f, g, u, \varepsilon) \geq Q(m, f, g, u, \varepsilon)$.

Proof. Let E be a subset of $f^m(K)$ such that $|E| = s(m, n, f, \varepsilon)$. Then for given $x \in f^m(K)$ the maximality of E implies that: there is $y \in E$ such that $d(g^i(x), g^i(y)) < \varepsilon$. So E is an (m, n, f, ε) spanning set. Hence $E_{m,n,f}(g, u, \varepsilon) \geq Q_{m,n,f}(g, u, \varepsilon)$. Thus $E(m, f, g, u, \varepsilon) \geq Q(m, f, g, u, \varepsilon)$. $\square$

Theorem 2.3 Let $\varepsilon > \varepsilon' > 0$, and let δ be a positive number such that $d(x, y) < \varepsilon$ implies $|u(x) - u(y)| < \delta$. Then $\delta + Q(m, f, g, u, \varepsilon') \geq E(m, f, g, u, 2\varepsilon)$.

Proof. Let D and E be two (m, n, f, ε') spanning and $(m, n, f, 2\varepsilon)$ separating sets respectively. Then we can associate to a given $x \in E$ a unique $y \in D$ such that the set $\{t \in f^m(K) : d(g^i(y), g^i(t)) < \varepsilon, i = 1, 2, \dots, n-1\}$ has no any other point of E . We y by $\varphi(x)$. Thus we have a one to one map φ from E to D . So

$$\begin{aligned} \sum_{y \in D} e^{(S_{n,m}u)(y)} &\geq \sum_{x \in E} e^{(S_{n,m}u)(\varphi(x))} \\ &\geq (\min_{x \in E} e^{(S_{n,m}u)(\varphi(x)) - (S_{n,m}u)(x)}) \sum_{x \in E} e^{(S_{n,m}u)(x)} \\ &\geq e^{-n\delta} \sum_{x \in E} e^{(S_{n,m}u)(x)}. \end{aligned}$$

Hence

$$e^{n\delta} Q_{m,n,f}(g, u, \varepsilon') \geq E_{m,n,f}(g, u, \varepsilon).$$

Thus

$$\delta + Q(m, f, g, u, \varepsilon') \geq E(m, f, g, u, 2\varepsilon). \quad \square$$

Theorem 2.2 and theorem 2.3 imply that

$$P(m, f, g, K, u) = \lim_{\varepsilon \rightarrow 0} E(m, f, g, u, \varepsilon).$$

3. MAIN RESULTS

We assume that (X, d) is a metric space and $f_1 : X \rightarrow X$ is a continuous map with a base K . Moreover we assume that $f_2 : X \rightarrow X$ is a continuous map conjugate to f_1 , i.e. there is a homeomorphism $\psi : X \rightarrow X$ such that $f_2 \circ \psi = \psi \circ f_1$. The topological conjugacy implies that f_2 is a map with the base $\psi(K)$.

Theorem 3.1 Let $g : X \rightarrow X$ be a continuous map and let $\psi(K) = K$. Then

$$P(f_1, g, K, u) = P(f_2, g, K, u).$$

Proof. Let $D \subseteq f_1^m(K)$ be a given (m, n, f, ε) separating set. Since K is a base for f_1 then there exists m_2 such that

$$f_1^m(K) \subseteq f_2^{m_2}(\psi(K)) = f_2^{m_2}(K).$$

Hence

$$E_{m,n,f_1}(g, u, \varepsilon) \leq E_{m_2,n,f_2}(g, u, \varepsilon).$$

So

$$E(m, f_1, g, u, \varepsilon) \leq E(m_2, f_2, g, u, \varepsilon).$$

Thus

$$P(m, f_1, g, K, u) \leq P(m_2, f_2, g, K, u).$$

Therefore

$$P(f_1, g, K, u) \leq P(f_2, g, K, u).$$

Now by replacing f_1 with f_2 we have

$$P(f_2, g, K, u) \leq P(f_1, g, K, u).$$

Thus

$$P(f_1, g, K, u) = P(f_2, g, K, u). \quad \square$$

We now consider $P(f, g, K, \cdot)$ as a pressure map. In fact we prove properties of dynamically defined topological pressure which show that: it has the properties of pressure.

We assume that (X, d) is a metric space, and $C(X, R)$ is the set of continuous functions from X to R . Moreover we assume $f : X \rightarrow X$ and $g : X \rightarrow X$ are continuous maps and f has a base K .

Theorem 3.2 If $u, v \in C(X, R)$ and $c \in R$ then:

- (i) if $u \leq v$ then $P(f, g, K, u) \leq P(f, g, K, v)$.
- (ii) $P(m, f, g, K, \cdot)$ is a real number or it is identically ∞ .
- (iii) If $c \geq 0$ then $P(f, g, K, u + c) \leq P(f, g, K, u) + c$.
- (iv) $P(f, g, K, u + v) \leq P(f, g, K, u) + P(f, g, K, v)$.
- (v) $P(f, g, K, cu) \leq cP(f, g, K, u)$ if $c \geq 1$,
and $P(f, g, K, cu) \geq cP(f, g, K, u)$ if $c \leq 1$.
- (vi) $|P(f, g, K, u)| \leq P(f, g, K, |u|)$.

Proof. (i) This follows from the fact

$$S_{n,m}u \leq S_{n,m}v.$$

- (ii) If $u_0 = u|_{f^m(K)}$, $a = \inf\{u_0(x), 0 \mid x \in f^m(K)\}$ and $b = \sup\{u_0(x), 0 \mid x \in f^m(K)\}$ then the inequalities

$$na \leq (S_{n,m}u)(x) \leq nb,$$

imply

$$r(m, n, f, \varepsilon)e^{na} \leq Q_{m,n,f}(g, u, \varepsilon) \leq s(m, n, f, \varepsilon)e^{nb}.$$

So

$$h(m, f, g) + a \leq P(m, f, g, K, u) \leq h(m, f, g) + b.$$

Thus

$P(m, f, g, K, u)$ is a real number if and only if $h(m, f, g)$ is a real number.

(iii) Since

$$(S_{n,m}(u + c))(x) \leq (S_{n,m}u)(x) + nc$$

then

$$Q_{m,n,f}(g, u + c, \varepsilon) \leq Q_{m,n,f}(g, u, \varepsilon)e^{nc}.$$

Hence

$$P(m, f, g, K, u + c) \leq P(m, f, g, K, u) + c.$$

Thus

$$P(f, g, K, u + c) \leq P(f, g, K, u) + c.$$

(iv) Since

$$S_{n,m}(u + v) = S_{n,m}(u) + S_{n,m}(v)$$

then

$$Q_{m,n,f}(g, u + v, \varepsilon) \leq Q_{m,n,f}(g, u, \varepsilon)Q_{m,n,f}(g, v, \varepsilon).$$

Thus

$$P(f, g, K, u + v) \leq P(f, g, K, u) + P(f, g, K, v).$$

(v) This follows from the following inequalities.

If D is a finite subset of X then

$$\sum_{x \in D} e^{c(S_{n,m}u)(x)} \leq \left(\sum_{x \in D} e^{(S_{n,m}u)(x)} \right)^c \text{ if } c \geq 1$$

and

$$\sum_{x \in D} e^{c(S_{n,m}u)(x)} \geq \left(\sum_{x \in D} e^{(S_{n,m}u)(x)} \right)^c \text{ if } c \leq 1.$$

(vi) This is a consequence of parts (i) and (v) and the inequalities $-|u| \leq u \leq |u|$. $\square$

4. CONCLUSION

When we have a hyperbolic fixed point p for a diffeomorphism h on a manifold M , with at least one negative eigenvalue, then the restriction of h on unstable manifold of p as the state space X is a continuous map with a base. The interesting case will happen if we put $f = h^{-1}|_X$ and $g = h|_X$. So the study of dynamically defined topological pressure in the case $f = g^{-1}$ can be a reasonable problem for further research.

Acknowledgement : This research has been supported financially by a research grant of Mahani Mathematical Research Center.

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