

DYNAMICALLY DEFINED TOPOLOGICAL ENTROPY

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ABSTRACT. In this paper by using a special kind of dynamics on a metric space, the notion of topological entropy is developed. The independence of "dynamically defined topological entropy" of the choice of base and semi-dynamics of the space, up to a topological conjugacy is proved. A relation between dynamically defined topological entropy of a map with its iterations is deduced.

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1. BASIC NOTIONS

The notion of topological entropy and its extensions [1, 2, 3, 4, 5, 6, 9, 10] play an important role in dynamical systems. This notion for relative semi-dynamical systems [8] has been extended in [7].

In the study of homoclinic points of a vector field or a diffeomorphism on a compact

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manifold in many cases we accept with non-compact unstable manifolds (see figure 1).

When we restrict the inverse dynamics on this manifold then the notion of topological entropy can be a good candidate in the direction of stability of dynamics. This notion can be useful in the case of hetroclinic points. So the extension of the notion of topological entropy for non-compact metric spaces by use of a semi-dynamics can be a reasonable topic for a further research.

For this goal we assume that (X, d) is a metric space and $f : X \rightarrow X$ is a continuous map with a base K . In fact a compact subset K of X is called a base of f [5] if for all compact set $C \subset X$ there exists a natural number $m(C)$ such that $f^n(K) \supseteq C$ for all $n \geq m(C)$.

The mapping f creates the semi-dynamical system $(X, \{f^n : n \in \mathbf{N}\})$. We would like to extend the notion of topological entropy for this semi-dynamical system. For this purpose let $g : X \rightarrow X$ be a continuous map. For given $m \in \mathbf{N}$ and $\varepsilon > 0$ we assume that $S_{f,\varepsilon}^m \subseteq f^m(K)$ is a set such that for all $x, y \in S_{f,\varepsilon}^m$ there is $0 \leq i \leq m-1$ so that $d(g^i(x), g^i(y)) \geq \varepsilon$. In this case $S_{f,\varepsilon}^m$ is called an (m, f, ε) separating set.

Theorem 1.1 For each $m \in \mathbf{N}$ and $\varepsilon > 0$ there is $k \in \mathbf{N}$ such that the cardinality of $S_{f,\varepsilon}^m$ is smaller than k .

Proof. Since g is a continuous map then there is $\varepsilon > \delta > 0$ such that $d(x, y) < \delta$ implies that $d(g^i(x), g^i(y)) < \varepsilon$ for all $0 \leq i \leq m-1$. Let k be the cardinal number of a finite subcover Θ of the open cover $\{B(x; \frac{\delta}{2}) : x \in X\}$ for the compact set $f^m(K)$. Since each member of Θ has at most one member of $S_{f,\varepsilon}^m$, then $|S_{f,\varepsilon}^m| < k$. $\square$

The previous theorem implies that there is a real number which is the maximal cardinal number of such $S_{f,\varepsilon}^m$, and we denote it by $s(m, f, \varepsilon)$.

Let $m \in \mathbf{N}$ and $\varepsilon > 0$ be given. Moreover let $R_{f,\varepsilon}^m \subseteq f^m(K)$ be a set such that for given $x \in f^m(K)$ there exists $y \in R_{f,\varepsilon}^m$ so that $d(g^i(x), g^i(y)) < \varepsilon$ for all $0 \leq i \leq m-1$. In this case $R_{f,\varepsilon}^m$ is called an (m, f, ε) spanning set.

Theorem 1.2 Let $S_{f,\varepsilon}^m$ be a set such that $|S_{f,\varepsilon}^m| = s(m, f, \varepsilon)$. Then it is an (m, f, ε) spanning set.

Proof. Let $x \in f^m(K)$ be given. Then there is $y \in S_{f,\varepsilon}^m$ such that $d(g^i(x), g^i(y)) < \varepsilon$ for all $0 \leq i \leq m-1$. Because if there is no such y then $S_{f,\varepsilon}^m \cup \{x\}$

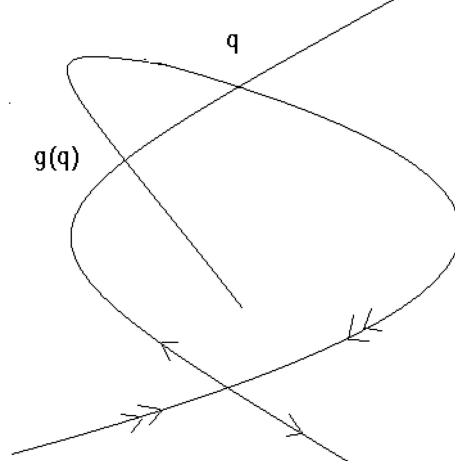


FIGURE 1. The dynamics of $g = f^{-1}$ on the unstable manifold of a hyperbolic fixed point of f .

is an (m, f, ε) separating set with the cardinality greater than $s(m, f, \varepsilon)$ which is a contradiction. $\square$

We denote the minimal cardinality of (m, f, ε) spanning sets by $r(m, f, \varepsilon)$. The previous theorem implies that $r(m, f, \varepsilon) \leq s(m, f, \varepsilon)$.

Theorem 1.3 If $\varepsilon > \varepsilon' > 0$, then $r(m, f, \varepsilon') \geq s(m, f, 2\varepsilon)$.

Proof. Let $R_{f, \varepsilon'}^m$ and $S_{f, 2\varepsilon}^m$ be two spanning and separating sets such that $|R_{f, \varepsilon'}^m| = r(m, f, \varepsilon')$ and $|S_{f, 2\varepsilon}^m| = s(m, f, 2\varepsilon)$. For given $x \in S_{f, 2\varepsilon}^m$ there is $y \in R_{f, \varepsilon'}^m$ such that the set $\{z \in X : d(g^i(y), g^i(z)) < \varepsilon' \text{ for all } i \in \{0, 1, 2, 3, \dots, m-1\}\}$ has no any other point of $S_{f, 2\varepsilon}^m$. Moreover if we replace x with $x \neq x' \in S_{f, 2\varepsilon}^m$ then the corresponding $y' \in R_{f, \varepsilon'}^m$ with the above property is not equal to y . Hence $|S_{f, 2\varepsilon}^m| \leq |R_{f, \varepsilon'}^m|$. $\square$

The previous theorem implies: if $\varepsilon > \varepsilon' > 0$ then

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log(s(m, f, \varepsilon')) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} \log(s(m, f, \varepsilon)) \text{ and} \\ \limsup_{m \rightarrow \infty} \frac{1}{m} \log(r(m, f, \varepsilon')) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} \log(r(m, f, \varepsilon)). \text{ Thus:}$$

$$\lim_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \frac{1}{m} \log(s(m, f, \varepsilon)) = \lim_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \frac{1}{m} \log(r(m, f, \varepsilon)).$$

This common limit denoted by $h_K^f(g)$, and it is called the dynamically defined topological entropy of g with the semi-dynamics f and the base K .

2. PROPERTIES OF DYNAMICALLY DEFINED TOPOLOGICAL ENTROPY

In this section we also assume that (X, d) is a metric space. Two continuous maps $f_1 : X \rightarrow X$ and $f_2 : X \rightarrow X$ are called topologically conjugate if there is a homeomorphism $\gamma : X \rightarrow X$ such that $f_2 \circ \gamma = \gamma \circ f_1$.

Theorem 2.1 If f_1 and f_2 are two conjugate continuous maps and f_1 is a map with the base K . Then f_2 is a map with the base $\gamma(K)$.

Proof. Let a compact set $C \subseteq X$ be given. Then $\gamma^{-1}(C)$ is a compact subset of X , and there is a natural number $N(\gamma^{-1}(C))$ such that $f_1^n(K) \supseteq \gamma^{-1}(C)$ for all $n \geq N(\gamma^{-1}(C))$. Thus for $n \geq N(\gamma^{-1}(C))$ we have

$$f_2^n \circ \gamma(K) = \gamma \circ f_1^n(K) \subseteq C.$$

Thus $\gamma(K)$ is a base for f_2 . $\square$

Theorem 2.2 If f_1 and f_2 are two conjugate continuous maps on X with basis K and L respectively then

$$h_K^{f_1}(g) = h_L^{f_2}(g)$$

where $g : X \rightarrow X$ is a continuous map.

Proof. At first we show that if K and L are two basis of f_1 then $h_K^{f_1}(g) = h_L^{f_1}(g)$. Since there is $l \in \mathbb{N}$ such that $L \subseteq f_1^l(K)$, then for all $m \in \mathbb{N}$, $f_1^m(L) \subseteq f_1^{l+m}(K)$. So $f_1^m(L) \subseteq f_1^{l(m+1)}(K)$ for all $m \in \mathbb{N}$. Thus $s_K(l(m+1), f_1, \varepsilon) \geq s_L(m, f_1, \varepsilon)$ where $\varepsilon > 0$. So $h_K^{f_1}(g) \geq \frac{1}{l} h_L^{f_1}(g)$. Hence $h_K^{f_1}(g) \geq h_L^{f_1}(g)$.

Similarly one can deduce that $h_L^{f_1}(g) \geq h_K^{f_1}(g)$. So $h_K^{f_1}(g) = h_L^{f_1}(g)$.

Now to complete the proof it is enough to show that

$$h_K^{f_1}(g) = h_{\gamma(K)}^{f_2}(g)$$

where $\gamma : X \rightarrow X$ is a homeomorphism such that $f_2 \circ \gamma = \gamma \circ f_1$. This follows from this point that if $S_{f_1, \varepsilon}^m$ and $R_{f_1, \varepsilon}^m$ are (m, f_1, ε) separating and spanning sets respectively then since γ is a homeomorphism then $\gamma(S_{f_1, \varepsilon}^m)$ and $\gamma(R_{f_1, \varepsilon}^m)$ are (m, f_2, ε') separating and spanning sets and $\varepsilon \rightarrow 0$ if and only if $\varepsilon' \rightarrow 0$. $\square$

By the use of previous theorem we can use of the notation $h^f(g)$ instead of $h_K^f(g)$. One can attention to this point that when X is a compact metric space and i is the identity mapping on it, then $h^i(g)$ is the usual topological entropy of g .

Theorem 2.3 If f is a mapping with a base and l is a natural number, then $h^f(g^l) = lh^f(g)$.

Proof. If K is a base for f and $|S_{f,\varepsilon}^m| = s_{g^l}(m, f, \varepsilon)$ then for all $x, y \in S_{f,\varepsilon}^m$ there is $0 \leq i \leq m-1$ such that $d(g^{li}(x), g^{li}(y)) \geq \varepsilon$. This implies that $d(g^j(x), g^j(y)) \geq \varepsilon$ for some $0 \leq j \leq lm-1$. Thus $s_{g^l}(m, f, \varepsilon) \geq s_g(lm, f, \varepsilon)$. Hence $\frac{1}{l}h_K^f(g^l) \geq h_K^f(g)$. If $R_{f,\varepsilon}^{lm} = r_g(lm, f, \varepsilon)$ then for given $x \in f^{lm}(K)$ there is $y \in R_{f,\varepsilon}^{lm}$ such that $d(g^j(x), g^j(y)) < \varepsilon$ for all $i \in \{0, 1, 2, \dots, lm-1\}$. This implies that $d(g^{li}(x), g^{li}(y)) < \varepsilon$ for all $0 \leq i \leq m-1$. Thus $r_{g^l}(m, f, \varepsilon) \leq r_g(lm, f, \varepsilon)$. Hence $\frac{1}{l}h_K^f(g^l) \leq h_K^f(g)$. Thus $h_K^f(g^l) = lh_K^f(g)$. So $h^f(g^l) = lh^f(g)$. $\square$

3. AN EXAMPLE OF DYNAMICALLY DEFINED TOPOLOGICAL ENTROPY

Let $g : \mathbf{R} \rightarrow \mathbf{R}$ be the mapping $g(x) = 4x$. Moreover let $f : \mathbf{R} \rightarrow \mathbf{R}$ be the mapping $f(x) = 3x$. We would like to calculate $h_K^f(g)$ where $K = [-1, 1]$.

Let $\varepsilon > 0$, and m , be given, and let $S_{f,\varepsilon}^m \subseteq f^m(K)$ be a set such that $|S_{f,\varepsilon}^m| = s(m, f, \varepsilon)$. Then for given $x, y \in S_{f,\varepsilon}^m$ there is $0 \leq i \leq m-1$ such that $|x - y|4^{m-1} \geq \varepsilon$. Let $f^m(u) = x$ and $f^m(v) = y$ where $u, v \in K$. Then $3^m|u - v|4^{m-1} \geq \varepsilon$. If we choose a partition of K with the length $\frac{\varepsilon}{3^m \cdot 4^{m-1}}$, then $s(m, n, f, \varepsilon) \frac{\varepsilon}{3^m \cdot 4^{m-1}} \leq 2$. So $s(m, f, \varepsilon) \leq \frac{2 \cdot 3^m \cdot 4^{m-1}}{\varepsilon}$. Thus $\frac{1}{m} \log s(m, f, \varepsilon) \leq \frac{1}{m} \log \left(\frac{2 \cdot 3^m \cdot 4^{m-1}}{\varepsilon} \right) = \frac{m-1}{m} \log 4 - \frac{1}{m} \log \varepsilon + \frac{m}{m} \log 3 + \frac{1}{m} \log 2$. Hence $\limsup_{m \rightarrow \infty} \frac{1}{m} \log s(m, f, \varepsilon) \leq \log 4 + \log 3$. So $h^f(g) \leq \log 4 + \log 3$.

If $R_{f,\varepsilon}^m \subseteq f^m(K)$ is a set such that $|R_{f,\varepsilon}^m| = r(m, f, \varepsilon)$, then for all $x \in f^m(K)$ there is $y \in R_{f,\varepsilon}^m$ such that $d(g^i(x), g^i(y)) \leq \varepsilon$ for all $0 \leq i \leq m-1$. So $4^{m-1}|x - y| < \varepsilon$. There exist $u, v \in K$ such that $x = f^m(u)$, and $y = f^m(v)$. Thus $3^m \cdot 4^{m-1}|u - v| < \varepsilon$. So $r(m, f, \varepsilon) \frac{\varepsilon}{3^m \cdot 4^{m-1}} \geq 2$. Hence $\limsup_{m \rightarrow \infty} \frac{1}{m} \log r(m, f, \varepsilon) \geq \lim_{m \rightarrow \infty} \frac{1}{m} \log \frac{2 \cdot 3^m \cdot 4^{m-1}}{\varepsilon} = \log 4 + \log 3$.

Thus $h^f(g) = \log 4 + \log 3$.

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